Solutions to sample exam

1 (a) $$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 0.25 \times 0.1 - y \times (0.1) = 0.25 - \frac{y}{1000}$$

$$\frac{dy}{dt} = \frac{25 - y}{1000}, \quad y(0) = 20$$

(b) $$\frac{dy}{250 - y} = \frac{dt}{1000} \implies -\ln(250 - y) = \frac{t}{1000} + C$$

$$\implies 25 - y = Ae^{-t/1000} \implies y = 250 - Ae^{-\frac{t}{1000}}$$

When $t = 0$, $y = 20$, $A = 25 - 20 = 5$

$$y = 25 - 20e^{-t/1000}$$

(c) $t = 5$, $$y = 25 - 20e^{-5/1000} = 25 - 20e^{-\frac{1}{200}} = 20.025\%$$

(d) $$\lim_{t \to \infty} y = \lim_{t \to \infty} 25 - 5e^{-t/1000} = 25$$

The oxygen level reaches 25 %.

(**Note**: in this problem total volume = 100, so there is no difference between %age and volume. In general, this is not true!)

2(a) $P' = kP\left(1 - \frac{P}{K}\right)$

(*) I think every one knows how to solve this.

(*) Solving we get,

$$P(t) = \frac{K}{1 + \frac{K - P_0}{P_0} e^{-kt}}$$, now $\leftrightarrow$ $t = 2009 - 1999 = 10$

$$P(10) = \frac{K}{1 + \frac{K-P_0}{P_0} e^{-0.6(10)}} = \frac{6}{1 + \frac{4}{2} e^{-6}} = 5.97 \times 10^8$$

(b) $$P(t) = 4 = \frac{6}{1 + 2e^{-0.6t}} \Rightarrow 4 + 8e^{-0.6t} = 6$$

$$8e^{-0.6t} = 2 \Rightarrow e^{-0.6t} = \frac{1}{4}$$

$$\Rightarrow t = \frac{\ln(4)}{0.6} = 2.31$$ years from 1999.

(c) Never happen If $P(t) = K$, then

$$\frac{\cancel{K}}{1 + 2e^{-0.6t}} = \cancel{K} \Rightarrow 1 + 2e^{-0.6t} = 1 \Rightarrow e^{-0.6t} = 0$$

this is not possible.

3(a) $\frac{dy}{dt} + \frac{3}{100+2t} y = 2$

$P = \frac{3}{100+2t} = \frac{3}{2} \frac{1}{50+t}$

$I = e^{\int \frac{3}{2} \frac{1}{50+t}} = e^{\frac{3}{2}\ln(50+t)} = (50+t)^{3/2}$

$\int \left(y(50+t)^{3/2}\right)' = \int 2(50+t)^{3/2}$

$y(50+t)^{3/2} = 2\left(\frac{2}{5}\right)(50+t)^{5/2} + C$

$\underline{\underline{t=0}}$ $\underline{\underline{y=100}}$, $100(50)^{3/2} = \frac{4}{5}(50)^{5/2} + C$

$C = 100(50)^{3/2} - \frac{4}{5}(50)^{5/2} = 50^{3/2}\left[100 - \frac{4}{\cancel{5}} \cdot \cancel{50}\right] = 20(50)^{3/2}$

$y = \frac{4}{5}(50+t) + \frac{20(50)^{3/2}}{(50+t)^{3/2}}$

(b) $xy' - y = x^2 \sin(x)$ $y(\pi) = 0$

$y' - \frac{y}{x} = x\sin(x)$.

$I(x) = e^{\int -\frac{1}{x}dx} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$

$\left(\frac{y}{x}\right)' = x \cdot \sin(x) \cdot \frac{1}{x}$

$\left(\frac{y}{x}\right)' = \sin(x) \Rightarrow \frac{y}{x} = -\cos(x) + C$

$\Rightarrow y = -x\cos(x) + cx$

Initial condition: $y(\pi)=0$

$$0=-\pi\cos(\pi)+c\pi \quad = \quad -\pi+c\pi \quad \Rightarrow \quad c=1$$

$$y(x) = -x\cos(x)+x.$$

(4) (a) $a_n = ne^{-n}$

$$f(x)=\frac{x}{e^{+x}}, \quad f'(x)=\frac{e^x-xe^x}{(e^x)^2} = \frac{e^x}{e^{2x}}(1-x) = \frac{1-x}{e^x}$$

$$f'(x)\leq 0 \iff 1-x\leq 0 \iff 1\leq x$$

So a_n is decreasing for $n\geq 1$.

(b)(i) $$\left|\frac{\sin(\sin(n))}{\sqrt{n+1}}\right| \leq \frac{1}{\sqrt{n+1}} \quad (\text{because } |\sin(x)|\leq 1)$$

$\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}}=0$, so <u>sandwich thm</u> tells us that

$$\lim_{n\to\infty} a_n = 0.$$

(ii) $a_n = \sin\left(\frac{n\pi}{2}\right)$

Consider $a_{2n} = \sin(n\pi)=0$

Consider $a_{4n+1} = \sin\left(2\pi+\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)=1$

Since $a_{2n}\to 0$, $a_{4n+1}\to 1$, the limit DNE

4(b) (iii) $a_n = \ln(n^2+1) - \ln(3n^2-2)$

$$= \ln\left(\frac{n^2+1}{3n^2-2}\right)$$

$\lim\limits_{n\to\infty} \dfrac{n^2+1}{3n^2-2} = \dfrac{1}{3}$, ln is continuous @ $x = \dfrac{1}{3}$

So $\lim\limits_{n\to\infty} a_n = \ln\left(\lim\limits_{n\to\infty} \dfrac{n^2+1}{3n^2-2}\right) = \ln\left(\frac{1}{3}\right) = -\ln(3)$.