Solutions to Exam 3 sample

① $3.\overline{417} = 3 + \frac{417}{10^3} + \frac{417}{10^6} + \cdots$

$$= 3 + \frac{\frac{417}{10^3}}{1 - \frac{1}{10^3}} = 3 + \frac{417}{999} = \frac{2997+417}{999}$$

$$= \frac{3414}{999}.$$

② $\sum_{n=1}^{\infty} \frac{2}{n(n+3)}$ $\qquad \frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$$S_n = \frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+3}\right) = \frac{2}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$\to \frac{2}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{\cancel{2}^{1} \times 11}{3 \times \cancel{6}3} = \frac{11}{9}.$$

③ (a) comparison to $\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\frac{n^2+n+1}{\sqrt{n^5+3n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^{5/2} + n^{3/2} + n^{1/2}}{\sqrt{n^5+3n}} = 1$$

by comparison test the series diverges, since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$.

3(b) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ Comparison to $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{n^{1+1/n}}{n} = \lim_{n\to\infty} n^{1/n} = 1$$

(why? $\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \frac{\ln(x)}{x}} \underset{LH}{=} e^{\lim_{x\to\infty} \frac{1}{x}} = e^0$)

by comparison test the series diverges, since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.

(c) Ratio test: $\frac{(n+1)}{e^{4(n+1)^2}} \cdot \frac{e^{4n^2}}{n} = \frac{n+1}{n} \frac{1}{e^{8n+4}} \to 0 < 1$

by ratio test the series converges.

4 (a) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$

abs conv $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ is a p-series, $p = 3/4 < 1$

not abs convergent.

Cond conv. A.S.T. ① $\frac{1}{n^{3/4}} \to 0$

② $f(x) = x^{-3/4}$, $f'(x) = -\frac{3}{4} x^{-7/4} < 0$

by A.S.T conditionally convergent.

4(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n} (-1)^n$

Abs conv: by ratio test:

$$\frac{3^{\cancel{n}+1}\cancel{(n+1)!}}{(n+1)^{n\cancel{+1}}} \cdot \frac{n^n}{\cancel{n!}\,\cancel{3^n}} = 3\left(\frac{n}{n+1}\right)^n$$

Now $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)}$

$$= e^{\lim_{x\to\infty}\frac{\ln(x)-\ln(x+1)}{\frac{1}{x}}} \underset{LH}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x+1}}{-\frac{1}{x^2}}} = e^{-1}$$

So the ratio test gives $\frac{3}{e} > 1$.

Divergent by the (secret) ratio test

(c) Note that $\displaystyle\lim_{n\to\infty}\frac{n}{\ln(n)} = \lim_{x\to\infty}\frac{x}{\ln(x)} = \lim_{x\to\infty}\frac{1}{\frac{1}{x}} = \infty \neq 0$

So the series diverges.

5 (a) alternating, starts with positive term

numerator increases by 4, denom by 3.

$$a_n = \frac{2 \cdot 6 \cdot 10 \cdots (4n-2)}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)} (-1)^{n-1}$$

(b) ratio test for abs convergence

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2 \cdot 6 \cdots \cdot (4n+2)}{5 \cdot 8 \cdots (3n+5)} \times \frac{5 \cdot 8 \cdot 11 \cdots (3n+2)}{2 \cdot 6 \cdots (4n-2)}$$

$$= \frac{4n+2}{3n+5} \to \frac{4}{3} > 1$$

diverges by ratio test (secret)

(6) Integral test.

$$\int_2^\infty \frac{1}{x \ln(x)^p} \; ; \quad \text{substitute } \ln(x) = u, \quad du = \frac{1}{x}$$

$$\int_2^t \frac{1}{x \ln(x)^p} dx = \int_{\ln(2)}^{\ln(t)} \frac{1}{u^p} du = \begin{cases} \left. \dfrac{u^{1-p}}{1-p} \right|_{\ln(2)}^{\ln(t)} & p \neq 1 \\ \left. \ln(u) \right|_{\ln(2)}^{\ln(t)} & p = 1 \end{cases}$$

$$= \begin{cases} \dfrac{\ln(t)^{1-p}}{1-p} - \dfrac{\ln(2)^{1-p}}{1-p}, & p \neq 1 \\ \ln(\ln(t)) - \ln(\ln(2)), & p = 1 \end{cases}$$

If $p=1$ $\quad \ln(\ln(t)) \longrightarrow +\infty$ $\quad$ divergent

$p<1$, $1-p>0$ $\quad \dfrac{\ln(t)^{1-p}}{1-p} \longrightarrow \infty$, divergent

$p>1$, $1-p<0$ $\quad \dfrac{\ln(t)^{1-p}}{1-p} \rightarrow 0$, convergent

Convergent for $p>1$, divergent $p \leq 1$