MATH 170, SECTION 01, EXAM 1
Friday, September 25th, 2009

Name (PRINT clearly): Solutions

You will receive full credit only if you show all work and explain all steps.
If you run out of space, please ask the instructor for additional sheets of paper. Number the sheets clearly and indicate the problem that is being solved. Scratch work will not be graded.
The maximum possible score is 150. Each question is worth 30 points.

Honor pledge

"I pledge on my honor that I have neither given nor received unauthorized aid on this assignment"

Sign here:

1. Find the length of the curve made up of the the graph of $y = x^2 - \frac{3}{4}$ that lies in the fourth quadrant.

$$\frac{dy}{dx} = 2x \qquad y=0 \text{ at } x = \pm\frac{\sqrt{3}}{2}$$

$$x=0 \text{ at } y = -3/4$$

$$L = \int_0^{\sqrt{3}/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\sqrt{3}/2} \sqrt{1+4x^2}\, dx$$

Set $2x = \tan(\theta)$ $\qquad dx = \frac{\sec^2(\theta)}{2}$

x	0	$\sqrt{3}/2$
θ	0	$\pi/3$

$$L = \frac{1}{2}\int_0^{\pi/3} \sqrt{1+\tan^2(\theta)} \cdot \sec^2(\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/3} \sec^3(\theta)\, d\theta$$

$$= \frac{1}{4}\left(\sec(\theta)\tan(\theta) + \ln(\sec(\theta)+\tan(\theta))\right)\Big|_0^{\pi/3} = \frac{1}{4}\left(2\sqrt{3} + \ln(\sqrt{3}+2)\right)$$

2. Find the area of the surface obtained by rotating the graph of $y = (1 + e^x)^{1/2}$ for $0 \le x \le 1$ about the x-axis.

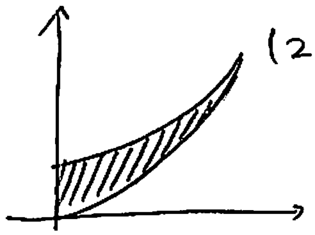
$$y = (1+e^x)^{1/2} \qquad \frac{dy}{dx} = \frac{1}{2}(1+e^x)^{-1/2} e^x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{4 + 4e^x + e^{2x}}{4(1+e^x)} = \frac{(e^x+2)^2}{4(1+e^x)}$$

$$S = 2\pi \int_0^1 (e^x+1)^{1/2} \sqrt{\frac{(e^x+2)^2}{4(1+e^x)}}\, dx$$

$$= \cancel{2}\pi \int_0^1 \frac{e^x+2}{\cancel{2}}\, dx = \pi \left[e^x + 2x \right]_0^1 = \pi (e+1)$$

3. Find the centroid of the region bounded by the curves $y = 2^x$, $y = x^2$ and the y-axis.



$$A = \int_0^2 2^x - x^2 \, dx = \frac{2^x}{\ln(2)} - \frac{x^3}{3} \Big|_0^2$$

$$= \frac{4}{\ln(2)} - \frac{8}{3} - \frac{1}{\ln(2)} = \frac{3}{\ln(2)} - \frac{8}{3}$$

$$\bar{x} = \frac{1}{A}\int_0^2 x \cdot 2^x - x^3 \, dx = \frac{x 2^x}{\ln(2)}\Big|_0^2 - \int_0^2 \frac{2^x}{\ln(2)} \, dx - \frac{x^4}{4}\Big|_0^2$$

$$= \frac{8}{\ln(2)} - \frac{2^x}{\ln(2)^2}\Big|_0^2 - \frac{16}{4}$$

$$= \left(\frac{8}{\ln(2)} - \frac{4}{\ln(2)^2} - 4\right) \Big/ A$$

$$\bar{y} = \frac{1}{2A}\int_0^2 (2^x)^2 - (x^2)^2 \, dx = \frac{1}{2A}\int_0^2 2^{2x} - x^4$$

$$= \frac{1}{2A}\left[\frac{2^{2x}}{2\ln(2)}\Big|_0^2 - \frac{x^5}{5}\Big|_0^2\right] = \frac{8}{\ln(2)} - \frac{1}{2\ln(2)} - \frac{32}{5}.$$

4. Suppose that we have $f(x) = ax(1-x^2)$ for $0 \le x \le 1$, and $f(x) = 0$ otherwise.

(a) Find the value of a that makes f a density function.

(b) Compute the mean.

(c) Find $P(X \le 1/3)$.

(d) Compute the median.

(a) $a\int_0^1 x(1-x^2)\,dx = a\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = a\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{a}{4} \overset{\text{density}}{=} 1$

So $a = 4$

(b) $\mu = \text{mean} = 4\int_0^1 x^2(1-x^2)\,dx = 4\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{8}{15}$

(c) $4\int_0^{1/3} x(1-x^2)\,dx = 4\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^{1/3} = 2x^2 - x^4\Big|_0^{1/3} = \frac{17}{81}$

(d) $8\int_0^m x(1-x^2)\,dx = 8\frac{m^2}{2} - 8\frac{m^4}{4} - 1 = 0$

So $4m^2 - 2m^4 - 1 = 0 \Rightarrow 2m^4 - 4m^2 + 1 = 0$

$m^2 = \dfrac{4 \pm \sqrt{16-8}}{4} = \dfrac{1 \pm \sqrt{8}}{4} = 1 \pm \dfrac{\sqrt{2}}{2}$

$m^2 = 1 - \dfrac{\sqrt{2}}{2}$ (the other point is not in $0 \le x \le 1$)

and so $m = \sqrt{1 - \dfrac{\sqrt{2}}{2}}$

5. Consider the differential equation

$$\frac{dx}{dt} = a(b^2 - x^2),$$

where a, b are positive constants.

(a) Solve the above differential equation for $x(t)$ with initial condition $x(0) = 0$.

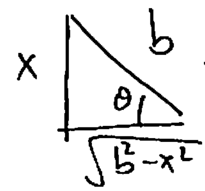
(b) Find the limit of $x(t)$ as $t \to \infty$.

(a) $$\int \frac{dx}{b^2-x^2} = \int \frac{b\cos(\theta)}{b^2\cos^2(\theta)}\,d\theta = \frac{1}{b}\int \sec(\theta)\,d\theta = \frac{1}{b}\ln\left(\sec(\theta)+\tan(\theta)\right)$$

$x = b\sin(\theta)$

$dx = b\cos(\theta)\,d\theta$

$$= \frac{1}{b}\ln\left|\frac{b}{\sqrt{b^2-x^2}} + \frac{x}{\sqrt{b^2-x^2}}\right|$$



$$\int a\cancel{b}\,dt = at + C$$ so we set

$$\frac{1}{b}\ln\left(\frac{b+x}{\sqrt{b^2-x^2}}\right) = at + C.$$

$$\ln\left(\frac{(b+x)^2}{b^2-x^2}\right)^{\frac{1}{2}} = abt + C \Rightarrow \frac{(b+x)^2}{b^2-x^2} = Ae^{2abt}.$$

$$\frac{(b+x)^2}{(b^2-x^2)} = \frac{(b+x)^{\cancel{2}}}{(b-x)\cancel{(b+x)}} = \frac{b+x}{b-x} = Ae^{2abt}.$$

$x(0) = 0$ gives $A = 1$.

$$\frac{b+x}{b-x} = y \Rightarrow y(b-x) = b+x \Rightarrow yb - b = x + xy = x(1+y)$$

$$x = -b\frac{(1-y)}{1+y}$$

So $$x(t) = -b\frac{(1-e^{2abt})}{1+e^{2abt}} \longrightarrow b \text{ as } t \to \infty.$$