MATH 170, SECTION 01, EXAM 3
Friday, November 20th, 2009

Name (PRINT clearly):

- Do not start the test until instructed to do so.
- No calculators.
- You will receive full credit only if you show all work and explain all steps.
- If you run out of space, please ask the instructor for additional sheets of paper. Number the sheets clearly and indicate the problem that is being solved. Scratch work will not be graded.
- The maximum possible score is **150**. Point values are indicated on the questions.
- The time limit is **50** minutes.
- There are a total of **6** problems.

Honor pledge

"I pledge on my honor that I have neither given nor received unauthorized aid on this assignment"

Sign here:

1. Write $2.\overline{135}$ as a ratio of integers. [10 points]

$$2.\overline{135} = 2 + \frac{135}{10^3} + \frac{135}{10^6} + \cdots$$

$$= 2 + \frac{135}{10^3} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{10^3}\right)^{n-1}$$

$$= 2 + \frac{135}{10^3} \cdot \frac{1}{1 - \frac{1}{10^3}} = 2 + \frac{135}{999} = \frac{2,133}{999} = \frac{79}{37}.$$

$$135 = 3^3 \times 5 \quad , \quad 999 = 3^3 \cdot 37$$

$$2 + \frac{5}{37} = \frac{79}{37}.$$

2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{3}{n^2+4n+3}$. [**20** points]

$$\frac{3}{(n+3)(n+1)} = \left(-\frac{1}{n+3} + \frac{1}{n+1}\right)\frac{3}{2}$$

$$s_n = \frac{3}{2}\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+3}\right) = \frac{3}{2}\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right] \to \frac{\cancel{3}}{2}\left[\frac{5}{\cancel{6}\,\cancel{3}\,2}\right] = \frac{5}{\cancel{4}}$$

$$\sum_{n=1}^{\infty} \frac{3}{n^2+4n+3} = \frac{5}{\cancel{4}}$$

3. Test the following series for convergence. [30 points]

(a) $\sum_{n=1}^{\infty} \frac{2n+4}{n^{7/2}+\sqrt{n}}$.

(b) $\sum_{n=1}^{\infty} \left(\frac{n+4}{2n+5}\right)^{n/3}$.

(a) $\sum_{n=1}^{\infty} \frac{2n+4}{n^{7/2}+\sqrt{n}}$ Limit comparison test with $n^{-5/2}$

$$\lim_{n\to\infty} \frac{\frac{2n+4}{n^{7/2}+\sqrt{n}}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{2n^{7/2}+4n^{5/2}}{n^{7/2}+\sqrt{n}} \to 2$$ which is non-zero and finite

the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges by the p-test, $p = 5/2 > 1$

and so $\sum_{n=1}^{\infty} \frac{2n+4}{n^{7/2}+\sqrt{n}}$ converges

(b) Root test.

$$\left(\left(\frac{n+4}{2n+5}\right)^{n/3}\right)^{1/n} = \left(\frac{n+4}{2n+5}\right)^{1/3} \longrightarrow \left(\frac{1}{2}\right)^{1/3} = \frac{1}{2^{1/3}} < 1$$

by Root test the series converges.

4. Decide whether the following series converge conditionally, converge absolutely, or diverge. [40 points]

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{1/n}}{n}$.

(b) $\sum_{n=1}^{\infty}(-1)^{n}\frac{n!}{(2n)!}$.

(c) $\sum_{n=1}^{\infty}(-1)^{n}\cos(1/n)$.

(a) $\left|\frac{e^{1/n}}{n}\right| \geq \frac{1}{n} \Rightarrow$ not absolutely convergent

A.S.T ① $\lim_{n\to\infty} \frac{e^{1/n}}{n} = 0$.

② $f(x) = \frac{e^{1/x}}{x} \Rightarrow f'(x) = \frac{-\frac{1}{x^2} e^{\frac{1}{x}} \cdot x - e^{1/x}}{x^2} = -\frac{e^{1/x}}{x^3}(1-x) < 0$

decreasing. by A.S.T conditionally convergent.

(b) Ratio test on $\frac{n!}{(2n)!}$

$a_{n+1} = \frac{(n+1)!}{2(n+1)!} \times \frac{(2n)!}{n!} = \frac{n}{(2n+2)(2n+1)} \to 0 < 1$

Converges absolutely by ratio test

(c) Note $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) \neq 0$, by divergence test

$\sum_{1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$ DNE

5. Consider the infinite series

$$-\frac{3}{4}+\frac{3\cdot 5}{4\cdot 7}-\frac{3\cdot 5\cdot 7}{4\cdot 7\cdot 10}+\frac{3\cdot 5\cdot 7\cdot 9}{4\cdot 7\cdot 10\cdot 13}-\cdots$$

(a) Based on the pattern give a formula for the n^{th} term of the series. [5 points]

(b) Test the series for convergence. [15 points]

(a) $$a_n = (-1)^n \frac{3\cdot 5\cdot \ldots \cdot (2n+1)}{4\cdot 7\cdot \ldots \cdot (3n+1)}$$

(b) Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{3\cdot 5\cdot \ldots}\,(2n+3)}{\cancel{4\cdot 7\cdot \ldots}\cdot(3n+4)} \times \frac{\cancel{4\cdot 7\cdots}\cdot(3n+1)}{\cancel{3\cdot 5\cdots}\,(2n+1)}$$

$$= \frac{2n+3}{3n+4} \longrightarrow \frac{2}{3} < 1$$

Converges absolutely by the ratio test, hence it converges.

6. Find the values of p for which the following series converges $\sum_{n=1}^{\infty} n(n^2+1)^p$. [30 points]

Comparison to n^{2p+1}

$$\lim_{n\to\infty} \frac{\cancel{n}(n^2+1)^p}{n^{2p+\cancel{1}}} = \lim_{n\to\infty} \frac{\cancel{n^{2p}}\left(1+\frac{1}{n^2}\right)^p}{\cancel{n^{2p}}} = 1$$

by comparison this converges precisely when $-(2p+1)>1$

So $-2p>2$ $\boxed{p<-1}$