

# A NOTE ON THE NORM AND SPECTRUM OF A FOGUEL OPERATOR

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ABSTRACT. We present two ways to compute the norm of a Foguel operator. This result gives a simpler proof of a recent result of Garcia [2].

## 1. INTRODUCTION

Let  $H$  be a Hilbert space. Given an isometry  $V \in B(H)$  and an operator  $T \in B(H)$ , the *Foguel operator* with symbol  $T$  is defined by

$$R_T = \begin{bmatrix} V^* & T \\ 0 & V \end{bmatrix}.$$

In [2], a formula is given for the norm of a Foguel operator  $R_T$ . The proof in [2] is based on an antilinear eigenvalue problem and certain properties of complex symmetric matrices. In this note, we give a more direct proof of this fact based on a direct computation of the inverse. We also show how the formula for the norm can be obtained as an application of the Schur complement.

Foguel operators have played a central role in counterexamples to similarity conjectures. The most famous of these is Pisier's counterexample [4] to the Halmos conjecture [3].

## 2. SPECTRUM AND NORM

In this section we derive a relationship between the spectrum of the operator  $R_T R_T^*$  and  $TT^*$ . The result is due to Garcia [2] but the proof given here is more direct.

In the proof we make use of the fact that if  $R$  is a selfadjoint operator on a Hilbert space, then  $R$  is invertible if and only if  $R$  is right-invertible. To see this, suppose that  $RS = I$ . Then  $S^* R^* = S^* R = I$ . Hence,  $R$  has a left-inverse, and is invertible.

**Theorem 1** (Garcia). *If  $\lambda > 0$  and  $\lambda \neq 1$ , then  $\lambda \in \text{spec}(R_T R_T^*)$  if and only if  $\lambda^{-1}(\lambda - 1)^2 \in \text{spec}(TT^*)$ . The norm of  $R_T$  is given by*

$$\|R_T\| = \frac{\|T\| + \sqrt{\|T\|^2 + 4}}{2}.$$

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*Proof.* Suppose that  $\lambda \notin \text{spec}(R_T R_T^*)$ . Since  $R_T R_T^* - \lambda I$  is invertible there exists a selfadjoint operator

$$S = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

such that  $R_T R_T^* S = \lambda S + I$ . We have

$$R_T R_T^* = \begin{bmatrix} V^*V + TT^* & TV^* \\ VT^* & VV^* \end{bmatrix} = \begin{bmatrix} I + TT^* & TV^* \\ VT^* & VV^* \end{bmatrix}.$$

Writing out the entries of the operator matrix equation  $R_T R_T^* S = \lambda S + I$  we get

$$(1) \quad (I + TT^*)A + TV^*X^* = \lambda A + I$$

$$(2) \quad (I + TT^*)X + TV^*B = \lambda X$$

$$(3) \quad VT^*A + VV^*X^* = \lambda X^*$$

$$(4) \quad VT^*X + VV^*B = \lambda B + I.$$

Using the fact that  $V^*V = I$  and multiplying equation (3) on the left by  $V^*$  we get  $T^*A + V^*X^* = \lambda V^*X^*$ , which gives

$$(5) \quad T^*A = (\lambda - 1)V^*X^*.$$

Multiplying (1) by  $(\lambda - 1)$  and substituting the expression from (5) we get

$$(6) \quad (\lambda - 1)(I + TT^*)A + TT^*A = (\lambda - 1)(\lambda A + I).$$

Rearranging and simplifying this last equation we get

$$((\lambda - 1)I + I)TT^*A + ((\lambda - I) + \lambda(\lambda - 1))A = (\lambda - 1)I$$

or

$$(\lambda TT^* - (\lambda - I)^2)A = (\lambda - 1)I.$$

Since we have assumed that  $\lambda \neq 0, 1$  we get

$$(7) \quad A = \frac{\lambda}{\lambda - 1} \left( TT^* - \frac{(\lambda - 1)^2}{\lambda} I \right)^{-1}.$$

Hence  $\lambda^{-1}(\lambda - 1)^2 \notin \text{spec}(TT^*)$ .

The operators  $X$  and  $B$  can also be computed explicitly. The choice of  $X = \frac{1}{\lambda - 1} A T V^*$  satisfies (5). Multiplying (4) by  $V^*$  we get  $T^*X + V^*B = \lambda V^*B + V^*$  and so

$$(8) \quad (\lambda - 1)V^*B = T^*X - V^*.$$

The choice

$$(9) \quad B = (\lambda - 1)^{-1}(V T^* X - V V^*)$$

is a solution of (8).

Now consider the case where  $\mu \notin \text{spec}(TT^*)$  and assume that  $\mu \neq 0$ . The equation  $\mu = (\lambda - 1)^2/\lambda$  has two positive solutions, choose  $\lambda$  to be either of these solutions. We choose  $A, X, B$  as in the equations above. It is a routine, but straightforward, calculation to show that this choice of  $A, B, X$  satisfies (1)–(4).

From the relationship between the spectrum of  $R_T R_T^*$  and the spectrum of  $TT^*$  we get that

$$(10) \quad \|T\|^2 = \sup\{\lambda^{-1}(\lambda - 1)^2 : \lambda \in \text{spec}(R_T R_T^*)\}.$$

The equation  $\|T\|^2 = \lambda^{-1}(\lambda - 1)^2$  has two solutions of the form  $\lambda, \lambda^{-1}$ . The function  $f(\lambda) = \lambda^{-1}(\lambda - 1)^2$  is increasing for  $\lambda \geq 1$  and  $\|T\|^2$  is the maximum value of the function  $f$  on the set  $\text{spec}(R_T R_T^*)$ . Since,  $\|R_T\| \geq \|V\| = 1$  we see that

$$(11) \quad \|T\|^2 = (\|R_T\|^2 - 1) / \|R_T\|.$$

This gives,

$$(12) \quad \|R_T\| = \frac{\|T\| + \sqrt{\|T\|^2 + 4}}{2}.$$

□

The proof of Theorem 1 actually gives us a little more information about the invertibility of  $R_T R_T^*$  and  $TT^*$ .

**Proposition 2.** *The operator  $R_T R_T^*$  is invertible if and only if  $V$  is unitary. The operator  $R_T R_T^* - I$  is invertible if and only if  $T$  is invertible.*

*Proof.* From (4) we have,  $V(T^*X + V^*B) = I$  and so  $VY = I$  for some  $Y$ . Hence,  $Y = (VV^*)Y = V^*(VY) = V^*$ . Which proves that  $VV^* = I$ .

On the other hand, if  $V$  is unitary, then  $\begin{bmatrix} V & -VT V^* \\ 0 & V^* \end{bmatrix}$  is the inverse of  $R_T R_T^*$ .

Now consider the case where  $R_T R_T^* - I$  is invertible. In this case from (1) we get  $TT^*A + TV^*X^* = I$ , which gives,  $TY = I$  for some  $Y$ . Multiply equation (4) by  $V^*$  we get  $T^*X = V^*$ . Now multiply by  $V$  to get  $T^*XV = I$ . Hence,  $ZT = I$  for some  $Z$  and so  $T$  is invertible. However, we now get from (5) that  $T^*A = 0$  and so  $A = 0$ . In this case, the inverse of  $R_T R_T^* - I$  is given by  $\begin{bmatrix} 0 & X \\ X^* & B \end{bmatrix}$ , where  $X = (T^{-1})^*V^*$  and  $B = -I$ .

Conversely, if  $T$  is invertible, then there exists  $Z$  such that  $ZT = TZ = I$  and we can write down the inverse of  $R_T R_T^* - I$  as above. □

As a final note in this section we point out that we can strengthen the power-bounded result in [2] to the case of polynomials in  $R_T$ .

**Proposition 3.** *Let  $A$  be a contraction and let  $T$  be an operator on  $H$ . Let  $R = \begin{bmatrix} A^* & T \\ 0 & A \end{bmatrix}$ . We have,*

$$\|R\| \leq \frac{\|T\| + \sqrt{\|T\|^2 + 4}}{2}$$

*Proof.* Let  $V_A$  denote the usual isometric Sz.-Nagy dilation of  $A$ , that is,

$$V_A = \begin{bmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & A^* \end{bmatrix}.$$

Let  $\tilde{T}$  denote the  $2 \times 2$  operator matrix that has  $T$  in its (1, 2) entry and is 0 otherwise. Let  $W = \begin{bmatrix} V_A^* & \tilde{T} \\ 0 & V_A \end{bmatrix}$ . Note that  $V_A^* = V_A^*$ . Since  $V_A$  is an isometry,  $W$  is a Foguel operator with symbol  $\tilde{T}$  and so  $\|W\| = \frac{1}{2}(\|T\| + (\|T\|^2 + 4)^{1/2})$ , since  $\|\tilde{T}\| = \|T\|$ . If we view  $W$  as a  $4 \times 4$  operator matrix, then the operator  $R$  is the compression of  $W$  to the first and fourth row and column. Hence,  $\|R\| \leq \|W\|$ . □

**Corollary 4.** *Let  $p(z) = a_0 + a_1z + \cdots + a_mz^m$  be a polynomial. Let  $\|p\|_\infty := \sup_{z \in \mathbb{D}} |p(z)|$  and let  $\tilde{p}(z) = |a_0| + |a_1|z + \cdots + |a_m|z^m$ . Let  $A$  and  $T$  be operators on  $H$  with  $\|A\| \leq 1$  and let*

$$R = \begin{bmatrix} A^* & T \\ 0 & A \end{bmatrix}.$$

We have,

$$\|p(R)\| \leq \frac{\|\tilde{p}'\|_\infty \|T\| + \sqrt{\|\tilde{p}'\|_\infty^2 \|T\|^2 + 4}}{2}$$

*Proof.* First note that we can assume that  $\|p\|_\infty \leq 1$ . A simple computation shows that

$$R^n = \begin{bmatrix} (A^*)^n & \sum_{j=0}^{n-1} (A^*)^j T A^{n-1-j} \\ 0 & A^n \end{bmatrix}$$

Let us denote the operator in the upper right corner of the above matrix by  $D_n(A, T)$  for  $n \geq 1$ .

We have,

$$p(R) = \begin{bmatrix} p(A)^* & \sum_{j=1}^m a_j D_j(A, T) \\ 0 & A^n \end{bmatrix}.$$

Since  $A$  is a contraction, von-Neumann's inequality tells us that  $\|p(A)\| \leq \|p\|_\infty \leq 1$ . Now,

$$\begin{aligned} \left\| \sum_{j=1}^m a_j D_j(A, T) \right\| &\leq \sum_{j=1}^m |a_j| \|D_j(A, T)\| \leq \sum_{j=1}^m |a_j| \sum_{i=0}^{j-1} \|(A^*)^i T A^{j-1-i}\| \\ &\leq \sum_{j=1}^m |a_j| \sum_{i=0}^{j-1} \|A\|^{j-1} \|T\| \leq \|T\| \sum_{j=1}^m j |a_j| \|A\|^{j-1} \\ &= \|T\| \tilde{p}'(\|A\|) \leq \|T\| \|\tilde{p}'\|_\infty \end{aligned}$$

The result now follows from Proposition 3.  $\square$

An application of the previous proposition with  $p(z) = z^n$  gives the result on the norm of  $R_T^n$  obtained in [2]. We have  $p = \tilde{p}$  and  $\|\tilde{p}'\|_\infty = n$ . Hence, we obtain the norm estimate

$$\|R_T^n\| \leq \frac{n \|T\| + \sqrt{n^2 \|T\|^2 + 4}}{2}.$$

### 3. A POSITIVITY PROOF OF THE NORM

In this section we present a short proof of the norm equality using the Schur complement.

We begin by giving a brief description of the Schur complement. Let  $R = \begin{bmatrix} P & X \\ X^* & Q \end{bmatrix}$ , with  $P \geq 0, Q > 0$ . The operator  $R$  is positive if and only if the matrix  $P - XQ^{-1}X^* \geq 0$ . The proof of this fact follows by first conjugating  $R$  by the positive matrix  $I \oplus Q^{-1/2}$  which gives  $R' = \begin{bmatrix} P & XQ^{-1/2} \\ Q^{-1/2}X^* & I \end{bmatrix}$ , and then using the fact that  $\begin{bmatrix} P & Y \\ Y^* & I \end{bmatrix} \geq 0$  if and only if  $P - YY^* \geq 0$ .

For an operator  $T \in B(H)$ ,  $\|T\| = \inf\{M : M^2I - TT^* > 0\}$ .

Consider the Foguel operator  $R_T$ . We know that  $\|R_T\| \geq 1$ . Let  $M > 1$ . We have

$$(13) \quad M^2I - R_T R_T^* = \begin{bmatrix} M^2I & 0 \\ 0 & M^2I \end{bmatrix} - \begin{bmatrix} I + TT^* & TV^* \\ VT^* & VV^* \end{bmatrix}$$

$$(14) \quad = \begin{bmatrix} (M^2 - 1)I - TT^* & -TV^* \\ -VT^* & M^2I - VV^* \end{bmatrix}$$

By applying the Schur complement we see that this operator is positive if and only if

$$(M^2 - 1)I - TT^* - TV^*(M^2I - VV^*)^{-1}VT^* \geq 0.$$

Since  $M > 1$  and  $V$  is a contraction we can expand  $(M^2I - VV^*)^{-1}$  in its Neumann series as  $M^{-2} \sum_{j=0}^{\infty} (VV^*)^j / (M^{2j})$ . Hence,

$$(15) \quad TV^*(M^2I - VV^*)^{-1}VT^* = T \left( M^{-2} \sum_{j=0}^{\infty} \frac{V(VV^*)^j V^*}{M^{2j}} \right) T^*$$

$$(16) \quad = M^{-2} TT^* \sum_{j=0}^{\infty} M^{-2j} = (M^2 - 1)^{-1} TT^*.$$

Hence, the positivity condition is

$$(M^2 - 1)I - TT^* - (M^2 - 1)^{-1} TT^* = (M^2 - 1)I - M^2(M^2 - 1)^{-1} TT^* \geq 0.$$

This happens if and only if

$$TT^* \leq M^{-2}(M^2 - 1)^2,$$

which happens if and only if

$$\|T\| \leq M^{-1}(M^2 - 1).$$

Hence,  $\|T\| = \|R_T\|^{-1} (\|R_T\|^2 - 1)$ , which is the same relation obtained in (11).

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