Spanning 2-trails from degree sum conditions

M. N. Ellingham*
Department of Mathematics, 1326 Stevenson Center
Vanderbilt University, Nashville, TN 37240, U. S. A.
mne@math.vanderbilt.edu

Xiaoya Zha†
Department of Mathematical Sciences
Middle Tennessee State University, Murfreesboro, TN 37132, U.S.A.
xzha@mtsu.edu

Yi Zhang
Tillinghast–Towers Perrin
175 Powder Forest Drive, Weatogue, CT 06089, U. S. A.
yi.zhang@tillinghast.com

17th December 2001; minor revisions 3rd September 2003

Abstract

Suppose G is a simple connected n-vertex graph. Let $\sigma_3(G)$ denote the minimum degree sum of three independent vertices in G (which is ∞ if G has no set of three independent vertices). A 2-*trail* is a trail that uses every vertex at most twice. Spanning 2-trails generalize hamilton paths and cycles. We prove three main results. First, if $\sigma_3(G) \geq n-1$, then G has a spanning 2-trail, unless $G \cong K_{1,3}$. Second, if $\sigma_3(G) \geq n$, then G has either a hamilton path or a closed spanning 2-trail. Third, if G is 2-edge-connected and $\sigma_3(G) \geq n$, then G has a closed spanning 2-trail, unless $G \cong K_{2,3}$ or $K_{2,3}^*$ (the 6-vertex graph obtained from $K_{2,3}$ by subdividing one edge). All three results are sharp. These results are related to the study of connected and 2-edge-connected factors, spanning k-walks, even factors, and supereulerian graphs. In particular, a closed spanning 2-trail may be regarded as a connected (and 2-edge-connected) even $[2,4]$-factor.

*Supported by NSF Grant DMS-0070613
†Supported by NSF Grant DMS-0070430

1 Introduction

We use Bondy and Murty's book [2] for terminology and notation not defined here, and consider finite simple graphs only.

A walk in a graph will be called a *k-walk* if each vertex is used no more than k times (if the walk is closed, so that the same vertex occurs as both first and last vertex, we count this as just one use of that vertex). A *k-trail* is a k-walk with no repeated edges. For the purposes of the current paper, a k-walk or k-trail need not be closed, and need not be spanning. Note that a closed spanning 1-walk or 1-trail is a hamilton cycle, and an open spanning 1-walk or 1-trail is a hamilton path. A graph with a closed spanning trail has a spanning Eulerian subgraph, and is sometimes called *supereulerian*.

The degree of a vertex v in the graph G will be denoted $\deg_G(v)$, or just $\deg(v)$. The set of vertices adjacent to v in G will be denoted $N_G(v)$ or just $N(v)$. The degree sum of independent sets of vertices of given cardinality has often been used to give sufficient conditions for the existence of hamilton paths or cycles, or other structures such as spanning trails. The following notation will be useful:

$$\sigma_k(G) = \min\{\Sigma_{i=1}^k \deg(v_i) \mid \{v_1, v_2, \ldots, v_k\} \text{ is independent in } G\}.$$

If there are no independent sets of cardinality k, $\sigma_k(G)$ is taken to be ∞. Our goal in this paper is to give conditions involving σ_3 for spanning 2-trails.

Spanning k-trails are related to many other types of spanning subgraphs. Some relationships are shown in Figure 1: a closed spanning k-trail is a generalization of a hamilton cycle, and a special case of several other types of spanning subgraphs that have been examined previously. In particular, the existence of a closed spanning k-trail is equivalent to the existence of a connected (or 2-edge-connected) even $[2, 2k]$-factor. It is also equivalent to the existence of a connected covering of the vertices of the graph by edge-disjoint cycles with each vertex appearing in at most k of the cycles. Spanning k-trails therefore seem to be a natural subject for investigation.

Degree sum results (involving σ_k, for some k) are common as sufficient conditions for the existence of spanning subgraphs obeying degree and connectivity restrictions. The first degree sum result was a condition for hamilton cycles by Ore, generalizing a well known result of Dirac based on the minimum degree.

Theorem A (Ore [17]**)** *If G is an n-vertex graph with $n \geq 3$ and $\sigma_2(G) \geq n$, then G has a hamilton cycle.*

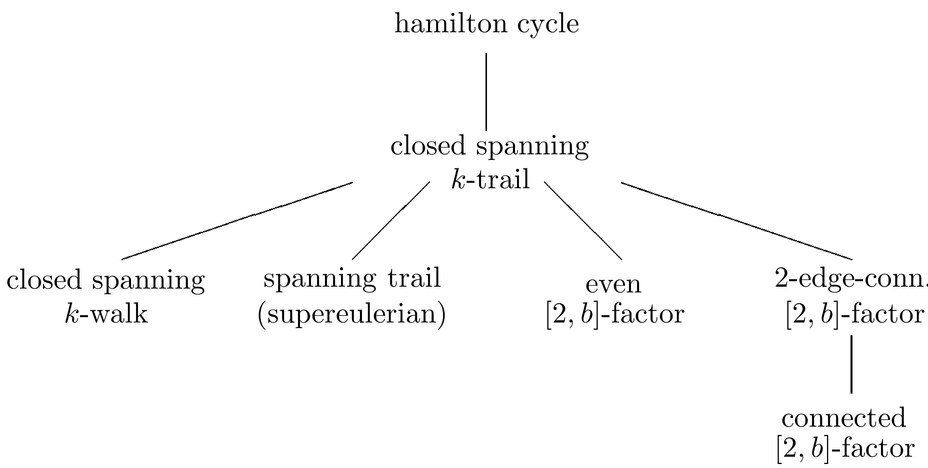

Figure 1: Hierarchy of spanning subgraphs

Corollary B *If G is an n-vertex graph with $\sigma_2(G) \geq n-1$, then G has a hamilton path.*

A result on k-walks that generalizes Theorem A was found by Jackson and Wormald.

Theorem C (Jackson and Wormald [12]**)** *If G is a connected n-vertex graph with $n \geq 3$ and $\sigma_{k+1}(G) \geq n$, where $k \geq 1$, then G has a closed spanning k-walk.*

For trails, which are the subject of this paper, there are several results involving σ_k that guarantee the existence of a spanning trail. However, none of these give a bound on how often each vertex is used. We begin with two results for σ_2.

Theorem D (Lesniak-Foster and Williamson [15]**)** *If G is an n-vertex graph with $n \geq 3$, $\delta(G) \geq 2$, and $\sigma_2(G) \geq n-1$, then G has a closed spanning trail.*

Since a graph with $\sigma_2 \geq n-1$ has $\delta \geq 2$ if and only if it is 2-edge-connected, the following is a strengthening of Theorem D.

Theorem E (Benhocine *et al.* [1]**)** *If G is a 2-edge-connected n-vertex graph with $n \geq 3$ and $\sigma_2(G) \geq (2n+3)/3$, then G has a closed spanning trail.*

These results have been improved further by Catlin [3] and Chen [5], provided n is taken to be sufficiently large.

The above results deal with closed trails. If the trails may be open, they are easier to find. Lesniak-Foster and Williamson [15] mention that if G is a connected n-vertex graph with $n \geq 5$ and $\sigma_2(G) \geq n-2$, then G contains a (possibly open) spanning trail. The following result involving σ_3 strengthens this.

Theorem F (Veldman [19]**)** *If G is a connected n-vertex graph with $n \geq 5$ and $\sigma_3(G) \geq n-1$, then G has a (possibly open) spanning trail.*

This has been further extended, as follows.

Theorem G (Z. Q. Chen and Y. F. Xue, cited in [6]**)** *If G is a connected n-vertex graph with $\sigma_3(G) \geq n-2$, then either G has a spanning trail (possibly open), or G is isomorphic to one of $K_{1,4}$, C_4^{++} (the 6-vertex graph obtained by attaching pendant edges to opposite vertices of a 4-cycle), or A_n with $n \geq 4$ (see the end of Section 2).*

Catlin [4] showed that some stronger conclusions could be reached if $\sigma_3 \geq n+1$.

Our results in this paper strengthen Theorems E and F, and we obtain results for 2-trails that parallel Theorem A and Corollary B. Our main theorems, Theorems 2.6, 3.3 and 4.1, are all sharp. With the exception of results by Gao and Wormald [11] for triangulations of the plane, projective plane, torus and Klein bottle, these are the first results that we are aware of that deal with spanning k-trails for $k \geq 2$.

As mentioned earlier, spanning k-trails are related to connected factors with degree conditions, and also to coverings of the vertices of a graph by cycles. Recent conditions based on degree sums for the existence of connected factors with degree restrictions include conditions involving σ_2 by Kouider and Mahéo [14] for 2-edge-connected $[2, b]$-factors, and by Nam [16] for connected $[a, b]$-factors. Conditions based on degree sums for coverings of the vertices of a graph by cycles, edges and vertices include work by Enomoto et al. [10] which was extended by Kouider and Lonc [13] and Saito [18]. Our results on closed 2-trails, Theorems 3.3 and 4.1, can be interpreted as results on the existence of connected even $[2, 4]$-factors, or as results on the existence of connected coverings of the vertices of a graph by edge-disjoint cycles with each vertex in at most 2 of the cycles.

2 Open 2-trails

We begin by examining the existence of open spanning 2-trails in graphs. The following assumption will be used heavily in this section, and also in Section 3.

Assumption 2.1 *Suppose G is a connected n-vertex graph, and let $T = u_1u_2\ldots u_p$ be a 2-trail in G that has the minimum number of edges among all 2-trails covering the maximum number of vertices.*

Lemma 2.2 *Suppose Assumption 2.1 holds. Then T is open, i.e., $u_1 \neq u_p$, and its ends u_1 and u_p are used only once.*

Proof. Suppose that $u_1 = u_j$ for some $j \geq 2$ (possibly $j = p$). Then $u_2u_3\ldots u_p$ is a 2-trail covering the same vertices as T but with fewer edges, contradicting Assumption 2.1. Thus, $u_1 \neq u_p$, and u_1 is used only once. Similarly, u_p is used only once. ∎

Lemma 2.3 *Suppose Assumption 2.1 holds. If $u \in V(T)$ is the initial vertex of an open 2-trail that covers $V(T)$, then $N(u) \subseteq V(T)$. In particular, $N(u_1) \subseteq V(T)$ and $N(u_p) \subseteq V(T)$.*

Proof. Suppose there exists $w \in N(u) \setminus V(T)$. Let $(u = v_1)v_2v_3\ldots v_q$ be the open 2-trail starting at u and covering $V(T)$. Since T has the maximum number of vertices, $v_1, v_2, \ldots, v_q \in V(T)$. Then $wv_1v_2\ldots v_q$ is an open 2-trail covering more vertices than T, contradicting Assumption 2.1. ∎

Define

$$N^-(u_1) = \{u_{i-1} \mid u_1u_i \in E(G)\},$$
$$N^+(u_p) = \{u_{i+1} \mid u_pu_i \in E(G)\}.$$

These are both well-defined since neither u_1 nor u_p is adjacent to itself. For any vertex v and $i = 1$ or 2, let $N_i(v)$ be the set of neighbours of v that are used i times on T.

Lemma 2.4 *Suppose Assumption 2.1 holds. If $u_k \in N^-(u_1)$ or $N^+(u_p)$ then u_k is used only once on T, and $N(u_k) \subseteq V(T)$.*

Proof. If $u_k \in N^-(u_1)$, then $T' = u_ku_{k-1}\ldots u_1u_{k+1}\ldots u_p$ is a 2-trail from u_k to u_p with the same number of vertices and edges as T. Therefore, T' satisfies Assumption 2.1. By Lemma 2.2, u_k is used only once on T' and hence only once on T, and by Lemma 2.3, $N(u_k) \subseteq V(T') = V(T)$. The argument when $u_k \in N^+(u_p)$ is similar. ∎

Lemma 2.5 *Suppose Assumption 2.1 holds. Then* $|N^-(u_1)| = |N(u_1)| + |N_2(u_1)|$, *and* $|N^+(u_p)| = |N(u_p)| + |N_2(u_p)|$.

Proof. $|N(u_1)| + |N_2(u_1)|$ is the number of indices i for which $u_i \in N(u_1)$. For each such i, $u_{i-1} \in N^-(u_1)$, and by Lemma 2.4 this counts every vertex in $N^-(u_1)$ exactly once. The proof for $|N^+(u_p)|$ is similar. ∎

The main result of this section, which strengthens Theorem F by adding the restriction that the spanning trail uses every vertex at most twice, is as follows.

Theorem 2.6 *Let* G *be a connected* n*-vertex graph with* $\sigma_3(G) \geq n-1$. *Then either* G *has a (possibly open) spanning* 2*-trail, or* $G \cong K_{1,3}$.

Proof. Suppose G does not have a spanning 2-trail. Let T satisfy Assumption 2.1. Since G is connected, there exists a vertex $w \in V(G) \setminus V(T)$ adjacent to a vertex of T.

Claim 2.6A There is no closed 2-trail T' with $V(T') = V(T)$. In particular, u_1 and u_p are nonadjacent.

Proof. Write $T' = v_1 v_2 \ldots v_q$, with $v_q = v_1$. Since $V(T') = V(T)$, w is adjacent to some v_i, $i \leq q-1$. Then $v_{i+1} v_{i+2} \ldots v_{q-1} (v_q = v_1) v_2 \ldots v_i w$ is a 2-trail covering more vertices than T' and T, contradicting Assumption 2.1. ∎

By Lemmas 2.2 and 2.3 and Claim 2.6A, u_1, u_p and w are distinct nonadjacent vertices, so $\deg(w) + \deg(u_1) + \deg(u_p) \geq n-1$. Let $C = N^-(u_1) \cup N^+(u_p) \cup N(w)$.

Claim 2.6B If $u_i \in N^-(u_1), u_j \in N^+(u_p), i < j$, and either $i \neq 1$ or $j \neq p$, then there exists k with $i < k < j$ for which at least one of the following holds:

(i) u_k is used only once on T, and $u_k \in V(T) \setminus C$, or

(ii) $u_k \in N_2(u_1)$ and u_k is the second occurrence of this vertex on T, or

(iii) $u_k \in N_2(u_p)$ and u_k is the first occurrence of this vertex on T.

Proof. We use induction on $j - i = d$.

Suppose $d = 1$. If $i = 1$ or $j = p$, then $u_1 u_p \in E(G)$. If $i > 1$ and $j < p$, then $u_1 u_j u_{j+1} \ldots u_p u_i u_{i-1} \ldots u_1$ is a closed 2-trail. In either case, Claim 2.6A is contradicted. So d cannot be 1, and the result holds vacuously in this case.

Now assume the result holds when $j - i < d$, and consider the case where $j - i = d \geq 2$. Since $i \neq 1$ or $j \neq p$, we may assume without loss of generality that $i \neq 1$.

Suppose u_{i+1} is used once on T. If $u_{i+1} \in N^-(u_1)$ or $N^+(u_p)$ then we may use induction. If $u_{i+1} \in N(w)$ then since $i+1 \neq 2$, $wu_{i+1}u_1u_2 \ldots u_p$ is a 2-trail covering more vertices than T, contradicting Assumption 2.1. So, $u_{i+1} \notin C$, and (i) holds.

Suppose u_{i+1} is used twice on T. Then $u_{i+1} \in N_2(u_1)$. If $d = 2$, then $u_{i+1} = u_{j-1} \in N(u_p)$, and either (ii) or (iii) holds depending on whether u_{i+1} is the second or first occurrence of this vertex, respectively. So, assume that $d \geq 3$. If u_{i+1} is being used for the second time, then (ii) holds, so suppose $u_{i+1} = u_l$ where $i+1 < l$. Then $T' = u_{i+2}u_{i+3} \ldots (u_l = u_{i+1})u_1u_2 \ldots (u_{i+1} = u_l)u_{l+1} \ldots u_p$ is a 2-trail covering $V(T)$ with the same number of edges as T, so Assumption 2.1 holds for T'. By Lemma 2.2, u_{i+2} is used only once on T', and hence only once on T. By Lemma 2.3, $N(u_{i+2}) \subseteq V(T') = V(T)$, so $u_{i+2} \notin N(w)$, and if $u_{i+2} \in N^-(u_1)$ or $N^+(u_p)$ we may use induction. Therefore, we may assume that $u_{i+2} \notin C$, and (i) holds. ∎

Claim 2.6C If either $\deg(u_1) \geq 2$ or $\deg(u_p) \geq 2$, then $|N_2(u_1)| + |N_2(u_p)| + |V(T) \setminus C| \geq |N^-(u_1) \cap N^+(u_p)| + 1$.

Proof. Write $N^-(u_1) \cap N^+(u_p) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_m}\}$, where $1 < i_1 < i_2 < \ldots < i_m < p$ and $m = |N^-(u_1) \cap N^+(u_p)|$. If $m \geq 1$, then applying Claim 2.6B to consecutive pairs of vertices in the sequence $u_1, u_{i_1}, u_{i_2}, \ldots, u_{i_m}, u_p$ we find $m+1$ distinct indices k satisfying (i), (ii) or (iii) of Claim 2.6B, of which there are at most $|V(T) \setminus C|$, $|N_2(u_1)|$, and $|N_2(u_p)|$, respectively.

If $m = 0$ and $\deg(u_1) \geq 2$, we apply Claim 2.6B to the last vertex u_i of $N^-(u_1)$ on T and $u_j = u_p$, and we use a similar argument if $\deg(u_p) \geq 2$. ∎

Now we may complete the proof of Theorem 2.6. Suppose $\deg(u_1) = 1$ and $\deg(u_p) = 1$. Then $\deg(w) \geq n-3$, but w is not adjacent to u_1, u_p or itself, so $\deg(w) = n-3$, and w is adjacent to all other vertices of G. Since w has a neighbor on T, $p \geq 3$. If $p \geq 4$, then w is adjacent to u_2 and u_3, and $u_1u_2wu_3u_4 \ldots u_p$ is a 2-trail with more vertices than T, a contradiction. Therefore $p = 3$ and $N(w) \cap V(T) = \{u_2\}$. If w has a neighbor x not on T, then u_1u_2wx is a 2-trail with more vertices than T, so $N(w) = \{u_2\}$. Hence, $1 = \deg(w) = n-3$, $n = 4$, and $G \cong K_{1,3}$.

Now suppose that $\deg(u_1) \geq 2$ or $\deg(u_p) \geq 2$. Then

$$\begin{aligned}
|C| &= |N(w)| + |N^-(u_1) \cup N^+(u_p)| \qquad \text{by Lemma 2.4} \\
&= |N(w)| + |N^-(u_1)| + |N^+(u_p)| - |N^-(u_1) \cap N^+(u_p)| \\
&= |N(w)| + |N(u_1)| + |N_2(u_1)| + |N(u_p)| + |N_2(u_p)| \\
&\qquad - |N^-(u_1) \cap N^+(u_p)| \qquad \text{by Lemma 2.5}
\end{aligned}$$

$$\begin{aligned} &\geq\ n - 1 + |N_2(u_1)| + |N_2(u_p)| - |N^-(u_1) \cap N^+(u_p)| \\ &\qquad \text{since } \sigma_3 \geq n-1 \\ &\geq\ n - |V(T) \setminus C| \qquad \text{by Claim 2.6C} \end{aligned}$$

and so $|C| + |V(T) \setminus C| \geq n$, which is impossible because C and $V(T) \setminus C$ are disjoint and w belongs to neither.

This concludes the proof of Theorem 2.6. ∎

The condition $\sigma_3(G) \geq n - 1$ in Theorem 2.6 is best possible. For $n \geq 4$, let A_n be the graph obtained from K_{n-3} by choosing a vertex v, then adding new vertices w, x_1, x_2 and edges vw, wx_1, wx_2. These graphs appear in Theorem G, above. For $n \geq 5$, A_n has $\sigma_3 = n-2$ and no spanning trail of any kind. Moreover, $K_{1,3}$, the exceptional graph in Theorem 2.6, is just A_4. As another family of examples, for $m \geq 1$ the graph $K_{m,2m+2}$ has $\sigma_3 = 3m = n - 2$ but no spanning 2-trail. Note also that the hypothesis that G is connected is necessary; it is easy to construct disconnected graphs with $\sigma_3 \geq n - 1$.

We also have the following straightforward corollary, which is sharp, at least when $n \equiv 2 \pmod 3$, as shown by $K_{m,2m+2}$. As usual, $\delta(G)$ denotes the minimum degree of G, and we use the facts that $\sigma_3(G) \geq 3\delta(G)$ and $\sigma_3(G) \geq 3\sigma_2(G)/2$.

Corollary 2.7 *Let G be a connected n-vertex graph. If $\delta(G) \geq (n-1)/3$ or $\sigma_2(G) \geq 2(n-1)/3$ then G has a (possibly open) spanning 2-trail.*

3 Closed 2-trails and hamilton paths

Now we consider the situation where we have a spanning 2-trail, and try to find a closed spanning 2-trail. The following fact will be used frequently.

Observation 3.1 *If $u_1u_2 \ldots u_p$ is a trail and $u_k = u_l$ with $k < l$, then $l \geq k + 3$.*

Lemma 3.2 *Suppose Assumption 2.1 holds, and $u_k = u_l$ with $k < l$.*

(i) *Suppose $k \leq i \leq l$, and u_1u_i or u_pu_i is in $E(G) \setminus E(T)$. If $i \geq k+1$ then u_{i-1} is used only once by T, and if $i \leq l - 1$ then u_{i+1} is used only once by T.*

(ii) *None of u_1u_{k+1}, u_pu_{k+1}, u_1u_{l-1} or u_pu_{l-1} is in $E(G) \setminus E(T)$.*

(iii) *$u_k = u_l$, u_{k-1}, u_{k+1}, u_{l-1}, and u_{l+1} all exist and are distinct vertices.*

Proof. (i) Suppose $k+1 \le i \le l$, and $u_p u_i \in E(G) \setminus E(T)$. Then there is a spanning 2-trail $T' = u_{i-1}u_{i-2}\dots u_{k+1}(u_k = u_l)u_{l+1}\dots u_p u_i u_{i+1}\dots u_{l-1}$ $(u_l = u_k)u_{k-1}\dots u_1$ with the same number of edges as T. Hence, by Lemma 2.2, T', and hence T, uses u_{i-1} only once. The other cases are similar.

(ii) If one of the given edges is in $E(G) \setminus E(T)$, then by (i) $u_k = u_l$ would be used only once by T, a contradiction.

(iii) Note that $k \ge 2$ and $l \le p-1$ by Lemma 2.2, so all these vertices exist. All of $u_{k-1}, u_{k+1}, u_{l-1}, u_{l+1}$ are joined to $u_k = u_l$ by distinct edges of T, so all five vertices are distinct. ∎

Given a trail $T = v_1 v_2 \dots v_k$, define $\rho_{\min}(T)$ to be the position of the first repeated vertex, i.e., the smallest i such that $v_i = v_j$ for some $j \ne i$, or ∞ if T has no repeated vertices. Similarly, define $\rho_{\max}(T)$ to be the largest i such that $v_i = v_j$ for some $j \ne i$, or $-\infty$ if T has no repeated vertices.

The main result of this section is the following. Note that the proof requires that we take care to distinguish between indices of vertices along T and the vertices themselves, and between elements of $E(G) \setminus E(T)$ and of $E(T)$.

Theorem 3.3 *Let G be a connected n-vertex graph with $\sigma_3(G) \ge n$. Then at least one of the following holds.*

(i) *G has a hamilton path.*

(ii) *G has a closed spanning 2-trail.*

Proof. Suppose G has neither a hamilton path nor a closed spanning 2-trail. Let $T = u_1 u_2 \dots u_p$ be a spanning 2-trail with (1) fewest edges; (2) subject to (1), smallest $\rho_{\min}(T)$; and (3) subject to (1) and (2), largest $\rho_{\max}(T)$. Assumption 2.1 holds for T. Since G has no hamilton path, $q = \rho_{\min}(T)$ is finite, and $u_q = u_t$ for some $t > q$.

If u_1 and u_p are adjacent, then there is a closed spanning 2-trail, so they are nonadjacent. We begin by finding a vertex u_{r+1} which will form the third vertex of an independent set with u_1 and u_p.

Claim 3.3A We may assume (possibly after a reordering of the vertices and edges of T that does not change $\rho_{\min}(T)$ or $\rho_{\max}(T)$) that there exists r, $2 \le r \le p-4$, such that

(a) $u_r = u_s$ for some s, $r+3 \le s \le p-1$.

(b) $r = q$ or $q+1$, so that each of $u_1, \dots, u_{r-2}$ is used only once on T.

(c) Either u_{r-1} is used only once on T, or $u_1 u_r$ and $u_p u_r$ are not in $E(G) \setminus E(T)$, or both.

(d) $u_{r+1} \ne u_2$ and $u_{r+1} \ne u_{p-1}$.

(e) There is no $i \geq r+1$ such that $u_{r+1}u_{i+1} \in E(T)$ and at least one of u_1u_i or u_pu_i is in $E(G) \setminus E(T)$.

Proof. Take $r = q$ (and $s = t$). This satisfies (a)–(c) above, since $u_1, \ldots, u_{q-1}$ are used only once on T. If $u_{q+1} = u_2$ then u_2 is used twice, so $q = 2$ giving $u_{q+1} = u_3 = u_2$, which is impossible. Thus, $u_{q+1} \neq u_2$. If $u_{q+1} = u_{p-1}$ then, by Lemma 3.2(iii), $u_{t-1} \neq u_{q+1} = u_{p-1}$. Therefore, by reversing the segment $u_qu_{q+1}\ldots u_{t-1}u_t$ of T, we may assume that $u_{q+1} \neq u_{p-1}$, and (d) also holds. We shall show that either (e) holds, or we can make another choice of r.

In our arguments we shall suppose that (e) does not hold. In doing this, we focus on the vertex u_{i+1} as described in (e). We say 'Suppose $x = u_{i+1}$ as in (e)' to mean 'Suppose there is $i \geq r+1$ so that $x = u_{i+1}$, $u_{r+1}u_{i+1} \in E(T)$ and at least one of u_1u_i or u_pu_i is in $E(G) \setminus E(T)$.'

Suppose $u_q = u_t = u_{i+1}$ as in (e). Since $i \geq r+1 = q+1$, $i = t-1$, which contradicts Lemma 3.2(ii) for $u_q \ldots u_t$.

Suppose $u_{q+2} = u_{i+1}$ as in (e). If $i = q+1$ then Lemma 3.2(ii) for $u_q \ldots u_t$ is contradicted. So $u_{q+2} = u_{i+1}$ for some $i \neq q+1$. By Observation 3.1, either $i+1 \leq q-1$ or $i+1 \geq q+5$. However, the former contradicts $q = \rho_{\min}(T)$, so $i+1 \geq q+4$ and in particular $i+1 > q+2$. Now u_1u_i or u_pu_i contradicts Lemma 3.2(ii) for $u_{q+2}\ldots u_{i+1}$.

If u_{q+1} is used only once by T, we are finished. So, assume that u_{q+1} is used twice, with $u_{q+1} = u_l$, where $l \geq q+4$ since $q = \rho_{\min}(T)$.

Suppose $u_{l+1} = u_{i+1}$ as in (e). If $i = l$ then $u_1u_l = u_1u_{q+1}$ or $u_pu_l = u_pu_{q+1}$ contradicts Lemma 3.2(ii) for $u_q \ldots u_t$. Therefore, $u_{l+1} = u_{i+1}$ where $i \neq l$. If $i > l$ then u_1u_i or u_pu_i contradicts Lemma 3.2(ii) for $u_{l+1}\ldots u_{i+1}$, so $i < l$. Since $q = \rho_{\min}(T)$, $i+1 \geq q$. By Lemma 3.2(iii), $u_l = u_{q+1}$, u_q and $u_{l+1} = u_{i+1}$ are distinct. Therefore, $i+1 \neq q$ or $q+1$, so $i+1 \geq q+2$, or $i \geq q+1$. Now applying Lemma 3.2(i) to u_i in $u_{q+1}\ldots u_l$, we see that u_{i+1} is used only once by T, a contradiction.

Suppose $u_{l-1} = u_{i+1}$ as in (e). If $i = l-2$ then we choose a new r (see below). So, suppose $i \neq l-2$. Then $u_{l-1} = u_{i+1}$ where $l-1 \neq i+1$. By an argument similar to the one for $u_{l+1} = u_{i+1}$, above, we obtain a contradiction to Lemma 3.2(i).

So, we satisfy conditions (a)–(e) with $r = q$, unless $u_{q+1} = u_l$ where u_1u_{l-2} or u_pu_{l-2} is in $E(G) \setminus E(T)$. In that case, reorder T as $T' = u_1u_2\ldots u_q(u_{q+1} = u_l)u_{l-1}\ldots u_{q+2}(u_{q+1} = u_l)u_{l+1}\ldots u_p = v_1\ldots v_p$, and take $r = q+1$. Then $v_r = u_{q+1} = u_l$, and neither $v_1v_r = u_1u_{q+1}$ nor $v_pv_r = u_pu_{q+1}$ are in $E(G) \setminus E(T)$ by Lemma 3.2(ii) on $u_q\ldots u_t$. Thus, (a)–(c) are satisfied. Note that by Lemma 3.2(i) for u_{l-2} on $u_{q+1}\ldots u_l$ we have that T uses $u_{l-1} = v_{r+1}$ only once, so T' also uses v_{r+1} only once. For (d), since $4 \le r+1 \le p-2$ and v_{r+1} is used only once, $v_{r+1} \neq v_2$ or v_{p-1}. For (e), since $v_{r+1} = u_{l-1}$ is used only once, we need only show that neither $v_r = v_l = u_l = u_{q+1}$ nor $v_{r+2} = u_{l-2}$ can be v_{i+1} as in (e).

Suppose $v_r = v_l = v_{i+1}$ as in (e) (applied to T'). Since $i \ge r+1$, $i = l-1$, which contradicts Lemma 3.2(ii) for $v_r \dots v_l$.

Suppose $v_{r+2} = v_{i+1}$ as in (e). If $i = r+1$ then Lemma 3.2(ii) for $v_r \dots v_l$ is contradicted. So $v_{r+2} = v_{i+1}$ for some $i \neq r+1$. Either $i+1 \le r-1 = q$ or $i+1 \ge r+5$. Since $q = \rho_{\min}(T)$, $i+1 \ge q$. If $i+1 = q = r-1$, then by Lemma 3.2(i) applied to $v_{r-1} = v_{r+2}$ on $v_{r-1} \dots v_{r+2}$, $v_r = u_{q+1}$ is used only once on T', and hence on T, a contradiction. Thus, $i+1 \ge r+5$ and in particular $i+1 > r+2$. Now $v_1 v_i$ or $v_p v_i$ contradicts Lemma 3.2(ii) for $v_{r+2} \dots v_{i+1}$.

The two reorderings of T mentioned above do not change $\rho_{\min}(T)$ or $\rho_{\max}(T)$, so this completes the proof of the claim. ∎

Now u_{r+1} is not adjacent to u_1 or u_p by an element of $E(T)$, by Claim 3.3A(d), or by an element of $E(G) \setminus E(T)$, by Lemma 3.2(ii) for $u_r \dots u_s$. Therefore, $\{u_1, u_{r+1}, u_p\}$ is an independent set.

For an arbitrary v, define the following sets of integers:

$$\begin{aligned} I(v) &= \{i \mid i \le r \text{ and } vu_i \in E(G) \setminus E(T)\}, \\ J(v) &= \{i \mid i \ge r+1 \text{ and } vu_i \in E(G) \setminus E(T)\}, \end{aligned}$$

Observe that

$$\begin{aligned} I(u_1) &\subseteq \{3, 4, \dots, r\}, \qquad & J(u_1) &\subseteq \{r+1, r+2, \dots, p-1\}, \\ I(u_p) &\subseteq \{2, 3, \dots, r\}, \qquad & J(u_p) &\subseteq \{r+1, r+2, \dots, p-2\}. \end{aligned}$$

Therefore, for $v = u_1$ or u_p, it makes sense to define

$$\begin{aligned} A^-(v) &= \{u_{i-1} \mid i \in I(v)\}, \\ B^+(v) &= \{u_{i+1} \mid i \in J(v)\}. \end{aligned}$$

Then, let

$$C = N(u_{r+1}) \cup A^-(u_1) \cup A^-(u_p) \cup B^+(u_1) \cup B^+(u_p).$$

Claim 3.3B $N(u_{r+1})$, $A^-(u_1) \cup A^-(u_p)$, and $B^+(u_1) \cup B^+(u_p)$ are pairwise disjoint.

Proof. Suppose first that v is in both $A^-(u_1) \cup A^-(u_p)$ and $B^+(u_1) \cup B^+(u_p)$. Then $v = u_{i-1} = u_{j+1}$ where $i \in I(u_1) \cup I(u_p)$, $j \in J(u_1) \cup J(u_p)$, so that

$i-1 <= r < j+1$. Then the edge u_1u_i or u_pu_i contradicts Lemma 3.2(ii) on $u_{i-1}\dots u_{j+1}$.

Now suppose that v is in both $B^+(u_1)\cup B^+(u_p)$ and $N(u_{r+1})$. Write $v = u_{i+1}$ where $i \in J(u_1)\cup J(u_p)$, so that $i \geq r+1$. By Claim 3.3A(e), $u_{r+1}u_{i+1} \notin E(T)$. Since $u_{r+1}u_s \in E(T)$, $i \neq s-1$. We show that in all cases a spanning 2-trail T' with fewer edges than T can be found, giving a contradiction. If $r+1 \leq i \leq s-2$ and $u_1u_i \in E(G)\setminus E(T)$, let

$$T' = u_pu_{p-1}\dots u_{s+1}(u_s = u_r)u_{r-1}\dots u_1u_iu_{i-1}\dots u_{r+1}u_{i+1}u_{i+2}\dots u_{s-1}.$$

A similar trail T' can be found if $r+1 \leq i \leq s-2$ and $u_pu_i \in E(G)\setminus E(T)$. If $i \geq s$ and $u_1u_i \in E(G)\setminus E(T)$, let

$$T' = u_pu_{p-1}\dots u_{i+1}u_{r+1}u_{r+2}\dots u_s\dots u_iu_1u_2\dots u_{r-1}.$$

If $i \geq s$ and $u_pu_i \in E(G)\setminus E(T)$, let

$$T' = u_1u_2\dots u_{r-1}(u_r = u_s)u_{s+1}\dots u_iu_pu_{p-1}\dots u_{i+1}u_{r+1}u_{r+2}\dots u_{s-1}.$$

Now suppose that v is in both $A^-(u_1)\cup A^-(u_p)$ and $N(u_{r+1})$. Write $v = u_{i-1}$, where $i \in I(u_1)\cup I(u_p)$, so that $i \leq r$. If $u_{r+1}u_{i-1} \in E(T)$, then u_{i-1} must be used twice on T, so by Claim 3.3A(b), $i = r$. However, since $u_{i-1} = u_{r-1}$ is used twice, by Claim 3.3A(c) $i = r \notin I(u_1)\cup I(u_p)$, which is a contradiction. Thus, $u_{r+1}u_{i-1} \in E(G)\setminus E(T)$, and the result follows by arguments similar to those of the previous paragraph for the case $i \geq s$.

This completes the proof of the claim. ∎

Claim 3.3C (a) The map $i \mapsto u_{i-1}$ provides bijections $I(u_1) \to A^-(u_1)$, $I(u_p) \to A^-(u_p)$, and $I(u_1)\cap I(u_p) \to A^-(u_1)\cap A^-(u_p)$.

(b) The map $i \mapsto u_{i+1}$ provides bijections $J(u_1) \to B^+(u_1)$, $J(u_p) \to B^+(u_p)$, and $J(u_1)\cap J(u_p) \to B^+(u_1)\cap B^+(u_p)$.

Proof. The arguments for $I(u_1) \to A^-(u_1)$, $I(u_p) \to A^-(u_p)$, $J(u_1) \to B^+(u_1)$ and $J(u_p) \to B^+(u_p)$ are similar; we prove the third. By definition of $B^+(u_1)$, $i \mapsto u_{i+1}$ maps $J(u_1)$ onto $B^+(u_1)$. We prove it is also one-to-one. If not, then there are $i, j \in J(u_1)$ with $i < j$ such that $u_{i+1} = u_{j+1}$. But now $u_1u_j \in E(G)\setminus E(T)$ contradicts Lemma 3.2(ii) for $u_{i+1}\dots u_{j+1}$.

The arguments for $I(u_1)\cap I(u_p) \to A^-(u_1)\cap A^-(u_p)$ and $J(u_1)\cap J(u_p) \to B^+(u_1)\cap B^+(u_p)$ are similar; we prove the latter. The map $i \mapsto u_{i+1}$ maps $J(u_1)\cap J(u_p)$ into $B^+(u_1)\cap B^+(u_p)$, but we must prove that this map is one-to-one and onto. It is one-to-one because it is a restriction of the one-to-one map $J(u_1) \to B^+(u_1)$. To show it is onto, consider any $v \in B^+(u_1)\cap B^+(u_p)$.

Since $v \in B^+(u_1)$, $v = u_{i+1}$ where $i \in J(u_1)$, and since $v \in B^+(u_p)$, $v = u_{j+1}$ where $j \in J(u_p)$. If $i < j$ then $u_p u_j \in E(G) \setminus E(T)$ contradicts Lemma 3.2(ii) for $u_{i+1} \dots u_{j+1}$, and if $i > j$ we get a similar contradiction. Therefore $i = j \in J(u_1) \cap J(u_p)$ maps to $v = u_{i+1}$. ∎

We now modify our definitions of $N(v)$ and $N_i(v)$ to get $N^*(v)$ and $N_i^*(v)$, where $N^*(v) = \{w \mid vw \in E(G) \setminus E(T)\}$, and $N_i^*(v)$ contains the vertices of $N^*(v)$ that are used i times on T, $i = 1$ or 2. Thus, $N^*(u_1) = N(u_1) \setminus \{u_2\}$, and $N^*(u_p) = N(u_p) \setminus \{u_{p-1}\}$.

Claim 3.3D If $v = u_1$ or u_p then $|I(v)| + |J(v)| = |N(v)| + |N_2^*(v)| - 1$,

Proof. In both cases $|N^*(v)| = |N(v)| - 1$, so

$$\begin{aligned} |I(v)| + |J(v)| &= |\{i \mid vu_i \in E(G) \setminus E(T)\}| \\ &= |N_1^*(v)| + 2|N_2^*(v)| = |N^*(v)| + |N_2^*(v)| \\ &= |N(v)| + |N_2^*(v)| - 1. \end{aligned}$$

∎

Claim 3.3E (a) If $i \in (I(u_1) \cap I(u_p)) \cup (J(u_1) \cap J(u_p))$ then $u_i \in N_2^*(u_1) \cap N_2^*(u_p)$.

(b) $|I(u_1) \cap I(u_p)| + |J(u_1) \cap J(u_p)| = 2|N_2^*(u_1) \cap N_2^*(u_p)|$.

Proof. (a) By definition, $u_i \in N^*(u_1) \cap N^*(u_p)$. If u_i is not used twice, then $u_1 u_2 \dots u_p u_i u_1$ is a closed spanning 2-trail, which is a contradiction.

(b) Each element of $N_2^*(u_1) \cap N_2^*(u_p)$ has the form u_i for exactly two numbers $i \in (I(u_1) \cap I(u_p)) \cup (J(u_1) \cap J(u_p))$. By (a), these numbers i include all elements of $(I(u_1) \cap I(u_p)) \cup (J(u_1) \cap J(u_p))$. ∎

Claim 3.3F $u_{r+1} \notin C$.

Proof. Clearly $u_{r+1} \notin N(u_{r+1})$. If $u_{r+1} \in A^-(u_1) \cup A^-(u_p)$, then $u_{r+1} = u_{i-1}$ where $i \leq r$. By Claim 3.3A(b), we must have $i - 1 = r - 1$, which contradicts Observation 3.1. If $u_{r+1} \in B^+(u_1) \cup B^+(u_p)$, then $u_{r+1} = u_{i+1}$ where $i \geq r + 1$. Then the edge $u_1 u_i$ or $u_p u_i$ in $E(G) \setminus E(T)$ contradicts Lemma 3.2(ii) for $u_{r+1} \dots u_{i+1}$. ∎

Claim 3.3G We have

$$|V(G) \setminus \{u_{r+1}\} \setminus C| + |N_2^*(u_p) \setminus N_2^*(u_1)| + |N_2^*(u_1) \setminus N_2^*(u_p)| \leq 1.$$

Proof. We have

$$
\begin{aligned}
|C| &= |N(u_{r+1})| + |A^-(u_1) \cup A^-(u_p)| + |B^+(u_1) \cup B^+(u_p)| \\
&\qquad\qquad \text{by Claim 3.3B} \\
&= |N(u_{r+1})| + |A^-(u_1)| + |A^-(u_p)| - |A^-(u_1) \cap A^-(u_p)| \\
&\qquad\qquad + |B^+(u_1)| + |B^+(u_p)| - |B^+(u_1) \cap B^+(u_p)| \\
&= |N(u_{r+1})| + |I(u_1)| + |I(u_p)| - |I(u_1) \cap I(u_p)| \\
&\qquad\qquad + |J(u_1)| + |J(u_p)| - |J(u_1) \cap J(u_p)| \qquad \text{by Claim 3.3C} \\
&= |N(u_{r+1})| + |N(u_1)| + |N(u_p)| + |N_2^*(u_1)| + |N_2^*(u_p)| - 2 \\
&\qquad\qquad - |I(u_1) \cap I(u_p)| - |J(u_1) \cap J(u_p)| \qquad \text{by Claim 3.3D} \\
&\geq n + |N_2^*(u_1)| + |N_2^*(u_p)| + |I(u_1) \cap I(u_p)| - |J(u_1) \cap J(u_p)| - 2 \\
&\qquad\qquad \text{since } \sigma_3 \geq n \\
&= n + |N_2^*(u_1)| + |N_2^*(u_p)| - 2|N_2^*(u_1) \cap N_2^*(u_p)| - 2 \\
&\qquad\qquad \text{by Claim 3.3E} \\
&= n + |N_2^*(u_p) \setminus N_2^*(u_1)| + |N_2^*(u_1) \setminus N_2^*(u_p)| - 2
\end{aligned}
$$

and the result follows since $|V(G) \setminus \{u_{r+1}\} \setminus C| = n - |C| - 1$, by Claim 3.3F. ∎

Now we complete the proof of Theorem 3.3. Recall that $q = \rho_{\min}(T) \geq 2$, and let $a = \rho_{\max}(T) \leq p-1$, so that $u_{a+1}, u_{a+2}, \ldots, u_p$ are used only once. Let $\alpha_1 = |\{u_1, u_2, \ldots, u_{q-1}\} \setminus C|$, $\alpha_2 = |\{u_{a+1}, u_{a+2}, \ldots, u_p\} \setminus C|$, $\beta_1 = |N_2^*(u_p) \cap \{u_2\}|$, and $\beta_2 = |N_2^*(u_1) \cap \{u_{p-1}\}|$. Because $u_2 \notin N^*(u_1)$ and $u_{p-1} \notin N^*(u_p)$, Claim 3.3G implies that $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leq 1$.

Suppose $\alpha_1 + \beta_1 = 0$. Then $\alpha_1 = 0$, so $u_1, u_2, \ldots, u_{q-1} \in C$. Since $u_1 \in C$ but $u_1 \notin N(u_{r+1})$, $A^-(u_1)$, $B^+(u_1)$ or $B^+(u_p)$, we get $u_1 \in A^-(u_p)$, and hence $u_2 \in N^*(u_p)$. Since $\beta_1 = 0$, u_2 must be used only once on T. Hence $q \geq 3$. We shall prove that $u_1, u_2, \ldots, u_{q-2} \in A^-(u_p)$. If $q = 3$ we are finished. Suppose that $2 \leq j \leq q-2$, and we have proved that $u_1, \ldots, u_{j-1} \in A^-(u_p)$. Since $u_{j-1} \in A^-(u_p)$, $u_j \in N^*(u_p)$. We know u_j is in C, but $u_j \notin B^+(u_1) \cap B^+(u_p)$. If $u_j \in A^-(u_1)$ then $u_{j+1} \in N^*(u_1)$ and hence

$$T' = u_1 u_2 \ldots u_j u_p u_{p-1} \ldots u_{j+1} u_1$$

is a closed 2-trail, a contradiction. If $u_j \in N(u_{r+1})$ then

$$T' = u_1 u_2 \ldots u_{r-1}(u_r = u_s) u_{s+1} \ldots u_p u_j u_{r+1} u_{r+2} \ldots u_{s-2}$$

has $\rho_{\min}(T') = j < q$, contradicting our choice of T. Therefore $u_j \in A^-(u_p)$. Repeating this argument, we get $u_{q-2} \in A^-(u_p)$ which means that $u_{q-1} \in N^*(u_p)$. But then, since $u_q = u_t$ with $t > q$,

$$T' = u_1 u_2 \ldots u_{q-1} u_p u_{p-1} \ldots u_t u_{t-1} \ldots u_{q+1}$$

has fewer edges than T, a contradiction.

Suppose now that $\alpha_1 + \beta_1 = 1$. Then $\alpha_2 + \beta_2 = 0$. By reasoning symmetric to the above we can show that $u_p \in B^+(u_1)$, so that $u_{p-1} \in N^*(u_1)$, and u_{p-1} is used only once, so that $a = \rho_{\max}(T) \leq p-2$. If $a = p-2$, we can use an argument symmetric to the one above, so suppose that $a \leq p-3$. We know that $u_{p-1} \in C$. By an argument symmetric to the one above, $u_{p-1} \notin B^+(u_p)$. Suppose that $u_{p-1} \in N(u_{r+1})$. Consider the trail

$$T' = u_{r-1} u_{r-2} \ldots u_1 u_{p-1} u_{r+1} u_{r+2} \ldots u_s \ldots u_{p-1} u_p.$$

Considering $(T')^{-1}$, the reverse of T', we have $\rho_{\min}((T')^{-1}) = 2$ and therefore $q = \rho_{\min}(T) = 2$ by choice of T. Also, by Lemma 2.2, u_{r-1} is used only once on T' and hence only once on T, so by Claim 3.3A(b), $r = q = 2$. Now we see that $\rho_{\min}(T') = 2 = \rho_{\min}(T)$ and $\rho_{\max}(T') = p-1 > \rho_{\max}(T)$. This contradicts our choice of T. Therefore $u_{p-1} \in B^+(u_1)$, so that $u_{p-2} \in N^*(u_1)$. But then the trail

$$T' = u_2 u_3 \ldots u_{p-2} u_1 u_{p-1} u_p$$

has $\rho_{\min}(T') = q-1 < \rho_{\min}(T)$, a contradiction.

This completes the proof of Theorem 3.3. ∎

The condition $\sigma_3(G) \geq n$ in Theorem 3.3 is best possible. For all $n_1, n_2, n_3 \geq 1$ the graph $K_1 + (K_{n_1} \cup K_{n_2} \cup K_{n_3})$ ('+' denotes join) has $\sigma_3 = n_1 + n_2 + n_3 = n-1$ but neither a hamilton path nor a closed spanning 2-trail. These graphs also appear as sharpness examples in [10]. Also, for every $m \geq 1$ the graph $K_{m,2m+1}$ has $\sigma_3 = 3m = n-1$ but neither a hamilton path nor a closed spanning 2-trail. Note also that the hypothesis that G is connected is necessary; it is easy to construct disconnected graphs with $\sigma_3 \geq n$.

We also have the following straightforward corollary, which is sharp, at least when $n \equiv 1 \pmod 3$, as shown by $K_{m,2m+1}$.

Corollary 3.4 *Let G be a connected n-vertex graph. If $\delta(G) \geq n/3$ or $\sigma_2(G) \geq 2n/3$ then G has either a hamilton path or a closed spanning 2-trail.*

It is also interesting to note the following, which follows from Theorem 3.3 because a graph with a cutedge cannot have a closed spanning trail.

Corollary 3.5 *If G is an n-vertex graph with a cutedge and $\sigma_3(G) \geq n$ then G has a hamilton path.*

4 Closed 2-trails in 2-edge-connected graphs

In this section we prove our third main result, as follows.

Theorem 4.1 *Let G be a 2-edge-connected n-vertex graph with $\sigma_3(G) \geq n$. Then either*

(i) *G is isomorphic to $K_{2,3}$, or the 6-vertex graph $K^*_{2,3}$ obtained by subdividing an edge of $K_{2,3}$, or*

(ii) *G has a closed spanning 2-trail.*

Proof. The result holds for $n = 1$ (whether or not one considers K_1 to be 2-edge-connected), so suppose that $n \geq 2$. Assume that G does not have a closed spanning 2-trail. By Theorem 3.3, G has a hamilton path $T = u_1u_2 \dots u_p$, where now $p = n$, and $u_1u_p \notin E(G)$. Since G is 2-edge-connected, $\deg(u_1), \deg(u_p) \geq 2$, so that $N^*(u_1), N^*(u_p) \neq \emptyset$, and $n \geq 4$. Define

$$\eta(T) = \min\{j - i \mid u_i \in N^*(u_1), u_j \in N^*(u_p)\},$$

which may be positive, zero or negative.

Claim 4.1A There exists a hamilton path T with $\eta(T) \leq 0$.

Proof. Suppose not. Let T be a hamilton path with $\eta(T)$ as small as possible. Then there exist q, t with $t - q = \eta(T) > 0$ such that $u_q \in N^*(u_1)$, $q \geq 3$, $N(u_1) \subseteq \{u_2, u_3, \dots, u_q\}$, $u_t \in N^*(u_p)$, $t \leq p - 2$, and $N(u_p) \subseteq \{u_t, u_{t+1}, \dots u_{p-1}\}$.

Suppose first that there exists some $r \leq q - 1$ such that $u_ru_s \in E(G)$ for some $s \geq q + 1$. Then $r \geq 2$. Choose r as large as possible. If $r = q - 1$ then

$$T' = u_{s-1}u_{s-2} \dots u_qu_1u_2 \dots u_{q-1}u_su_{s+1} \dots u_p$$

has $\eta(T') \leq t - s < \eta(T)$, a contradiction. So $r \leq q - 2$. Define

$$\begin{aligned} A^-(u_1) &= \{u_{i-1} \mid i \leq r, u_1u_i \in E(G)\}, \\ B^+(u_1) &= \{u_{i+1} \mid i \geq r + 1, u_1u_i \in E(G)\}. \end{aligned}$$

Clearly $|A^-(u_1)| + |B^+(u_1)| = |N(u_1)|$.

Now we claim that $A^-(u_1)$, $B^+(u_1)$, $N(u_{r+1})$ and $N^+(u_p)$ are pairwise disjoint. Since $A^-(u_1) \subseteq \{u_1, \dots u_{r-1}\}$, $B^+(u_1) \subseteq \{u_{r+2}, \dots u_{q+1}\}$, and $N^+(u_p) \subseteq \{u_{t+1}, \dots, u_p\}$, these three sets are disjoint; furthermore, u_{r+1} belongs to none of them. Therefore we need only consider intersections of $N(u_{r+1})$ with the other three sets. If there is $u_{i-1} \in A^-(u_1) \cap N(u_{r+1})$, then

$$T' = u_{s-1}u_{s-2} \dots u_{r+1}u_{i-1}u_{i-2} \dots u_1u_iu_{i+1} \dots u_ru_su_{s+1} \dots u_p$$

has $\eta(T') \leq t - s < \eta(T)$, a contradiction (this works even if $i = 2$). If there is $u_{i+1} \in B^+(u_1) \cap N(u_{r+1})$, then because $r + 1 \leq q - 1$ and we chose r as large as possible, $i \leq q - 1$, and

$$T' = u_{s-1}u_{s-2} \dots u_{i+1}u_{r+1}u_{r+2} \dots u_iu_1u_2 \dots u_ru_su_{s+1} \dots u_p$$

has $\eta(T') \leq t - s < \eta(T)$, a contradiction. If there is $u_{i+1} \in N^+(u_p) \cap N(u_{r+1})$ then because $r + 1 \leq q - 1$, we contradict our choice of r as large as possible.

Since $u_1 \in A^-(u_1)$ and $u_p \in N^+(u_p)$, the above shows us that $\{u_1, u_{r+1}, u_p\}$ is independent. We therefore have

$$\begin{aligned} n &\leq |N(u_1)| + |N(u_p)| + |N(u_{r+1})| \\ &= |A^-(u_1)| + |B^+(u_1)| + |N^+(u_p)| + |N(u_{r+1})| \end{aligned}$$

which is impossible, since these four sets are disjoint and u_{r+1} belongs to none of them.

Therefore, there is no number r as described above, and u_q is a cutvertex. Making the same argument with T reversed, which does not change $\eta(T)$, we find that u_t is also a cutvertex. If $t = q+1$ then u_qu_t is a cutedge, which is a contradiction. Thus, $t \geq q + 2$. Now $\{u_1, u_{q+1}, u_p\}$ is independent, and it is easy to see that $\deg(u_1) + \deg(u_{q+1}) + \deg(u_p) \leq n - 1$, again a contradiction.

Hence, there must exist T with $\eta(T) \leq 0$. ∎

Now for each hamilton path T define

$$\theta(T) = \min\{i - j \mid u_j \in N^*(u_p), u_i \in N^*(u_1), j \leq i\},$$

which is finite when $\eta(T) \leq 0$, and ∞ (the minimum of the empty set) when $\eta(T) > 0$. Let T be a hamilton path with $\theta(T)$ as small as possible. By Claim 4.1A, $\theta(T)$ is finite, so there exist r, s with $s - r = \theta(T)$, $u_r \in N^*(u_p)$ and $u_s \in N^*(u_1)$. If $\theta(T) = 0$, we have a closed spanning 2-trail $u_1u_2 \dots u_pu_{r=s}u_1$, and if $\theta(T) = 1$, then we have a hamilton cycle $u_1u_2 \dots u_ru_pu_{p-1} \dots u_{r+1=s}u_1$.

So we may assume that $\theta(T) \geq 2$, i.e., $s \geq r+2$. None of $u_{r+1}, u_{r+2}, \ldots u_{s-1}$ are adjacent to u_1 or u_p, by definition of $\theta(T)$. For $v = u_1$ or u_p, define

$$\begin{aligned} A^-(v) &= \{u_{i-1} \mid i \leq r, vu_i \in E(G)\}, \\ B^+(v) &= \{u_{i+1} \mid i \geq s, vu_i \in E(G)\}. \end{aligned}$$

so that $|A^-(v)| + |B^+(v)| = |N(v)|$. Also, let

$$C = N(u_{r+1}) \cup A^-(u_1) \cup A^-(u_p) \cup B^+(u_1) \cup B^+(u_p).$$

Claim 4.1B $A^-(u_1)$, $A^-(u_p)$, $B^+(u_1)$, $B^+(u_p)$ and $N(u_{r+1})$ are pairwise disjoint, except that possibly $u_1 \in A^-(u_1) \cap A^-(u_p)$, and possibly $u_p \in B^+(u_1) \cap B^+(u_p)$.

Proof. Since $A^-(u_1), A^-(u_p) \subseteq \{u_1, \ldots u_{r-1}\}$ and $B^+(u_1), B^+(u_p) \subseteq \{u_{s+1}, \ldots u_p\}$, the former two are disjoint from the latter two. If there is a vertex other than u_1 in $A^-(u_1) \cap A^-(u_p)$, or other than u_p in $B^+(u_1) \cap B^+(u_p)$, then there is $u_j \in N^*(u_1) \cap N^*(u_p)$ and

$$T' = u_1 u_2 \ldots u_p u_j u_1$$

is a closed 2-trail, a contradiction.

Now consider the intersections of $N(u_{r+1})$ with the other sets. If there is $u_{i-1} \in A^-(u_1) \cap N(u_{r+1})$ then

$$T' = u_1 \ldots u_{i-1} u_{r+1} u_{r+2} \ldots u_p u_r u_{r-1} \ldots u_i u_1$$

is a hamilton cycle, a contradiction (this works even when $i = 2$). If there is $u_{i-1} \in A^-(u_p) \cap N(u_{r+1})$ then

$$T' = u_1 u_2 \ldots u_{i-1} u_{r+1} u_{r+2} \ldots u_s \ldots u_p u_i u_{i+1} \ldots u_r$$

has $\theta(T') \leq s - r - 1 < \theta(T)$ because of the edges $u_1 u_s$, $u_r u_{r+1}$, which is a contradiction. If there is $u_{i+1} \in B^+(u_1) \cap N(u_{r+1})$, then

$$T' = u_1 u_2 \ldots u_r u_p u_{p-1} \ldots u_{i+1} u_{r+1} u_{r+2} \ldots u_i u_1$$

is a hamilton cycle, a contradiction. If there is $u_{i+1} \in B^+(u_p) \cap N(u_{r+1})$ then

$$T' = u_1 u_2 \ldots u_i u_p u_{p-1} \ldots u_{i+1}$$

has $\theta(T') \leq s - r - 1 < \theta(T)$ because of the edges $u_1 u_s$, $u_{i+1} u_{r+1}$, which is a contradiction (this works even when $i = p - 1$). Thus, $N(u_{r+1})$ is disjoint from the other four sets.

This ends the proof of the claim. ∎

Claim 4.1C $|V(G) \setminus C| \leq |A^-(u_1) \cap A^-(u_p)| + |B^+(u_1) \cap B^+(u_p)|$.

Proof. By Claim 4.1B, $N(u_{r+1})$, $A^-(u_1) \cup A^-(u_p)$ and $B^+(u_1) \cup B^+(u_p)$ are pairwise disjoint, so

$$\begin{aligned} |C| &= |N(u_{r+1})| + |A^-(u_1) \cup A^-(u_p)| + |B^+(u_1) \cup B^+(u_p)| \\ &= |N(u_{r+1})| + |A^-(u_1)| + |A^-(u_p)| - |A^-(u_1) \cap A^-(u_p)| \\ &\qquad + |B^+(u_1)| + |B^+(u_p)| - |B^+(u_1) \cap B^+(u_p)| \\ &= |N(u_{r+1})| + |N(u_1)| + |N(u_p)| - |A^-(u_1) \cap A^-(u_p)| \\ &\qquad - |B^+(u_1) \cap B^+(u_p)| \\ &\geq n - |A^-(u_1) \cap A^-(u_p)| - |B^+(u_1) \cap B^+(u_p)| \qquad \text{since } \sigma_3 \geq n \end{aligned}$$

and the result follows. ∎

Let $\gamma = |V(G) \setminus C|$, $\alpha = |A^-(u_1) \cap A^-(u_p)|$ and $\beta = |B^+(u_1) \cap B^+(u_p)|$. Claim 4.1C says that $\gamma \leq \alpha + \beta$. Since $u_{r+1} \in V(G) \setminus C$, we know that $\gamma \geq 1$. By Claim 4.1B, $\alpha = 1$ if $u_2 u_p \in E(G) \setminus E(T)$, and 0 otherwise, while $\beta = 1$ if $u_1 u_{p-1} \in E(G) \setminus E(T)$, and 0 otherwise.

Claim 4.1D $\theta(T) = 2$, i.e., $s = r + 2$.

Proof. Suppose that $\theta(T) \geq 3$, i.e., $s \geq r + 3$. If $u_s \in N(u_{r+1})$, then

$$T' = u_1 u_2 \ldots u_r u_p u_{p-1} \ldots u_{r+1} u_s u_1$$

is a closed 2-trail, a contradiction. Thus, $u_s \notin N(u_{r+1})$ and hence $u_s \in V(G) \setminus C$ and $\gamma \geq 2$. Therefore $\alpha = \beta = 1$, so that $u_2 u_p, u_1 u_{p-1} \in E(G) \setminus E(T)$. But now

$$T' = u_3 u_4 \ldots u_p u_2 u_1$$

is a hamilton path with $\theta(T') \leq 2 < \theta(T)$ because of the edges $u_3 u_2$, $u_1 u_{p-1}$, which is a contradiction. ∎

Claim 4.1E Suppose $1 \leq k \leq r - 1$, $u_k \in A^-(u_p)$, and $u_{k+1}, \ldots, u_{r-1} \in C$. Then for $k \leq j \leq r - 1$, we have $u_j \in A^-(u_p)$, and for $k + 1 \leq j \leq r - 1$ we have $u_j \notin N(u_{r+1}) \cup A^-(u_1)$.

Proof. We show that $u_j \in A^-(u_p)$ for $k \leq j \leq r - 1$ by induction on j, proving the rest along the way. We are given $u_k \in A^-(u_p)$. Suppose now that $k + 1 \leq j \leq r - 1$. By induction, $u_{j-1} \in A^-(u_p)$, so $u_j \in N^*(u_p)$. If $u_j \in N(u_{r+1})$, then

$$T' = u_1 u_2 \ldots u_{r+1} u_j u_p u_{p-1} \ldots u_{r+2=s} u_1$$

is a closed spanning 2-trail, a contradiction. If $u_j \in A^-(u_1)$ then

$$T' = u_1u_2 \ldots u_ju_pu_{p-1} \ldots u_{j+1}u_1$$

is a hamilton cycle, a contradiction. Therefore $u_j \notin N(u_{r+1}) \cup A^-(u_1)$, but since $u_j \in C$, we must have $u_j \in A^-(u_p)$. ∎

Claim 4.1F If $\alpha = 1$ and $u_2, \ldots, u_{r-1} \in C$, then
(a) $u_1, u_2, \ldots, u_{r-1} \in A^-(u_p)$, and consequently $u_2, u_3, \ldots, u_r \in N^*(u_p)$;
(b) $N^*(u_1) \subseteq \{u_{r+2}, u_{r+4}, u_{r+5}, \ldots, u_{p-1}\}$;
(c) $N^*(u_{r+1}) \subseteq \{u_{r+4}, \ldots, u_{p-1}\}$.

Proof. Since $\alpha = 1$, we apply Claim 4.1E with $k = 1$.

(a) This follows immediately.

(b) Clearly $u_1, u_2 \notin N^*(u_1)$. By Claim 4.1E, $u_2, u_3, \ldots, u_{r-1} \notin A^-(u_1)$, so $u_3, \ldots, u_r \notin N^*(u_1)$. By definition of $\theta(T)$, $u_{r+1} \notin N^*(u_1)$. If $u_{r+3} \in N^*(u_1)$ then

$$T' = u_2u_3 \ldots u_{r+2}u_1u_{r+3}u_{r+4} \ldots u_pu_21$$

is a hamilton cycle, a contradiction. We know that $u_p \notin N^*(u_1)$.

(c) We know that $u_1, u_p \notin N^*(u_{r+1})$. By Claim 4.1E, $u_2, u_3, \ldots, u_{r-1} \notin N^*(u_{r+1})$. Clearly $u_r, u_{r+1}, u_{r+2} \notin N^*(u_{r+1})$. If $u_{r+3} \in N^*(u_{r+1})$, then

$$T' = u_1u_2 \ldots u_ru_pu_{p-1} \ldots u_{r+3}u_{r+1}u_{r+2}u_1$$

is a hamilton cycle, a contradiction. ∎

Now we complete the proof of Theorem 4.1. Since $\theta(T) = 2$, i.e., $s = r + 2$, we have $r + 1 = s - 1$. This means that symmetric arguments may be applied from the two ends of our path T. We examine two cases.

(1) $\alpha = 0$ and $\beta = 1$, or $\alpha = 1$ and $\beta = 0$. By symmetry we may suppose that $\alpha = 1$ and $\beta = 0$. Then $\gamma = 1$, so u_{r+1} is the only vertex that does not belong to C. By Claim 4.1F(b) and the fact that $\beta = 0$, $N^*(u_1) \subseteq \{u_{r+2}, u_{r+3}, \ldots, u_{p-2}\}$, and since $N^*(u_1) \neq \emptyset$, $r \leq p - 4$.

Suppose that $r \leq p - 5$.

First, we show that $u_{r+4}, \ldots, u_p \notin B^+(u_1)$. We know $u_p \notin B^+(u_1)$ since $\beta = 0$. If there is $u_k \in B^+(u_1)$ with $r + 4 \leq k \leq p - 1$, then by Claim 4.1E (using symmetry) we get $u_{r+3}, \ldots, u_k \in B^+(u_1)$. In particular, since $k \geq r + 4$, $u_{k-1} \in B^+(u_1)$, and

$$T' = u_2u_3 \ldots u_{k-2}u_1u_{k-1}u_k \ldots u_pu_2$$

is a hamilton cycle, a contradiction. From Claim 4.1F(b), we now know that $N^*(u_1) = \{u_{r+2}\}$.

Second, we show that if $r \le p-6$, then $u_{r+4}, \ldots, u_{p-2} \notin N(u_{r+1})$. Suppose there is k, $r+4 \le k \le p-2$, with $u_k \in N(u_{r+1})$. If $u_{k+1} \in B^+(u_p)$ then

$$T' = u_1 u_2 \ldots u_{r+1} u_k u_p u_{p-1} \ldots u_k \ldots u_{r+1} u_1$$

is a closed 2-trail, a contradiction. From above, $u_{k+1} \notin B^+(u_1)$. Since $u_{k+1} \in C$, we must have $u_{k+1} \in N(u_{r+1})$. Now

$$T' = u_1 u_2 \ldots u_r u_p u_{p-1} \ldots u_{k+1} u_{r+1} u_k u_{k-1} \ldots u_{r+2} u_1$$

is a hamilton cycle, a contradiction. From Claim 4.1F(c), we now know that $N^*(u_{r+1}) \subseteq \{u_{p-1}\}$.

Third, we show that $u_{p-1} \notin N(u_{r+1})$. Suppose $u_{p-1} \in N(u_{r+1})$. If $r \ge 3$, then, by Claim 4.1F(a), $u_2, u_3 \in N^*(u_p)$, and

$$T' = u_1 u_2 u_p u_3 \ldots u_{r+1} u_{p-1} u_{p-2} \ldots u_{r+2} u_1$$

is a hamilton cycle, a contradiction. Therefore, $r = 2$. If $r \le p-6$, then from above u_{r+4} must be in $B^+(u_p)$ since it is not in $B^+(u_1) \cup N(u_{r+1})$, and

$$T' = u_1 u_2 \ldots u_r u_p u_{r+3} u_{r+4} \ldots u_{p-1} u_{r+1} u_{r+2} u_1$$

is a hamilton cycle, a contradiction. Therefore, $r = p-5$. Thus, $p = n = 7$. Besides T, G contains the edges u_1u_4, u_2u_7 and u_3u_6. There are no further edges incident with u_1 or $u_{r+1} = u_3$. If this is all of G, then $\{u_1, u_5, u_7\}$ is an independent set with degree sum $6 < n = 7$, a contradiction. Therefore, there is an additional edge incident with at least one of u_5 or u_7, either to increase the degree sum or destroy the independence of this set. The only possible extra edges are u_5u_2, u_5u_7 and u_7u_4. The edge u_5u_2 gives the closed 2-trail $u_2u_3u_4u_1u_2u_5u_6u_7u_2$, the edge u_5u_7 gives a hamilton cycle $u_1u_2u_3u_6u_7u_5u_4u_1$, and the edge u_7u_4 gives the closed 2-trail $u_1u_2 \ldots u_7u_4u_1$, all of which provide contradictions.

So, now we know that $u_{r+4}, \ldots, u_{p-1} \in B^+(u_p)$ and $N^*(u_{r+1}) = \emptyset$. Thus, $\{u_1, u_{r+1}, u_{p-1}\}$ is an independent set. Since $\deg(u_1) = \deg(u_{r+1}) = 2$, $\deg(u_{p-1} \ge n-4$, so u_{p-1} is nonadjacent to at most three vertices, and hence adjacent to at least one of u_r or u_{r+2}. If $u_{p-1}u_r \in E(G)$, then

$$T' = u_r u_{r+1} u_{r+2} u_1 u_2 \ldots u_r u_p u_{r+3} u_{r+4} \ldots u_{p-1} u_r$$

is a closed 2-trail, a contradiction. If $u_{p-1}u_{r+2} \in E(G)$, then

$$T' = u_1u_2 \dots u_{r+2} \dots u_{p-2}u_pu_{p-1}u_{r+2}u_1$$

is a closed 2-trail, a contradiction. Thus, we do not have $r \leq p-5$.

Therefore, $r = p-4$. Then $N(u_1) = \{u_2, u_{r+2}\}$. By Claim 4.1F(c), $N^*(u_{r+1}) = \emptyset$ and so $N(u_{r+1}) = \{u_r, u_{r+2}\}$. Since $\beta = 0$, $u_1 \notin N^*(u_{p-1})$. Suppose $u_j \in N^*(u_{p-1})$ where $2 \leq j \leq r$. By Claim 4.1F(a), $u_j \in N^*(u_p)$, so

$$T' = u_1u_2 \dots u_ju_{p-1}u_pu_ju_{j+1} \dots u_{r+2=p-2}u_1$$

is a closed 2-trail, a contradiction. We know $u_{r+1} \notin N^*(u_{p-1})$. Therefore, $N(u_{p-1}) = \{u_{r+2=p-2}, u_p\}$. Now $\{u_1, u_{r+1}, u_{p-1}\}$ is an independent set with degree sum 6, so $p = n \leq 6$. Since $r = p-4 \geq 2$, we have $p = 6$ and $r = 2$. Besides T, G contains the edges u_1u_4 and u_2u_6. If this is all of G, then $G \cong K^*_{2,3}$. Otherwise, since $\{u_1, u_{r+1}, u_p\} = \{u_1, u_3, u_6\}$ and $\{u_1, u_{r+1}, u_{p-1}\} = \{u_1, u_3, u_5\}$ are independent, G contains the edge u_2u_4, and has a closed spanning 2-trail $u_1u_2u_3u_4u_5u_6u_2u_4u_1$, a contradiction.

(2) $\alpha = \beta = 1$. Since there is at most one vertex of $V(G) \setminus C$ other than u_{r+1}, at least one of $D_1 = \{u_1, u_2, \dots, u_{r-1}\}$ or $D_2 = \{u_{s+1}, u_{s+2}, \dots, u_p\}$ is a subset of C. By symmetry we may suppose that $D_1 \subseteq C$.

If $r \geq 3$, then by Claim 4.1F(a) we have $u_2, u_3 \in N^*(u_p)$, and

$$T' = u_1u_2u_pu_3u_4 \dots u_{p-1}u_1$$

is a hamilton cycle, a contradiction.

Therefore, $r = 2$. If $u_3u_{p-1} \notin E(G)$, then

$$T' = u_3u_2u_1u_4u_5 \dots u_p$$

is a hamilton path with $2 = \theta(T) \leq \theta(T') \leq 2$ that falls under case (1), so we are finished. If $u_3u_{p-1} \in E(G) \setminus E(T)$ then

$$T' = u_1u_{p-1}u_3u_4 \dots u_{p-1}u_pu_2u_1$$

is a closed 2-trail, a contradiction. Therefore, $u_3u_{p-1} \in E(T)$ and $p = 5$. Besides T, G contains the edges u_1u_4 and u_2u_5. If this is all of G, then $G \cong K_{2,3}$. Otherwise, since $\{u_1, u_3, u_5\}$ is independent, G contains the edge u_2u_4 and has a closed spanning 2-trail $u_1u_2u_3u_4u_5u_2u_4u_1$, a contradiction.

This concludes the proof of Theorem 4.1. ∎

The condition $\sigma_3(G) \geq n$ in Theorem 4.1 is best possible. Suppose $n \geq 7$. Take K_{n-4}, choose (possibly identical) vertices v_1, v_2, then add new vertices w_1, w_2, x_1, x_2 and edges $v_1w_1, v_2w_2, w_1x_1, w_1x_2, w_2x_1, w_2x_2$. We get two graphs, depending on whether v_1 is equal to v_2 or not. Both are 2-edge-connected (the one with $v_1 \neq v_2$ is actually 2-connected), have $\sigma_3 = n - 1$, and do not have a closed spanning 2-trail. Also, for every $m \geq 2$ the graph $K_{m,2m+1}$ is 2-edge-connected (actually, m-connected), has $\sigma_3 = 3m = n-1$, and has no closed spanning 2-trail. Note also that the hypothesis that G is 2-edge-connected is clearly necessary.

We also have the following straightforward corollary, which is sharp, at least when $n \equiv 1 \pmod 3$, as shown by $K_{m,2m+1}$. This shows that Theorem 4.1 strengthens Theorem E.

Corollary 4.2 *Let G be a 2-edge-connected n-vertex graph. If $\delta(G) \geq n/3$ or $\sigma_2(G) \geq 2n/3$ then G has a closed spanning 2-trail unless $G \cong K_{2,3}$ or $K^*_{2,3}$.*

Results for graphs embedded on surfaces can also be obtained from minimum degree conditions. An argument similar to that of Duke [8], using Euler's formula to show that $\delta \geq n/3$, gives the following. We omit the details. Similar results can be obtained from Corollaries 2.7 and 3.4.

Corollary 4.3 *Suppose G is a 2-edge-connected graph embeddable on a surface (compact, without boundary, orientable or nonorientable) of Euler characteristic $\chi < 0$, such that $\delta(G) \geq 3 + \sqrt{9 - 2\chi}$. Then G has a closed spanning 2-trail.*

Perhaps Theorem 4.1 can be extended to spanning k-trails for $k \geq 3$. The natural conjecture would be that $\sigma_{k+1} \geq n$, together with sufficient edge-connectivity, guarantees the existence of a closed spanning k-trail. Another interesting question is whether a toughness condition, together with sufficient edge-connectivity, guarantees the existence of a closed spanning k-trail. Similar results are known for closed spanning k-walks for $k \geq 2$ [9, 12], and a result for k-trails might shed some light on Chvátal's well known conjecture [7] that sufficiently tough graphs have a hamilton cycle.

Acknowledgement

The authors thank Guantao Chen for pointing us to references [10, 13, 18].

References

[1] A. Benhocine, L. Clark, N. Köhler and H. J. Veldman, On circuits and pancyclic line graphs, *J. Graph Theory* 10 (1986) 411-425.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, American Elsevier, New York, 1976.

[3] P. A. Catlin, Contractions of graphs with no spanning eulerian subgraphs, *Combinatorica*, 8 (1988) 313-321.

[4] P. A. Catlin, Spanning eulerian subgraphs and matchings, *Discrete Math.* 76 (1989) 95-116.

[5] Z.-H. Chen, A degree condition for spanning eulerian subgraphs, *J. Graph Theory* 17 (1993) 5-21.

[6] Z.-H. Chen, Supereulerian graphs, independent sets, and degree-sum conditions, *Discrete Math.* 179 (1998) 73-87.

[7] V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* 5 (1973) 215-228.

[8] Richard A. Duke, On the genus and connectivity of hamiltonian graphs, *Discrete Math.* 2 (1972) 199-206.

[9] M. N. Ellingham and Xiaoya Zha, Toughness, trees and walks, *J. Graph Theory* 33 (2000) 125-137.

[10] Hikoe Enomoto, Atsushi Kaneko, Mekkia Kouider and Zsolt Tuza, Degree sums and covering cycles, *J. Graph Theory* 20 (1995) 419-422.

[11] Zhicheng Gao and Nicholas C. Wormald, Spanning Eulerian subgraphs of bounded degree in triangulations, *Graphs Combin.* 10 (1994) 123-131.

[12] Bill Jackson and Nicholas C. Wormald, k-walks of graphs, *Australas. J. Combin.* 2 (1990) 135-146.

[13] Mekkia Kouider and Zbigniew Lonc, Covering cycles and k-term degree sums, *Combinatorica* 16 (1996) 407-412.

[14] M. Kouider and M. Mahéo, Two-edge-connected $[2, k]$-factors in graphs, *J. Combin. Math. Combin. Comput.* 35 (2000) 89-95.

[15] L. Lesniak-Foster and J. E. Williamson, On spanning and dominating circuits in graphs. *Canad. Math. Bull.* 20 (1977) 215-220.

[16] Yunsun Nam, Ore-type condition for the existence of connected factors, preprint, 2000.

[17] O. Ore, Note on Hamilton circuits, *Amer. Math. Monthly* 67 (1960) 55.

[18] Akira Saito, Degree sums and graphs that are not covered by two cycles, *J. Graph Theory* 32 (1999) 51-61.

[19] H. J. Veldman, Existence of spanning and dominating trails and circuits, *J. Graph Theory* 10 (1986) 23-31.