Exam 4, Sample, Solutions

1(a) $\sum_{n=1}^{\infty} 2^{n+1} x^n$

Apply the ratio test

$$\left|\frac{2^{n+2}\, x^{n+1}}{2^{n+1}\, x^n}\right| = 2|x| \to 2|x|$$

Converges if $2|x|<1$, $|x|<\frac{1}{2}$ $\qquad R = \frac{1}{2}$

diverges if $2|x|>1$, $|x|>\frac{1}{2}$.

Endpoints: $\underline{x=\frac{1}{2}}$ $\quad \sum_{n=1}^{\infty} 2^{n+1}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 2$, diverges since $2 \not\to 0$

$\underline{x=-\frac{1}{2}}$ $\quad \sum_{n=1}^{\infty} 2^{n+1}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n 2$; diverges $(-1)^n 2 \not\to 0$.

Interval $= \left(-\frac{1}{2}, \frac{1}{2}\right)$

1(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Apply the ratio test

$$\left|\frac{(n+1)!\, x^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!\, x^n}\right| = |x|\left(\frac{n}{n+1}\right)^n \to \frac{|x|}{e}$$

converges if $|x|<e$

diverges if $|x|>e$.

$R=e$.

The interval of convergence is too hard to find.

I won't ask you such a hard problem!

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n+1}$

Ratio test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2n+3} \times \frac{2n+1}{x^n} \right| = \lim_{n\to\infty} |x| \frac{2n+1}{2n+3} = |x|$$

$$\left.\begin{array}{l} \text{Converges if } |x|<1 \\ \text{diverges if } |x|>1 \end{array}\right\} R = 1$$

Endpoints:

$x=1$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$ is divergent, compare to a p-series $p=1$.

$x=-1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ is an alternating series.

① $\frac{1}{2n+1}$ is decreasing ② $\frac{1}{2n+1} \to 0$

Converges by A.S.T.

② $f(x) = \dfrac{2+x}{2-x}$

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

$$\frac{2+x}{2-x} = 2\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{x^n}{2^n} + \sum_{n=0}^{\infty} \underbrace{\frac{x^{n+1}}{2^{n+1}}}_{\text{n+1 starts from 1}}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{2^n} + \sum_{n=1}^{\infty} \frac{x^n}{2^n} = 1 + \sum_{n=\cancel{0}}^{\infty} 2\frac{x^n}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{2^{n-1}}$$

Since a geometric series $\frac{1}{1-\frac{x}{2}}$ has a radius $= 2$ and interval $= (-2,2)$ we see that $R = 2$ and & the interval $= (-2,2)$.

(3) $\sum_{n=0}^{\infty} \frac{n}{1+n^2} x^n$

Ratio test

$$\left| \frac{n+1}{1+(n+1)^2} x^{n+1} \times \frac{x^n (1+n^2)}{n} \right| = |x| \frac{n+1}{n} \frac{1+n^2}{1+(n+1)^2}.$$

$$\Longrightarrow |x| \qquad \left.\begin{array}{l}\text{converges if } |x|<1 \\ \text{diverges if } |x|>1\end{array}\right\} R = 1$$

End points: $\underline{x = 1}$ $\quad \sum_{n=0}^{\infty} \frac{n}{1+n^2}$ divergent compare to $\sum \frac{1}{n}$

$\underline{x = -1}$ $\quad \sum_{n=0}^{\infty} \frac{n}{1+n^2} (-1)^n$ Alternating series

(1) $\lim_{n\to\infty} \frac{n}{1+n^2} = 0$

(2) decreasing $\frac{d}{dx} \frac{x}{1+x^2} = \frac{(1+x^2) - x(2x)}{(1+x^2)^2}$

$$= \frac{1-x^2}{(1+x^2)^2} < 0 \text{ if } x > 1.$$

Converges by AST

Interval $= [-1, 1)$

$$f'(x) = \frac{d}{dx} \sum_{0}^{\infty} \frac{n}{1+n^2} x^n = \sum_{0}^{\infty} \frac{n^2 x^{n-1}}{1+n^2}$$

Radius in the same $= 1$

Endpoints: $\underline{x=1}$ $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$ diverges, $\frac{n^2}{1+n^2} \to 1 \neq 0$

$\underline{x=-1}$ $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2} (-1)^n$ diverges, $\frac{n^2}{1+n^2} \not\to 0$

$$F(x) = \sum_{n=0}^{\infty} \int \frac{n}{1+n^2} x^n = C + \sum_{n=0}^{\infty} \frac{n}{(1+n^2)} \frac{x^{n+1}}{(n+1)}$$

$R = 1$

Endpoint: $x = 1$ $\sum_{n=1}^{\infty} \frac{n}{(n+1)(1+n^2)}$ is convergent compare with $\frac{1}{n^2}$.

$\underline{x=-1}$ $\sum_{n=1}^{\infty} \frac{n \; (-1)^{n+1}}{(n+1)(1+n^2)}$ converges (absolutely) compare with $\frac{1}{n^2}$.

(or use $x=1$).

(4) $f(x) = \sin(2x)$

$$T_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \quad \text{(from the Maclaurin series)}$$

$$\text{Error} = \frac{M|x|^{n+1}}{(n+1)!} \quad \text{where} \quad M = \max_{-x \le t \le x} |f^{(n+1)}(t)|$$

$$|f^{(n+1)}(x)| = |2^{n+1} \sin(2x)| \quad \text{or} \quad |2^{n+1} \cos(2x)|.$$

So $M \le 2^{n+1}$

$$\text{Error} \le 2^{n+1} \frac{|x|^{n+1}}{(n+1)!}$$

$\sin(1) = \sin(2 \times \frac{1}{2})$ so $x = \frac{1}{2}$.

$$\text{Error} \le \frac{2^{n+1} \left(\frac{1}{2}\right)^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}$$

when using T_8 we have $n = 8$, $\text{Error} \le \frac{1}{9!}$

4 decimal places is 0.0001.

$$\text{Error} = \frac{1}{(n+1)!} < 0.0001 = \frac{1}{10{,}000}$$

$10{,}000 < (n+1)!$ which is true for $n+1 = 8$ $(8! \simeq 40{,}000)$

So $T_7(x)$ works as approximant.