

A characterisation of cubic parity graphs (with complete details)

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Abstract

A graph is \mathbf{Z}_m -well-covered if all maximal independent sets have the same cardinality modulo m . \mathbf{Z}_m -well-covered graphs generalise *well-covered* graphs, those in which all independent sets have the same cardinality. \mathbf{Z}_2 -well-covered graphs are also called *parity* graphs. A characterisation of cubic well-covered graphs was given by Campbell, Ellingham and Royle. Here we extend this to a characterisation of cubic \mathbf{Z}_m -well-covered graphs for all integers $m \geq 2$; the most interesting case is $m = 2$, cubic parity graphs. Our main technique involves minimal non-well-covered graphs, and allows us to build our characterisation as an extension of the existing characterisation of cubic well-covered graphs.

I. Introduction

All graphs in this paper are simple and finite.

A graph is *well-covered* if all maximal independent sets have the same cardinality, and \mathbf{Z}_m -*well-covered*, for some $m \geq 2$, if all maximal independent sets have the same cardinality modulo m . Well-covered graphs were introduced by Plummer [11], and Plummer has written a useful survey paper [12]. \mathbf{Z}_2 -well-covered graphs were introduced by Finbow and Hartnell [8, 9] under the name of *parity* graphs.

Both well-covered and \mathbf{Z}_m -well-covered graphs are special cases of a more general concept [4]. Given a graph whose vertices are weighted by elements of an abelian group A , we say the A -weighted graph is *well-covered* if all maximal independent sets have the same total weight. If A is a ring with identity (regarded as an additive group) then a graph is *A-well-covered* if it is well-covered when we give every vertex the weight 1 from A . Since only the additive subgroup of A generated by 1 matters here, we usually take A to be a cyclic group \mathbf{Z} or \mathbf{Z}_m . ‘ \mathbf{Z} -well-covered’ just means well-covered in Plummer’s original sense, and this definition of ‘ \mathbf{Z}_m -well-covered’ is equivalent to the previous one.

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Cubic well-covered graphs were characterised by Campbell, Ellingham and Royle [2], following earlier work by Campbell and Plummer [1, 3]. In this paper we build on the characterisation of cubic well-covered graphs to obtain a characterisation of cubic \mathbf{Z}_m -well-covered graphs for all $m \geq 2$. Most of the work is for the case $m = 2$, so our main result is a characterisation of cubic parity graphs. This research is of interest from three different angles: the original question, the techniques used, and the results obtained.

First, characterising \mathbf{Z}_m -well-covered cubic graphs is a natural question. Determining whether a graph is well-covered is a co-NP-complete problem [7, 14], even for $K_{1,4}$ -free graphs [6]. The same proof shows that for each cyclic group A , determining whether a graph is A -well-covered is co-NP-complete for $K_{1,4}$ -free graphs. Therefore, it is unlikely that a general characterisation of A -well-covered graphs will be found, so it is of interest to find special classes of graphs where the A -well-covered graphs can be characterised. When $A = \mathbf{Z}$, two simple restrictions that allow characterisations involve the girth or the vertex degrees. When $A = \mathbf{Z}_m$, girth restrictions have been investigated [5, 9], but degree restrictions have not. (It is known that for given d and A there is a polynomial time algorithm to determine whether a graph of maximum degree at most d is A -well-covered [4], but this does not give the overall structure of the A -well-covered graphs.) Cubic graphs are the simplest interesting class of graphs with a degree restriction, so they are a natural subject for investigation.

Second, the proofs in this paper demonstrate the usefulness of the technique of ‘minimal non-well-covered graphs.’ This idea, developed in [4, 13, 15], and equivalent to the ‘critical nongreedy hypergraphs’ of [6], provides a ‘minimal forbidden structure’ approach to well-covered graphs. Minimal forbidden structures are an important general concept in graph theory. The most famous example is Robertson and Seymour’s theory of graph minors, where many classes of graphs are characterised by forbidden minors. Another example is the characterisation of line graphs in terms of forbidden induced subgraphs. We feel that minimal non-well-covered graphs are an important tool that should be more widely used. In this paper they allow us to characterise cubic \mathbf{Z}_m -well-covered graphs by building on the known characterisation of cubic well-covered graphs, rather than developing this characterisation independently.

Third, the results in this paper shed some light on the difference between well-covered graphs and \mathbf{Z}_m -well-covered graphs. The interesting case for our results is $m = 2$. Connected cubic well-covered graphs, with six exceptions, are made by joining together three basic building blocks (see Theorem II.7). Since every well-covered graph is also \mathbf{Z}_2 -well-covered, one might expect that the cubic \mathbf{Z}_2 -well-covered graphs extend the cubic well-covered graphs by using a few more building blocks together with the known ones. However, this turns out not to be so. Cubic \mathbf{Z}_2 -well-covered graphs seem to divide sharply into those that are well-covered and those that are not, with different structures in each case.

In Section II we introduce minimal non-well-covered graphs, deduce some general results using them, and apply these results to cubic graphs. This allows us to dispose of the problem of characterising cubic \mathbf{Z}_m -well-covered graphs for $m \geq 3$ in Theorem II.8, and gives us some information for the case $m = 2$. In Section III we introduce three families of graphs and one special graph, which will turn out to be exactly the connected cubic graphs that are \mathbf{Z}_2 -well-covered but not well-covered. In Section IV we prove this, using a fairly lengthy case analysis. Our main results are stated as Theorem IV.3 and Corollary IV.4. In Section V we give some concluding remarks.

II. Minimal non-well-covered graphs

We begin by summarising some ideas from Section 2 of [4], to which we refer the reader for details and history. As in Section I, A denotes a cyclic group \mathbf{Z} or \mathbf{Z}_m , $m \geq 2$.

Let S be a set of vertices in a graph G . The subgraph of G induced by S is denoted $G[S]$. The *neighbourhood* of S is the set $N_G(S) = \{u : uv \in E(G) \text{ for some } v \in S\}$. The *closed neighbourhood* of S is the set $N_G[S] = S \cup N_G(S)$. We often abbreviate $N_G(S)$ to $N(S)$, $N_G[S]$ to $N[S]$, $N(\{v\})$ to $N(v)$ and $N[\{v\}]$ to $N[v]$. We say that S *dominates* another set of vertices T if $T \subseteq N[S]$. Given a subgraph H of G , we say G *encloses* H , and write $H \leq G$, if there is an independent set I of vertices of G so that $H = G - N_G[I]$. In practice we often use a different but equivalent definition: $H \leq G$ if there is an independent set I' such that H is a component of $G - N_G[I']$. The relation ' \leq ' is a partial order on graphs. We sometimes abuse notation by writing $H' \leq G$ when we mean that H' is isomorphic to some H with $H \leq G$; it should be clear when we are doing this.

The importance of this partial order is that if G is A -well-covered and $H \leq G$, then H is also A -well-covered. We use this as a tool to examine the structure of an A -well-covered graph. If H is *not* A -well-covered, then an A -well-covered graph G cannot contain any configuration that would imply that $H \leq G$. In fact, we can restrict the graphs H that we use here. H is a *minimal non- A -well-covered* graph if H is not A -well-covered, but if $J \leq H$ and $J \neq H$, then J is A -well-covered. If $A = \mathbf{Z}$, we just refer to H as *minimal non-well-covered*.

Observation II.1 (special case of [4, Observation 2.2]). *A graph G is not A -well-covered if and only if there is a minimal non- A -well-covered graph H with $H \leq G$.*

This observation is more useful if we know something about the structure of minimal non- A -well-covered graphs. The following theorem provides some important information. It generalises results for $A = \mathbf{Z}$ obtained independently by Caro, Sebő and Tarsi [6] (who used a hypergraph formulation), Ramey [13], and Zverovich [15]. Here $+$ denotes join, and $\alpha(G)$ denotes the cardinality of a maximum independent set in G . Two integers p, q are *equal in A* if they have equal images under the homomorphism $\mathbf{Z} \rightarrow A$ that maps $1 \in \mathbf{Z}$ to $1 \in A$. Otherwise they are *distinct in A* . 'Equal in \mathbf{Z}_m ' means equivalent modulo m , and 'equal in \mathbf{Z} ' just means equal.

Theorem II.2 (special case of [4, Theorem 2.3]). *A graph H is minimal non- A -well-covered if and only if there exist A -well-covered graphs H_1, H_2, \dots, H_k such that $H = H_1 + H_2 + \dots + H_k$ and $\alpha(H_i)$ and $\alpha(H_j)$ are distinct in A for some i and j .*

Now we apply Observation II.1 and Theorem II.2 to prove some results for graphs of bounded degree. As usual, $\Delta(G)$ denotes the maximum degree of the graph G .

Lemma II.3. *Suppose $\Delta(H) \leq m$, where $m \geq 2$. Then H is minimal non- \mathbf{Z}_m -well-covered if and only if H is minimal non-well-covered.*

Proof. If H cannot be expressed as a join, then by Theorem II.2 H is neither minimal non- \mathbf{Z}_m -well-covered nor minimal non-well-covered, and the lemma holds. So, suppose that $H = H_1 + \dots + H_k$ for some $k \geq 2$. Since $\Delta(H) \leq m$, each graph H_i must have at most m vertices. But then each H_i is \mathbf{Z}_m -well-covered if and only if it is (\mathbf{Z} -)well-covered, and moreover $\alpha(H_i)$ and $\alpha(H_j)$ are distinct in \mathbf{Z}_m if and only if they are distinct in \mathbf{Z} . Therefore, H has a join decomposition satisfying Theorem II.2 with $A = \mathbf{Z}_m$ if and only if it has one with $A = \mathbf{Z}$, and the lemma follows. ■

Corollary II.4. Suppose $\Delta(G) \leq m$, where $m \geq 2$. Then G is \mathbf{Z}_m -well-covered if and only if G is well-covered.

Proof. For any $H \leq G$, $\Delta(H) \leq m$, and so by Lemma II.3 H is minimal non- \mathbf{Z}_m -well-covered if and only if it is minimal non-well-covered. The result then follows by applying Observation II.1 with $A = \mathbf{Z}_m$ and $A = \mathbf{Z}$. ■

Lemma II.5. Suppose $\Delta(H) \leq m + 1$, and H is a minimal non-well-covered graph that is \mathbf{Z}_m -well-covered. Then $H \cong K_{1,m+1}$.

Proof. By Theorem II.2 we can write $H = H_1 + \dots + H_k$ where $k \geq 2$, each H_i is well-covered, and $\alpha(H_i) \neq \alpha(H_j)$ for some i, j . Since $\Delta(H) \leq m + 1$, each H_i has at most $m + 1$ vertices. If each H_i has no more than m vertices then H is minimal non- \mathbf{Z}_m -well-covered by the argument from the proof of Lemma II.3, a contradiction. Therefore we may assume that H_1 has $m + 1$ vertices. Since $\Delta(H) \leq m + 1$, we must have $k = 2$ and H_2 must have no edges. Moreover, if H_2 has $m + 1$ vertices then $\Delta(H) \leq m + 1$ would mean that H_1 has no edges and $H \cong K_{m+1,m+1}$, which is well-covered. Therefore, H_2 has at most m vertices. Now $\alpha(H_1)$ and $\alpha(H_2)$ must be distinct in \mathbf{Z} since H is minimal non-well-covered, but equal in \mathbf{Z}_m since H is not minimal non- \mathbf{Z}_m -well-covered. Thus, one must be 1 and the other must be $m + 1$, which implies that $H \cong K_{1,m+1}$. ■

Corollary II.6. Suppose $\Delta(G) = m + 1$, where $m \geq 2$. If G is \mathbf{Z}_m -well-covered but not well-covered then $K_{1,m+1} \leq G$.

Proof. By Observation II.1 there is a minimal non-well-covered H with $H \leq G$. Since H is \mathbf{Z}_m -well-covered and $\Delta(H) \leq \Delta(G) = m + 1$, by Lemma II.5 we have $H \cong K_{1,m+1}$. ■

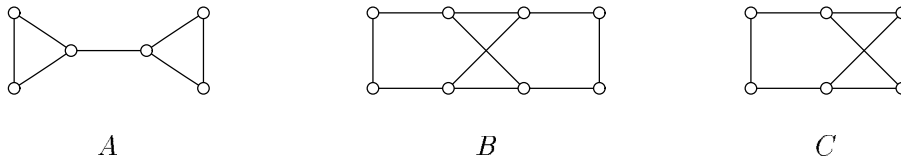


Figure II.1

Now we apply the above results to cubic graphs. First we recall the characterisation of well-covered cubic graphs from [2]. Let A , B and C be the graphs depicted in Figure II.1. Define a *terminal pair* to be a pair of adjacent degree two vertices. Let \mathcal{W} denote the class of cubic graphs constructed as follows. Given a collection of copies of A , B and C , join every terminal pair by two edges to a terminal pair in another (possibly the same) graph, so that the result is cubic. Then we have the following characterisation (which is stated incorrectly in the survey paper [12, p. 272]).

Theorem II.7 ([2, Theorem 6.1]). *Let G be a connected cubic graph. Then G is well-covered if and only if one of the following is true.*

- (i) $G \in \mathcal{W}$; or
- (ii) G is one of six exceptional graphs: K_4 , $K_{3,3}$, $K_{3,3}^*$, $C_5 \times K_2$, Q^{**} or P_{14} (see [2] for details).

Corollary II.4 yields the following.

Theorem II.8. Suppose $m \geq 3$. Then a cubic graph is \mathbf{Z}_m -well-covered if and only if it is well-covered. Therefore, G is a connected \mathbf{Z}_m -well-covered cubic graph if and only if (i) or (ii) of Theorem II.7 holds.

For cubic \mathbf{Z}_2 -well-covered graphs we will make use of the following two results. The first is a special case of Corollary II.6, and the second follows easily from Observation II.1 and Theorem II.2.

Lemma II.9. If a cubic graph G is \mathbf{Z}_2 -well-covered but not well-covered, then $K_{1,3} \leq G$.

Lemma II.10. The minimal non- \mathbf{Z}_2 -well-covered graphs with maximum degree at most 3 are $P_3 = K_{1,2} = K_1 + 2K_1$, $L = K_1 + (K_2 \cup K_1)$, $K_{1,1,2} = K_1 + K_1 + 2K_1$, and $K_{2,3} = 2K_1 + 3K_1$. Thus, a cubic graph is \mathbf{Z}_2 -well-covered if and only if it encloses none of these four graphs.

III. Families of cubic parity graphs

Every well-covered graph is also \mathbf{Z}_m -well-covered for all m , and in particular is \mathbf{Z}_2 -well-covered. Therefore, the cubic \mathbf{Z}_2 -well-covered graphs include all the cubic well-covered graphs. In this section we introduce three families of cubic graphs that (except for some small members) are \mathbf{Z}_2 -well-covered but not well-covered. We also introduce one special graph with this property.

The first family consists of the *prisms* $C_m \times K_2$, $m \geq 3$. The second family consists of the *twisted prisms* or *Möbius ladders* M_{2m} , $m \geq 2$. M_{2m} can be defined as a cycle $v_0v_1v_2 \dots v_{2m-1}v_0$ plus chords v_iv_{i+m} , $0 \leq i \leq m-1$. The third family consists of graphs X_{4k} , $k \geq 2$, consisting of two cycles $w_0w_1w_2 \dots w_{2k-1}w_0$ and $x_0x_1x_2 \dots x_{2k-1}x_0$ together with the edges $w_{2i}x_{2i+1}$ and $w_{2i+1}x_{2i}$, $0 \leq i \leq k-1$. In Figure III.1 we show the 16-vertex members of all three families.

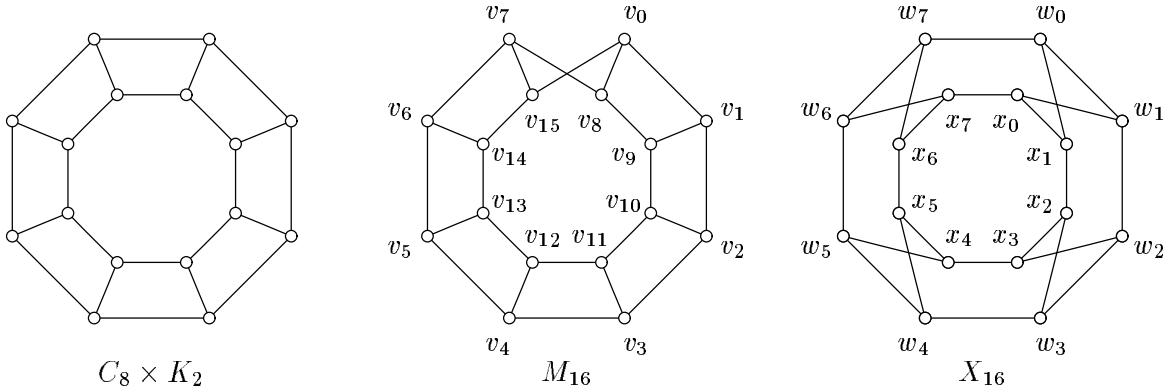


Figure III.1

Theorem III.1. (i) The following graphs are well-covered (where we use the notation of [2] for graphs in \mathcal{W}): $C_3 \times K_2 \cong -A- \in \mathcal{W}$, $C_5 \times K_2$, $M_4 \cong K_4$, $M_6 \cong K_{3,3}$, $M_8 \cong -B- \in \mathcal{W}$.

(ii) The following graphs are \mathbf{Z}_2 -well-covered but not well-covered: $C_4 \times K_2 \cong X_8$, $C_m \times K_2$ for $m \geq 6$, M_{2m} for $m \geq 5$, and X_{4k} for $k \geq 3$.

Proof. (i) can be checked easily against Theorem II.7.

(ii) It is easy to verify that the maximal independent sets have cardinalities 2 and 4 in $C_4 \times K_2 \cong X_8$, and 3 and 5 in M_{10} , so these two graphs satisfy the theorem. We may therefore restrict our attention to the graphs with at least 12 vertices.

To prove they are \mathbf{Z}_2 -well-covered we need only show that these graphs enclose none of the four graphs of Lemma II.10: P_3 , $L = K_1 + (K_2 \cup K_1)$, $K_{1,1,2}$ and $K_{2,3}$. However, it is clear that none of the graphs in (ii) contain induced subgraphs isomorphic to L , $K_{1,1,2}$ or $K_{2,3}$, so it suffices to show that these graphs do not enclose P_3 . To prove they are not well-covered it suffices to prove that they do enclose the non-well-covered graph $K_{1,3}$.

Let G be either $C_m \times K_2$ or M_{2m} where $m \geq 6$. Then G has the subgraph shown in Figure III.2(a), which is induced except that G possibly has edges ak, bl or al, bk . All P_3 's in G are equivalent by an automorphism of G to either dfh or efh . Suppose that $dfh \leq G$, so that $G - N[I]$ has dfh as a component for some independent set I . Now I cannot contain c, e or g , but must contain a vertex other than f adjacent to e , which is impossible. Suppose that $efh \leq G$, so that $G - N[I]$ has efh as a component for some I . Then I cannot contain c or d , but must contain a neighbour of c other than e and a neighbour of d other than f . Therefore I contains both a and b , which is impossible. Thus, G does not enclose P_3 . On the other hand, by taking $I = \{a, i\}$ we see that G does enclose a $K_{1,3}$ with edges df, ef, hf .



Figure III.2

Now let G be X_{4k} where $k \geq 3$. Then G has the subgraph shown in Figure III.2(b), which is induced except that G possibly has edges ak, bl . All P_3 's in G are equivalent by an automorphism of G to either dfh or gfh . Suppose that $dfh \leq G$, so that dfh is a component of $G - N[I]$ for some independent set I . Now I cannot contain e, g or j . I must contain a neighbour of g other than f , so $i \in I$, but I must also contain a neighbour of j other than h , so either k or l must be in I , which is impossible. Suppose that $gfh \leq G$, so that gfh is a component of $G - N[I]$ for some independent set I . Now I cannot contain d or e . I must contain a neighbour of e other than g or h , so $c \in I$, but I must also contain a neighbour of d other than f , so either a or b must be in I , which is impossible. Thus, G does not enclose P_3 . On the other hand, by taking $I = \{c, k\}$ we see that G does enclose a $K_{1,3}$ with edges df, gf, hf . ■

There is one other cubic graph that is \mathbf{Z}_2 -well-covered but not well-covered that we mention here. Let R_{12} be the graph of Figure III.3. It is not difficult to show that the maximal independent sets in R_{12} have cardinalities 4 and 6.

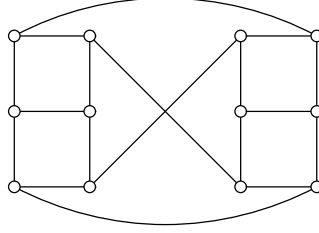


Figure III.3: R_{12}

IV. Characterisation of cubic parity graphs

In this section we characterise the cubic \mathbf{Z}_2 -well-covered graphs that are not well-covered. The two main tools are Lemmas II.9 and II.10. Lemma II.9 guarantees that a \mathbf{Z}_2 -well-covered graph that is not well-covered encloses a $K_{1,3}$. Define a *star vertex* of G to be the degree 3 vertex of some $K_{1,3}$ enclosed in G . Our main argument is a case-by-case analysis to determine the structure near a star vertex.

Theorem IV.1. *Suppose G is a connected cubic graph that is \mathbf{Z}_2 -well-covered but not well-covered. Let v be a star vertex of G . Then either G is isomorphic to one of $C_4 \times K_2 \cong X_8$, M_{10} , or R_{12} , or else v is one of two types:*

- (A) *There is an induced subgraph as shown in Figure IV.1(a) containing v in the specified position.*
- (B) *There is an induced subgraph as shown in Figure IV.1(b) containing v in the specified position.*

Moreover, in both cases every neighbour of v is also a star vertex of the same type as v .

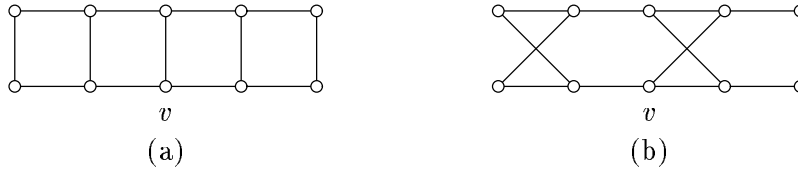


Figure IV.1

Proof. First we introduce some notation. N_i denotes the set of vertices at distance i from v , and $n_i = |N_i|$. m_i denotes the number of edges with both ends in N_i , and m_{ij} denotes the number of edges with one end in N_i and the other in N_j . Since v is a star vertex, $G[N[v]]$ is isomorphic to $K_{1,3}$, so that $m_1 = 0$, i.e., the neighbours of v are pairwise nonadjacent. Moreover, there is an independent set I such that $G[N[v]]$ is a component of $G - N[I]$. We choose such an I with a minimum number of elements. $I \cap N_3$ must dominate N_2 , and minimality implies that $I \subseteq N_3$.

Write $I = \{u_1, u_2, \dots, u_p\}$. Since p is minimum, each $u_i \in I$ must have at least one *private neighbour* in N_2 : a vertex of N_2 adjacent to u_i but to no other element of I . Let $P(u_i)$ denote the set of private neighbours of u_i in N_2 , i.e., $P(u_i) = N_2 - \bigcup_{j \neq i} N[u_j]$. $P(u_i)$ is nonempty and disjoint from $P(u_j)$, $j \neq i$. Let $\mathcal{P} = \{P(u_1), P(u_2), \dots, P(u_p)\}$.

Lemma IV.2. *We cannot have $P(u_i) = \{t\}$ where t has exactly one neighbour t' in N_1 . We also cannot have $P(u_i) = \{s, t\}$ where s and t both have exactly one neighbour in N_1 , which is the same vertex t' in each case.*

Proof. Suppose otherwise. Then, by definition of $P(u_i)$, $I' = I - \{u_i\} \cup \{t\}$ is an independent set that dominates all of N_2 , apart from possibly s in the second case. It also dominates $t' \in N_1$, but no other elements of N_1 . Therefore, $G - N[I']$ has $G[N[v]] - t' \cong P_3$ as a component, i.e., G encloses P_3 , which cannot happen since P_3 is not \mathbf{Z}_2 -well-covered. ■

We use the following facts throughout the rest of the proof of Theorem IV.1, often without explicit mention. Every vertex of N_2 is adjacent to a vertex of $I \subseteq N_3$, and to a vertex of N_1 . Consequently, every vertex of N_2 is adjacent to at most one other vertex of N_2 . We have $m_{23} \geq n_2$, $m_{12} = 6 \geq n_2$, and $m_{12} + 2m_2 + m_{23} = 6 + 2m_2 + m_{23} = 3n_2$. Consequently, $6 + 2m_2 \leq 2n_2$. It follows that $3 \leq n_2 \leq 6$. Counting edges from I to N_2 and using the fact that each $u_i \in I$ has at least one private neighbour in N_2 , we obtain $n_2/3 \leq p \leq n_2$.

To first prove that v is of type (A) or (B), we divide the proof into cases identified by the value of n_2 . So, our top level cases are labelled (3) to (6).

(3) Suppose that $n_2 = 3$. Since $6 + 2m_2 \leq 2n_2 = 6$, $m_2 = 0$. Consequently, $m_{23} = 3$ and every vertex of N_2 is adjacent to one vertex of N_3 and two vertices of N_1 . The subgraph $G[N_1 \cup N_2]$ is bipartite and 2-regular, so it must be a 6-cycle. Thus, G contains the induced subgraph of Figure IV.2(a).

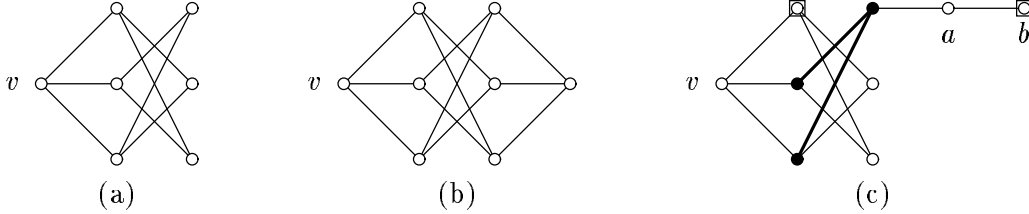


Figure IV.2

If $p = |I| = 1$ then G is the graph of Figure IV.2(b), which is the cube $C_4 \times K_2 \cong X_8$. If $p \geq 2$ then some vertex a of I has a neighbour b that does not belong to N_2 . Then G encloses a P_3 as shown in Figure IV.2(c) (where it does not matter if b has neighbours in N_2), so this cannot happen. (Note that we frequently obtain a contradiction by showing that G encloses P_3 , L , $K_{1,1,2}$ or $K_{2,3}$. Usually we draw a figure in which an independent set I' is indicated by open circles surrounded by squares, vertices of $N[I'] - I'$ by open circles, vertices not in $N[I']$ by solid circles, and edges of the enclosed graph by heavy lines. Occasionally a vertex whose exact status is unknown and does not matter will be indicated by a circle that is half solid and half open.)

(4) Suppose that $n_2 = 4$. Then we can write $N_2 = \{a_1, a_2, b_1, b_2\}$ where a_1 and a_2 have one neighbour in N_1 and b_1 and b_2 have two neighbours in N_1 . Since b_1 and b_2 are adjacent to a vertex of $I \subseteq N_3$, the only possible edge with both ends in N_2 is a_1a_2 . There are two subcases.

(4.1) Suppose that a_1 and a_2 have a common neighbour in N_1 , which we call w . Then b_1 and b_2 are both adjacent to the other neighbours of N_1 , which we call x_1 and x_2 . See Figure IV.3(a).

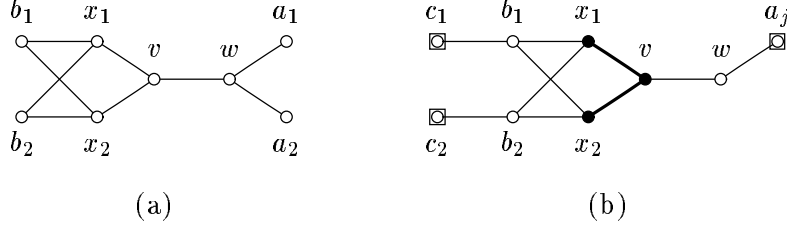


Figure IV.3

Let c_i be the neighbour of b_i in I for $i = 1, 2$; possibly $c_1 = c_2$. If some a_j , $j = 1, 2$, is adjacent to neither c_1 nor c_2 , then G encloses P_3 as shown in Figure IV.3(b). This must be the case if $c_1 = c_2$, so now we may assume that $c_1 \neq c_2$, and both a_1 and a_2 are adjacent to at least one of c_1 and c_2 . Without loss of generality we may assume that $a_1 c_1 \in E(G)$.

If $c_1 c_2 \in E(G)$, then G encloses P_3 as shown in Figure IV.4(a), so we may assume that $c_1 c_2 \notin E(G)$. Since c_2 is not adjacent to w or c_1 , c_2 has at least one neighbour $d \neq b_2$ that is not adjacent to a_1 (possibly $d = a_1$ or a_2). Then G encloses $K_{2,3}$ as shown in Figure IV.4(b). This concludes case (4.1).

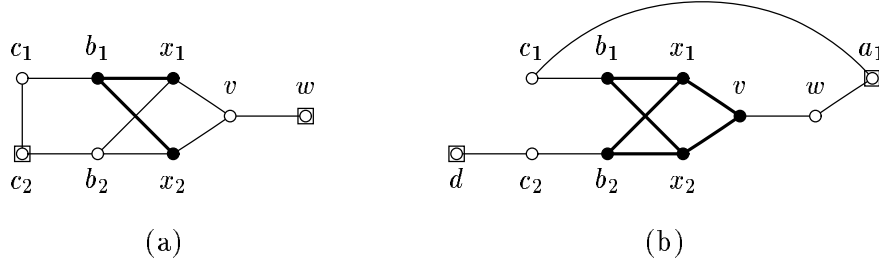


Figure IV.4

(4.2) Suppose that a_1 and a_2 do not have a common neighbour in N_1 . Let w_i be the neighbour of a_i in N_1 for $i = 1, 2$, and let x be the third vertex of N_1 . Without loss of generality, we may assume that b_i is adjacent to w_i and x for $i = 1, 2$. See Figure IV.5.

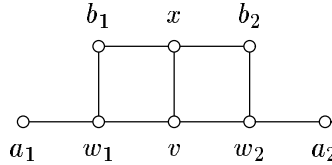


Figure IV.5

Now $n_2/3 = 4/3 \leq p \leq n_2 = 4$, so $2 \leq p \leq 4$. If $p = |I| = 4$ then we must have $\mathcal{P} = \{\{a_1\}, \{a_2\}, \{b_1\}, \{b_2\}\}$, but $P(u_i) = \{a_1\}$ violates Lemma IV.2, so $p \neq 4$.

(4.2.1) Suppose that $p = 2$. Suppose b_1 and b_2 are adjacent to the same vertex $d = u_1$ of I . Then d cannot also be adjacent to a_1 or a_2 , or we would have $P(u_2) = \{a_1\}$ or $\{a_2\}$, violating Lemma IV.2. Therefore a_1 and a_2 are both adjacent to the other vertex $c = u_2$ of I , and G encloses P_3 as shown in Figure IV.6(a). Thus, b_1 and b_2 have distinct neighbours in I : suppose $d_i \in I$ is adjacent to b_i , $i = 1, 2$, so that $I = \{d_1, d_2\}$. If both d_1 and d_2 are adjacent to both a_1 and a_2 , then G is the graph of Figure IV.6(b), which is isomorphic to M_{10} . If one d_i , say d_1 , is adjacent to both a_1 and a_2 while the other, d_2 , is not, then d_2 has a neighbour $f_2 \notin \{a_1, a_2, b_2\}$, and G encloses P_3 as shown in Figure IV.7(a). So, we may assume that neither d_1 nor d_2 is adjacent to both a_1 and a_2 , and thus each of d_1 and d_2 is adjacent to one of a_1 or a_2 .

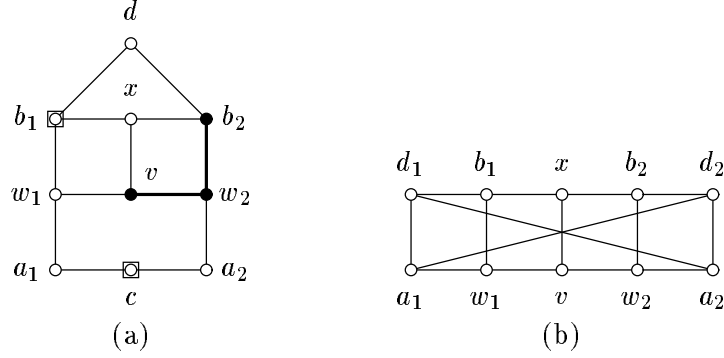


Figure IV.6

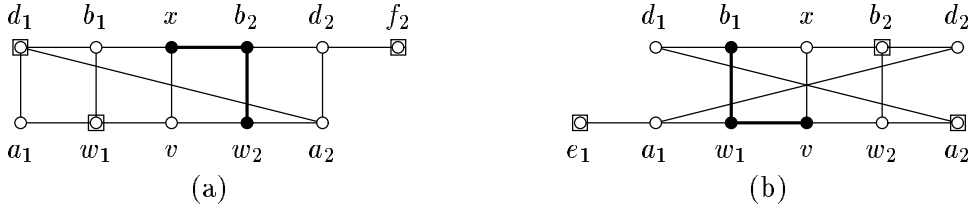


Figure IV.7

Suppose first that $a_1d_2, a_2d_1 \in E(G)$. If a_1 and a_2 do not have a common neighbour, let $e_1 \neq w_1, d_2$ be a neighbour of a_1 . Then G encloses P_3 as shown in Figure IV.7(b). Therefore a_1 and a_2 have a common neighbour e , and by a symmetric argument d_1 and d_2 have a common neighbour f . If $ef \notin E(G)$ then G encloses P_3 as shown in Figure IV.8(a). If $ef \in E(G)$ then G is the graph of Figure IV.8(b), which is isomorphic to R_{12} .

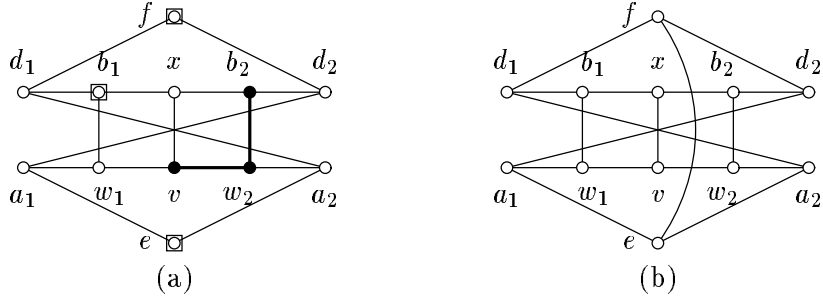


Figure IV.8

Thus, we may suppose that $a_1d_1, a_2d_2 \in E(G)$. Since $d_1, d_2 \in I$, we know that $d_1d_2 \notin E(G)$. Let $f_1 \neq a_1, b_1$ be a neighbour of d_1 . If $a_1a_2 \in E(G)$ then G encloses P_3 as shown in Figure IV.9(a). Therefore, $a_1a_2 \notin E(G)$, we have the induced subgraph shown in Figure IV.9(b), and v is a star vertex of type (A) as in the statement of this theorem. This concludes case (4.2.1).

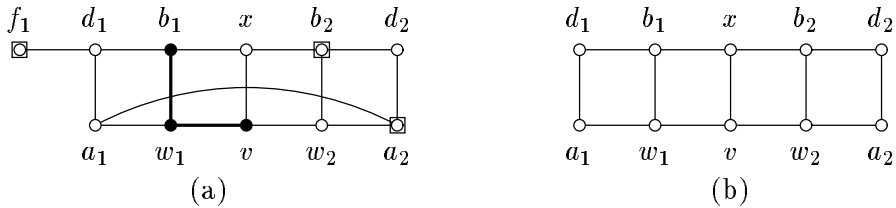


Figure IV.9

(4.2.2) Suppose that $p = 3$. By Lemma IV.2, no $P(u_i)$ can be $\{a_1\}$ or $\{a_2\}$, so the only possibility for \mathcal{P} is $\{\{a_1, a_2\}, \{b_1\}, \{b_2\}\}$. Then I contains a vertex c adjacent to both a_1 and a_2 , and a vertex d_i adjacent to b_i for $i = 1, 2$; there are no other adjacencies between I and N_2 . Now d_1 has two neighbours other than b_1 , and since neither is a_1 or a_2 , at most one is adjacent to c . Thus, d_1 has a neighbour $f \neq b_1$ not adjacent to c . But then G encloses P_3 as shown in Figure IV.10. This concludes cases (4.2.2), (4.2) and (4).

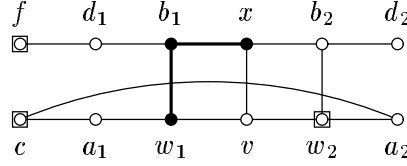


Figure IV.10

(5) Suppose that $n_2 = 5$. Then one vertex c of N_2 must be adjacent to two vertices of N_1 , and the remaining four vertices of N_2 must each be adjacent to one vertex of N_1 . Thus, G contains the subgraph shown in Figure IV.11(a), where $d = u_1$ is the vertex of I adjacent to c , so that $c \in P(d)$. Here we may have edges between a_1, a_2, b_1, b_2 .

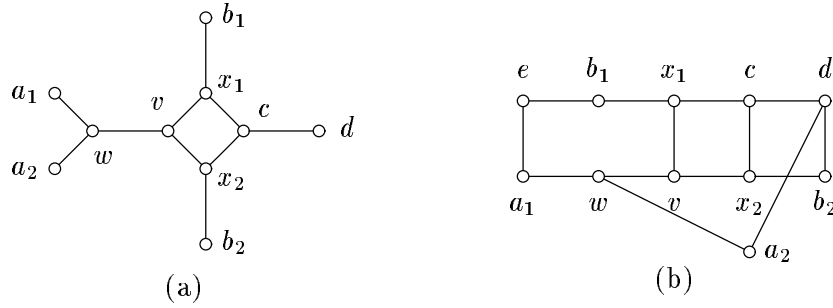


Figure IV.11

(5.1) Suppose that $p = 2$. Write $e = u_2$. By Lemma IV.2, $|P(e)| \neq 1$, and $P(e) \neq \{a_1, a_2\}$. So, up to symmetry there are four possibilities for $P(e)$: $\{a_1, b_1\}$, $\{b_1, b_2\}$, $\{a_1, a_2, b_1\}$ and $\{a_1, b_1, b_2\}$.

(5.1.1) Suppose that $P(e) = \{a_1, b_1\}$. Then d must be adjacent to a_2 and b_2 as well as c ; e may possibly be adjacent to one of a_2 or b_2 also. Thus, G has the subgraph shown in Figure IV.11(b). Let g_1 be the neighbour of b_1 other than e or x_1 ; possibly $g_1 = a_1, a_2$ or b_2 . If $g_1 = a_2$ or b_2 then $a_2 b_2 \notin E(G)$ and G encloses P_3 as shown in Figure IV.12(a). So, we may assume $g_1 \neq a_2, b_2$. If $a_1 g_1 \notin E(G)$ (including the case $g_1 = a_1$) then G encloses P_3 as shown in Figure IV.12(b). Therefore, g_1 is a vertex other than a_2, b_2 adjacent to a_1 . If one of e or g_1 is adjacent to neither a_2 nor b_2 , then G encloses P_3 as shown in Figure IV.13(a). Finally, if $a_2 g_1, b_2 e \in E(G)$ or $a_2 e, b_2 g_1 \in E(G)$, then G is the graph of Figure IV.13(b), which is isomorphic to R_{12} . This concludes case (5.1.1).

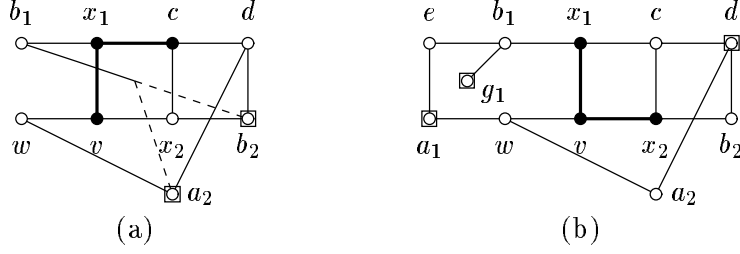


Figure IV.12

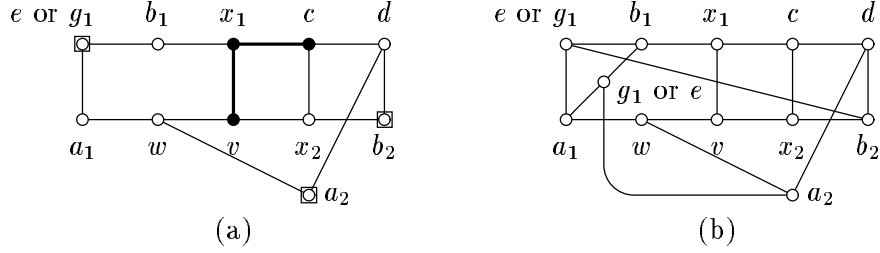


Figure IV.13

(5.1.2) Suppose that $P(e) = \{b_1, b_2\}$. Then d must be adjacent to a_1 and a_2 as well as c ; e may possibly be adjacent to one of a_1 or a_2 also. Thus, G has the subgraph shown in Figure IV.14(a). To show that v is a star vertex of type (B), we must show that G does not contain the edges a_1a_2 , b_1b_2 , or $a_i b_j$ where $i, j = 1, 2$. If $a_1a_2 \in E(G)$ then G encloses $K_{1,1,2}$ as shown in Figure IV.14(b). If $b_1b_2 \in E(G)$ then G encloses P_3 as shown in Figure IV.15(a). If an edge $a_i b_j \in E(G)$, then by symmetry we may assume it is $a_1 b_1$. Then $a_1 b_2 \notin E(G)$, and G encloses P_3 as shown in Figure IV.15(b). Therefore, v is a type (B) vertex as in the statement of this theorem. This concludes case (5.1.2).

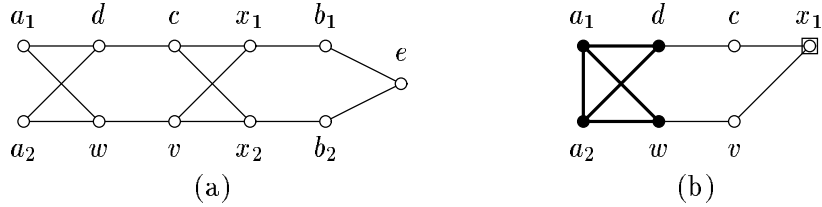


Figure IV.14

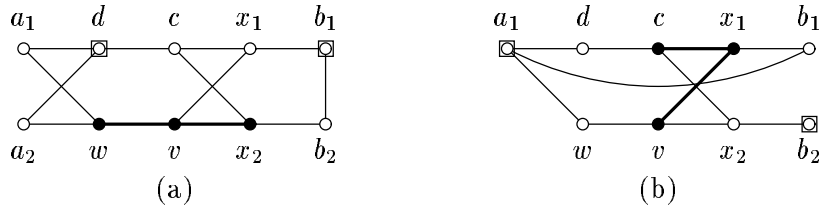


Figure IV.15

(5.1.3) Suppose that $P(e) = \{a_1, a_2, b_1\}$. Then $P(d) = N(d) \cap N_2 = \{b_2, c\}$ and G is as shown in Figure IV.16(a). Let g_1 be the neighbour of b_1 other than e or x_1 ; possibly $g_1 = a_1, a_2$ or b_2 . If $dg_1 \in E(G)$ (including the case $g_1 = b_2$) then G encloses a P_3 as shown in Figure IV.16(b), so $g_1 \neq b_2$ and $dg_1 \notin E(G)$. If g_1 is not adjacent to some a_i (including $g_1 = a_i$), say $a_i = a_1$, then G encloses P_3 as shown in Figure IV.17(a). Finally, if $a_1 g_1, a_2 g_1 \in E(G)$, then G encloses $K_{2,3}$ as shown in Figure IV.17(b). This concludes case (5.1.3).

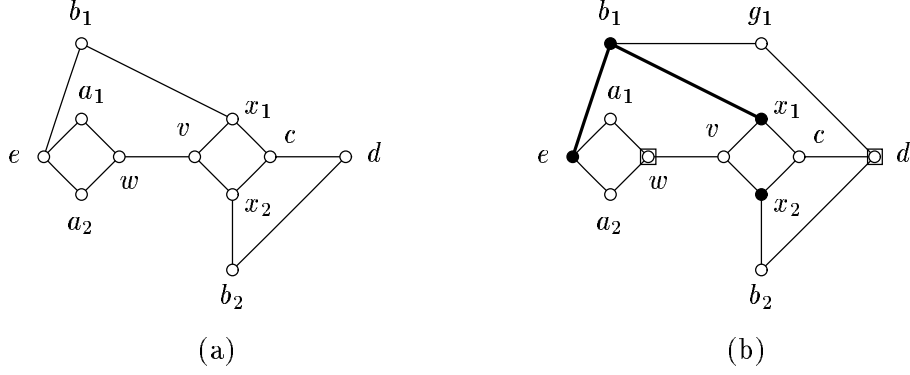


Figure IV.16

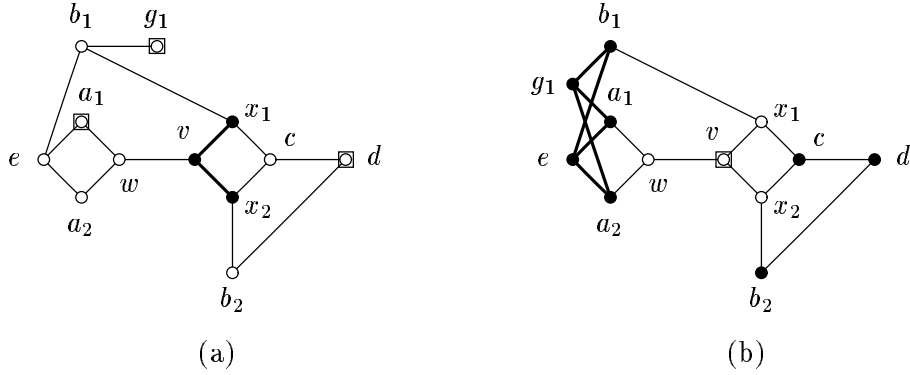


Figure IV.17

(5.1.4) Suppose that $P(e) = \{a_1, b_1, b_2\}$. Then $P(d) = N(d) \cap N_2 = \{a_2, c\}$. Let f be the third neighbour of d . Then G encloses P_3 , as shown in Figure IV.18(a). This concludes cases (5.1.4) and (5.1).

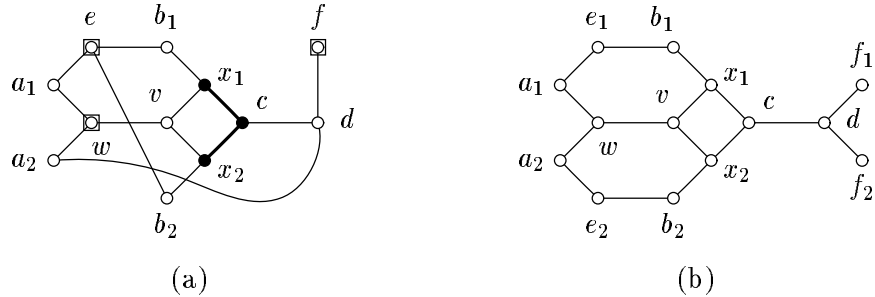


Figure IV.18

(5.2) Suppose that $p = 3$. By Lemma IV.2 neither $P(u_2)$ nor $P(u_3)$ has cardinality 1, and neither is equal to $\{a_1, a_2\}$. Therefore, without loss of generality we may write $u_2 = e_1$ and $u_3 = e_2$ where $P(e_i) = \{a_i, b_i\}$ for $i = 1, 2$. The only vertices of N_2 adjacent to each $u_i \in \{d, e_1, e_2\}$ are those in $P(u_i)$. Let f_1, f_2 be the neighbours of d other than c . Thus, G has the subgraph of Figure IV.18(b). If some f_i , say f_1 , is adjacent to neither e_1 nor e_2 , then G encloses P_3 as shown in Figure IV.19. Therefore, without loss of generality we may suppose that $e_1 f_1, e_2 f_2 \in E(G)$.

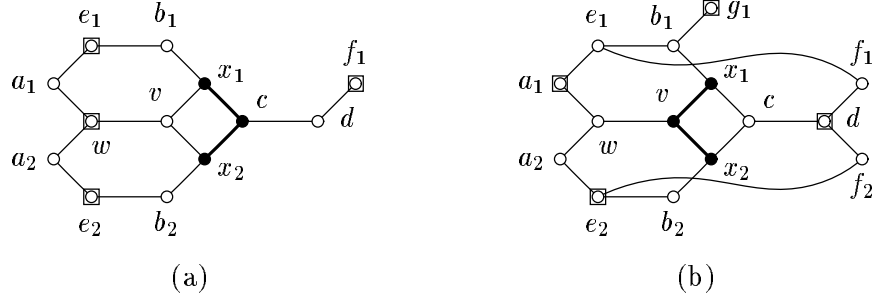


Figure IV.19

Let g_1 be the neighbour of b_1 other than e_1 or x_1 . Possibly g_1 is one of a_1, a_2, b_2, f_1, f_2 . If g_1 is none of these five vertices and $a_1g_1 \notin E(G)$, or if $g_1 = a_1$, then G encloses P_3 as shown in Figure IV.19(b). If g_1 is none of the five vertices, but $a_1g_1 \in E(G)$, then G encloses P_3 as shown in Figure IV.20(a). If g_1 is one of a_2, b_2, f_1, f_2 , then G encloses P_3 as shown in Figure IV.20(b): whatever the exact location of g_1 , it belongs to $N[I'] - I'$. This concludes cases (5.2) and (5).

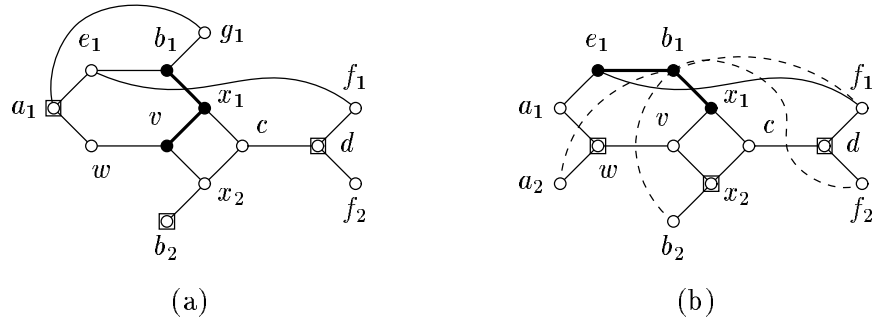


Figure IV.20

(6) Suppose that $n_2 = 6$. Then every vertex of N_2 has exactly one neighbour in N_1 . It follows from Lemma IV.2 that no $u_i \in I$ has $|P(u_i)| = 1$. Therefore $|P(u_i)| = 2$ or 3 for each $u_i \in I$, and $p = |I| = 2$ or 3 .

(6.1) Suppose that $p = 2$. Then each vertex of I must be adjacent to 3 vertices of N_2 . Up to symmetry, there are two ways this can happen: either some vertex of I is adjacent to two vertices of N_2 that have a common neighbour in N_1 , or not. The two situations are shown in Figure IV.21, where $I = \{c_1, c_2\}$. In both cases there may exist additional edges with both ends in N_2 .

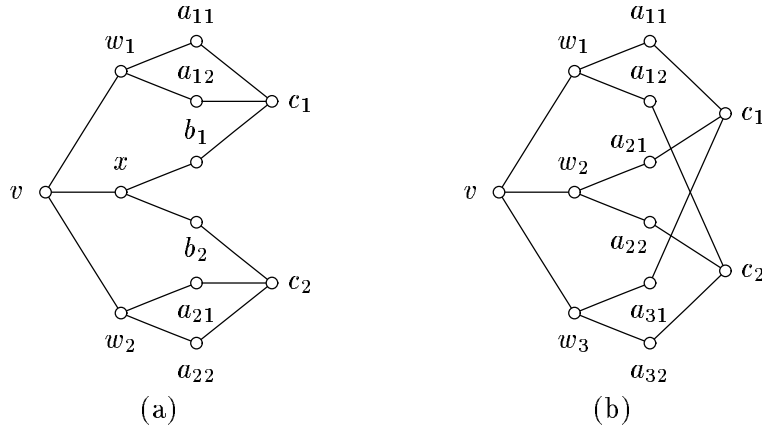


Figure IV.21

(6.1.1) Suppose that some vertex of I is adjacent to two vertices of N_2 that have a common neighbour in N_1 , as in Figure IV.21(a). Let e_1 denote the neighbour of b_1 other than x or c_1 ; possibly $e_1 \in N_2 - \{b_1\}$. If $e_1 = b_2$, choose a_{1i}, a_{2j} so that $a_{1i}a_{2j} \notin E(G)$, and then G encloses L as shown in Figure IV.22(a). If $e_1 = a_{21}$ or a_{22} , say $e = a_{21}$, then G encloses P_3 as shown in Figure IV.22(b). If $e_1 = a_{1i}$, or $e_1 \in N_3$ and $a_{1i}e_1 \notin E(G)$, for either i , then G encloses P_3 as shown in Figure IV.23(a). Finally, if $e_1 \in N_3$ and $a_{11}e_1, a_{12}e_1 \in E(G)$, then G encloses $K_{2,3}$ as shown in Figure IV.23(b). This concludes case (6.1.1).

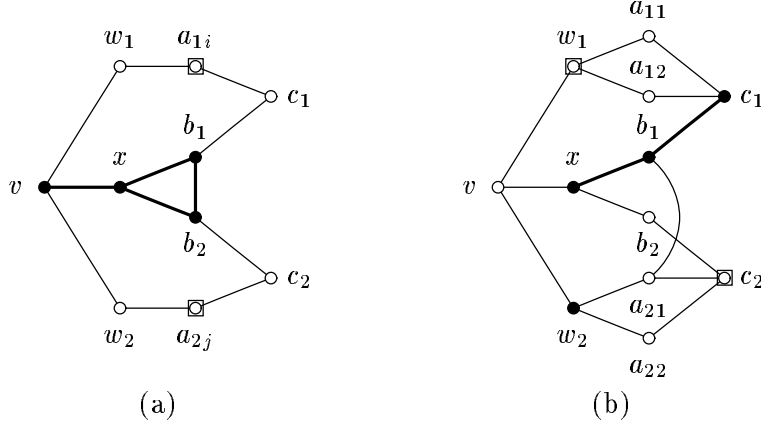


Figure IV.22

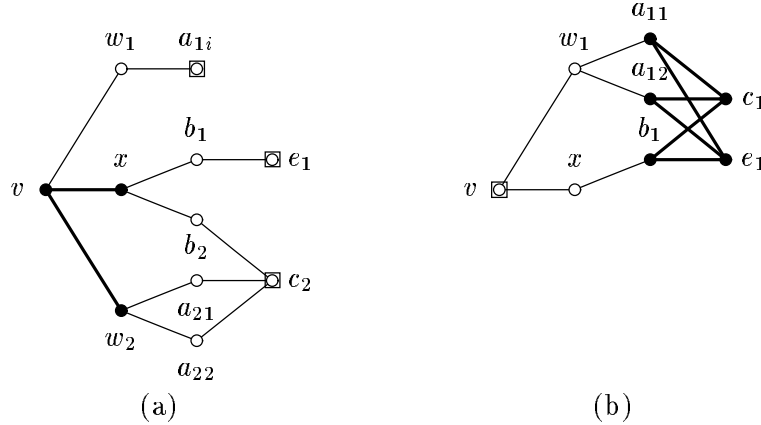


Figure IV.23

(6.1.2) Suppose that neither vertex of I is adjacent to two vertices of N_2 that have a common neighbour in N_1 , as in Figure IV.21(b). Let d_{ij} be the neighbour of a_{ij} other than w_i or c_j , for $i = 1, 2, 3$ and $j = 1, 2$; possibly $d_{ij} \in N_2 - \{a_{ij}\}$. Consider d_{11} . If $d_{11} \in N_2 - \{a_{11}\}$ or $d_{11} \in N_3$ and d_{11} has a neighbour $f \notin N_2$, then G encloses P_3 as shown in Figure IV.24(a). Therefore, we may assume that d_{11} is in N_3 , but all neighbours of d_{11} are in N_2 . By symmetry, the same holds for all d_{ij} . Thus, there are exactly two distinct vertices of the form d_{ij} , say $d_1 = d_{11}$ and $d_2 \neq d_{11}$, each with three neighbours in N_2 . Instead of $I = \{c_1, c_2\}$, we could take $I = \{d_1, d_2\}$. Thus, if $d_{i1} = d_{i2}$ for some i , we would have case (6.1.1), so we may assume that $d_{i1} \neq d_{i2}$ for all $i = 1, 2, 3$. Thus, each d_i is adjacent to neighbour of each w_j , $i = 1, 2$ and $j = 1, 2, 3$. If $N(c_i) = N(d_j)$ for some i and j , then without loss of generality we have the situation of Figure IV.24(b) and G encloses $K_{2,3}$. Otherwise, we must have $|N(c_i) \cap N(d_j)| = 1$ for some i, j , say $i = j = 1$, and we have the situation in IV.24(c), and G encloses P_3 . This concludes cases (6.1.2) and (6.1).

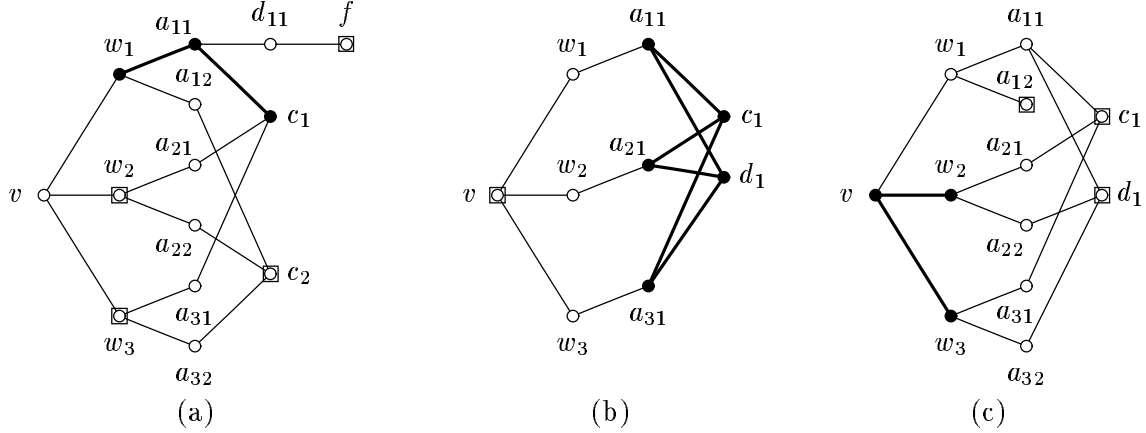


Figure IV.24

(6.2) Suppose that $p = 3$. From Lemma IV.2, each $P(u_i)$, $u_i \in I$, must have exactly two elements, so each element of I must be adjacent to exactly two elements of N_2 . By Lemma IV.2, this must happen as shown in Figure IV.25, where $I = \{c_1, c_2, c_3\}$. We will frequently refer to the following 12-cycle:

$$C = a_{11}w_1a_{12}c_3a_{21}w_2a_{22}c_1a_{31}w_3a_{32}c_2a_{11}.$$

For each $i = 1, 2, 3$ and $j = 1, 2$, the vertex a_{ij} is adjacent to w_i , c_k (where $k \equiv i + j \pmod{3}$), and a third vertex that we denote by d_{ij} . Each d_{ij} may be a vertex of N_2 or of N_3 . We divide our argument into several steps.

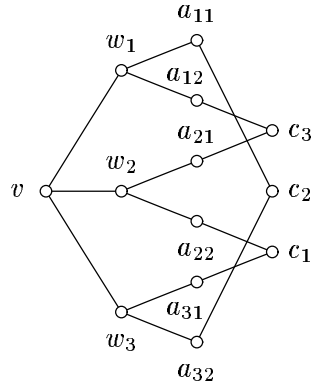


Figure IV.25

(A) First we show that if $d_{ij} \in N_3$, then $c_i d_{ij} \in E(G)$. It suffices to consider d_{11} . Suppose that $d_{11} \in N_3$. If $a_{32}d_{11}, c_1d_{11}, c_3d_{11} \notin E(G)$, then G encloses P_3 as shown in Figure IV.26(a). If $a_{32}d_{11} \in E(G)$, then G encloses P_3 as shown in Figure IV.26(b). Figure IV.26(b) also applies if $c_3d_{11} \in E(G)$ provided $a_{22}a_{32} \notin E(G)$. So suppose that $c_3d_{11}, a_{22}a_{32} \in E(G)$. Let e_2 be the neighbour of c_2 distinct from a_{11}, a_{32} , which belongs to N_3 or N_4 . If $e_2 = c_1$ then G encloses P_3 as shown in Figure IV.26(c). If $e_2 = d_{11}$ then G encloses L as shown in Figure IV.27(a). If $e_2 = d_{31}$ then G encloses P_3 as shown in Figure IV.27(b). If $e_2 \neq c_1, d_{11}, d_{31}$ then G encloses P_3 as shown in Figure IV.27(c). So we have shown that $a_{32}d_{11}, c_3d_{11} \notin E(G)$, so it must be the case that $c_1d_{11} \in E(G)$.

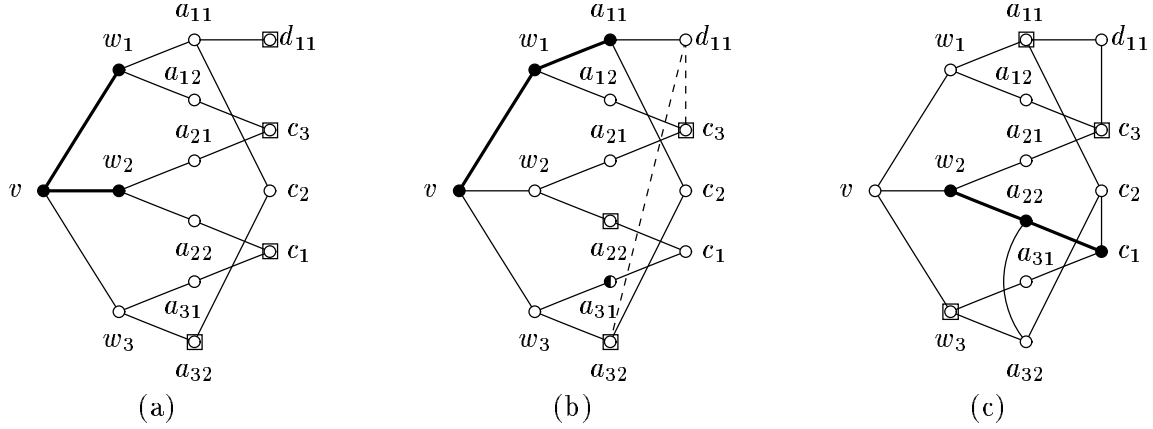


Figure IV.26

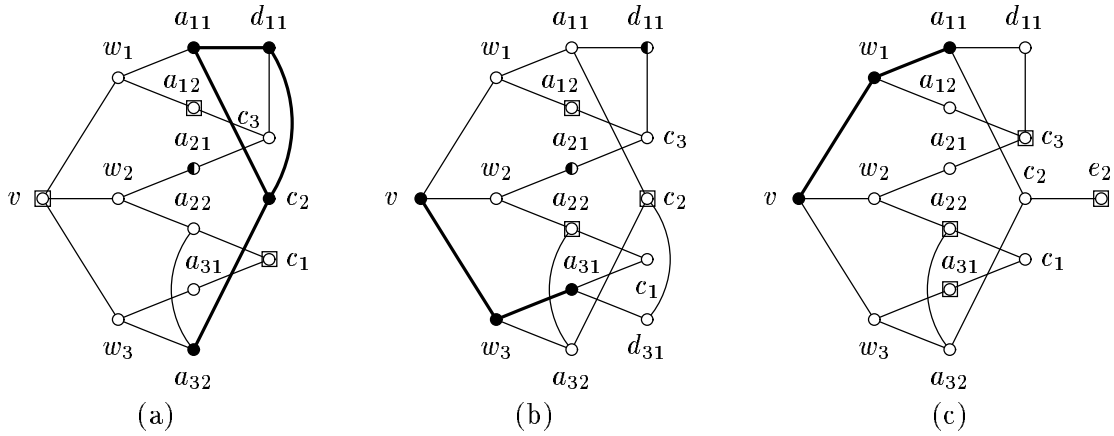


Figure IV.27

(B) Next we show that if $d_{ij} \in N_2$, then d_{ij} is not at distance 2 from a_{ij} on C . It suffices to consider d_{11} : we must show that $d_{11} \neq a_{32}, a_{12}$. If $d_{11} = a_{32}$ then G encloses P_3 as shown in Figure IV.26(a) (identifying d_{11} with a_{32}). Suppose that $d_{11} = a_{12}$. If $a_{21}a_{32} \in E(G)$ then G encloses P_3 as shown in Figure IV.28(a), and if $a_{21}a_{32} \notin E(G)$ then G encloses L as shown in Figure IV.28(b).

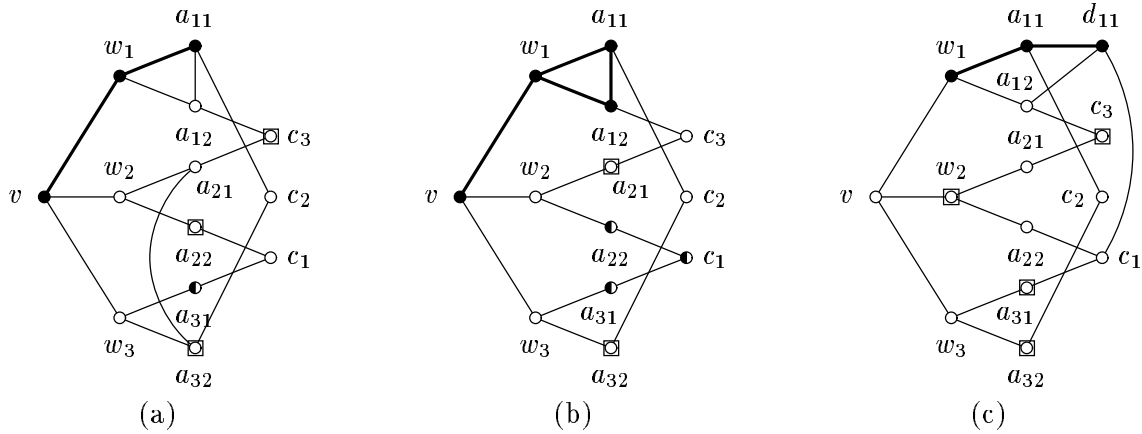


Figure IV.28

(C) Now we show that if $d_{ij} \in N_3$, then $d_{i,3-j} \in N_2$. It suffices to consider $i = j = 1$. Suppose $d_{11} \in N_3$ and $d_{12} \in N_3$ also. Since both d_{11} and d_{12} must be adjacent to c_1 , $d_{11} = d_{12}$. By (B), $a_{31}a_{32} \notin E(G)$, and G encloses P_3 as shown in Figure IV.28(c).

Note that if $d_{ij} \in N_2$, then $d_{ij} = a_{kl}$ for some k, l , so that $d_{kl} = a_{ij} \in N_2$ also. Therefore, an even number of the six d_{ij} 's belong to N_2 , and hence an even number belong to N_3 . By (C), at most three of the d_{ij} 's belong to N_3 , so either two are in N_3 and four are in N_2 , or all six are in N_2 .

(D) Suppose that there are two d_{ij} 's in N_3 . By symmetry, we may assume that $d_{11} \in N_3$, and by (A), $c_1d_{11} \in E(G)$. First we prove that the other d_{ij} in N_3 is d_{22} , and that $d_{11} \neq d_{22}$. If $d_{22} \in N_2$, or if $d_{22} = d_{11}$, then G encloses P_3 as shown in Figure IV.29(a), where d_{22} , the third neighbour of a_{22} , is a vertex of $N[I'] - I'$ in either case. Thus $d_{22} \in N_3$ and $d_{11} \neq d_{22}$. By (A), $c_2d_{22} \in E(G)$. Since the other four d_{ij} 's must belong to N_2 , and since $a_{31}a_{32} \notin E(G)$ by (b), there are two possibilities: $a_{12}a_{31}, a_{21}a_{32} \in E(G)$, so that G encloses P_3 as in Figure IV.29(b), or $a_{12}a_{32}, a_{21}a_{31} \in E(G)$, so that G encloses P_3 as in Figure IV.29(c). Therefore, all d_{ij} 's must belong to N_2 .

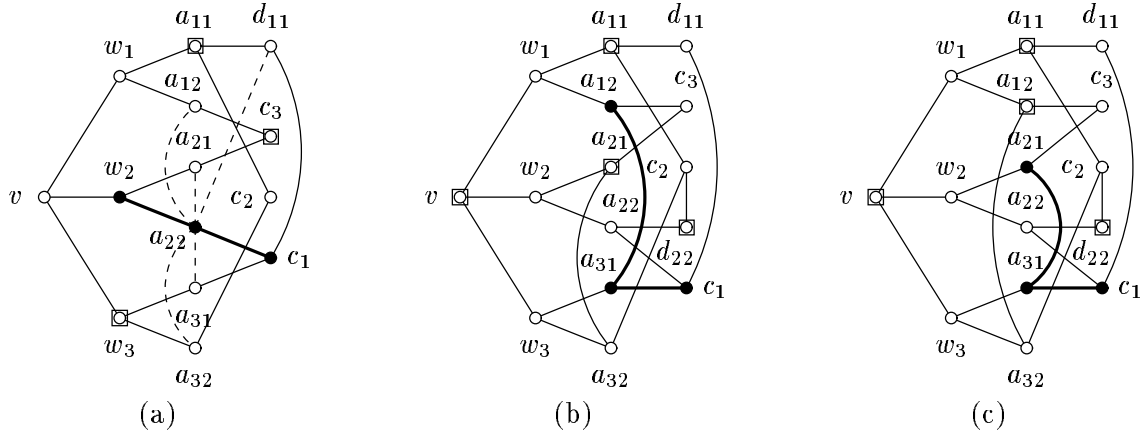


Figure IV.29

(E) Finally, we show that all d_{ij} 's cannot belong to N_2 . If they do, there are three edges of the form $a_{ij}a_{kl}$. By (B), a_{ij} and a_{kl} cannot be distance 2 apart in C , so they must be distance 4 or 6. There are only two essentially different situations: each edge $a_{ij}a_{kl}$ joins vertices at distance 6 in C , or one such edge joins vertices at distance 6 while the other two join vertices at distance 4. In the first situation we have $a_{11}a_{22}, a_{12}a_{31}, a_{21}a_{32} \in E(G)$. Let e_2 be the vertex other than a_{11} and a_{32} adjacent to c_2 . If $c_3e_2 \notin E(G)$, then G encloses P_3 as shown in Figure IV.30(a), and a similar result holds by symmetry if $c_1e_2 \notin E(G)$. Thus, $c_1e_2, c_3e_2 \in E(G)$, and G is the graph of Figure IV.30(b), which encloses a P_3 as shown. In the second situation, we may assume by symmetry that $a_{12}a_{31} \in E(G)$, and then we must also have $a_{11}a_{21}, a_{22}a_{32} \in E(G)$. Then G encloses P_3 as in Figure IV.30(c). This concludes cases (6.2) and (6).

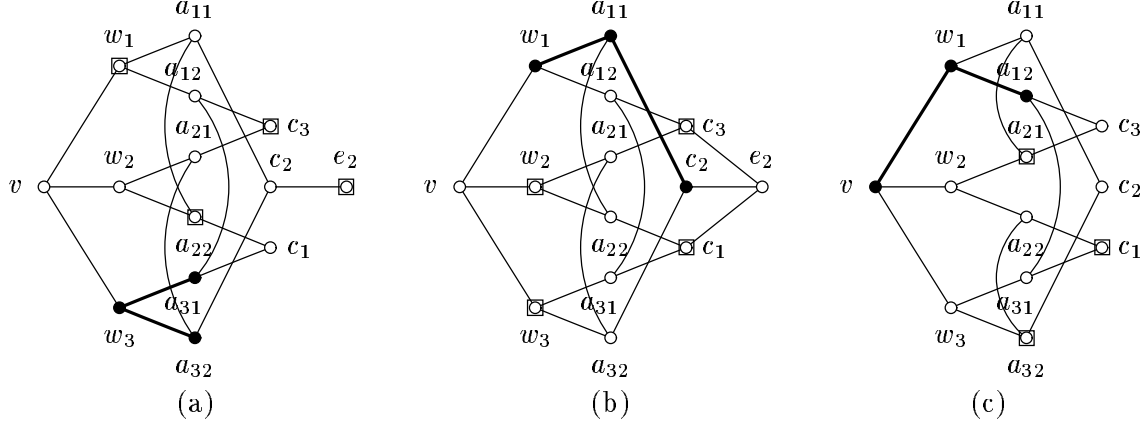


Figure IV.30

We have now shown that every star vertex is of type (A), when $n_2 = 4$, or type (B), when $n_2 = 5$. We must now prove that every neighbour of a star vertex is also a star vertex of the same type.

Suppose v is a star vertex of type (A). We may assume that v is part of an induced subgraph labelled as in Figure IV.9(b). From $I' = \{a_1, a_2\}$ we see that x is also a star vertex with $n_2 = 4$, so x is a star vertex of type (A). Now consider w_1 . Let $e_1 \neq d_1, w_1$ be the third neighbour of a_1 , and $f_1 \neq a_1, b_1$ be the third neighbour of d_1 . Both of these are (possibly equal) new vertices that do not appear in Figure IV.9(b). If $e_1 f_1 \notin E(G)$ (including the case $e_1 = f_1$) then G encloses a P_3 as in Figure IV.31(a). Therefore, e_1 and f_1 are distinct and adjacent. Now Figure IV.31(b) shows that w_1 is a star vertex with $n_2 = 4$, so w_1 is a star vertex of type (A). The argument for w_2 is symmetric.



Figure IV.31

Now suppose v is a star vertex of type (B). We may assume that v is part of a subgraph labelled as in Figure IV.14(a), which is induced except that e may be adjacent to one of a_1 or a_2 . Let $g_i \neq d, w$ be the third neighbour of a_i , $i = 1, 2$. Suppose first that e is adjacent to some a_i ; by symmetry, we may assume $a_1 e \in E(G)$. Then g_2 is a new vertex, not shown in Figure IV.14(a). If g_2 is not adjacent to some b_i , say $i = 2$ by symmetry, then G encloses P_3 as shown in Figure IV.32(a). Therefore, $b_1 g_2, b_2 g_2 \in E(G)$, and G is the graph of Figure IV.32(b), which is isomorphic to X_{12} , in which every vertex is a star vertex of type (B).

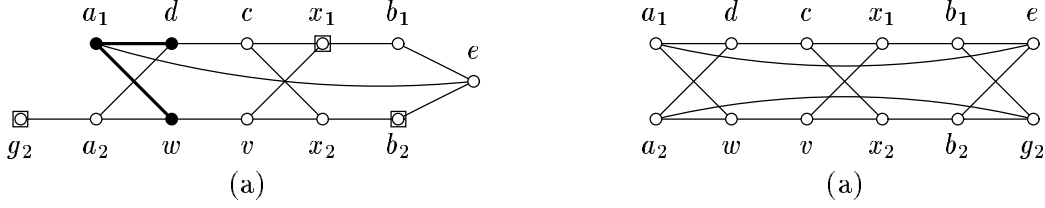


Figure IV.32

Suppose now that $a_1e, a_2e \notin E(G)$. We first show that w is a star vertex of type (B). Now g_1, g_2 are (possibly equal) new vertices, which do not appear in Figure IV.14(a). Let h denote a neighbour of g_1 distinct from a_1 and (if $g_1 = g_2$) also from a_2 ; possibly $h \in \{b_1, b_2, e, g_2\}$. Suppose that $g_1 = g_2$. Let j be any neighbour of h distinct from $g_1 = g_2$; possibly $j \in \{b_1, b_2, e, x_1, x_2\}$. Since there is no vertex adjacent to both x_1 and x_2 other than v and $j \neq v$, $jx_i \notin E(G)$ for some x_i (including the possibility that $j = x_i$); by symmetry we may assume that $x_i = x_1$. Then G encloses $K_{2,3}$ as shown in Figure IV.33(a). Therefore, $g_1 \neq g_2$. Since $h \neq v$, $hx_i \notin E(G)$ for some x_i ; by symmetry we may take $x_i = x_1$. If $g_2h \notin E(G)$ (including the case $h = g_2$) then G encloses P_3 as shown in Figure IV.33(b). Therefore, $g_2h \in E(G)$. Then Figure IV.34 shows that w is a star vertex with $n_2 = 5$, i.e., a star vertex of type (B).

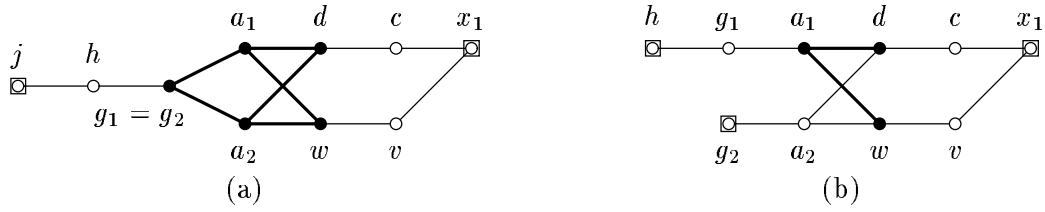


Figure IV.33

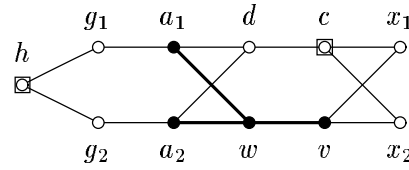


Figure IV.34

Now we must show that x_1 and x_2 are star vertices of type (B). Let $f_i \neq e, x_i$ denote the third neighbour of b_i for $i = 1, 2$. Then f_1, f_2 are (possibly equal) new vertices, that do not appear in Figure IV.14(a). If $f_1 \neq f_2$, then a_1 cannot be adjacent to both f_1, f_2 , so $a_1f_i \notin E(G)$ for some f_i ; by symmetry we may take $f_i = f_1$. Then G encloses P_3 as shown in Figure IV.35(a). Thus, $f_1 = f_2$. Then Figure IV.35(b) shows that x_1 is a star vertex with $n_2 = 5$, i.e., a star vertex of type (B). The argument for x_2 follows by symmetry.

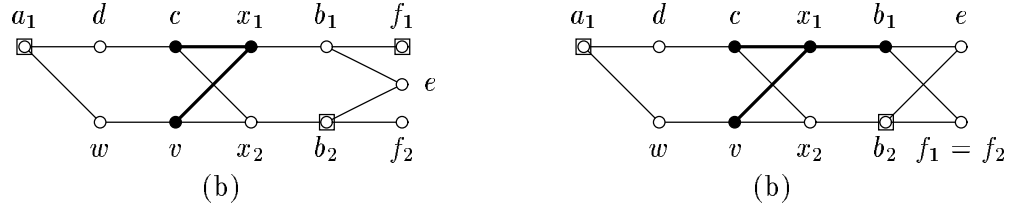


Figure IV.35

This concludes the proof of Theorem IV.1. ■

Theorem IV.3. *Let G be a connected cubic graph. Then G is \mathbf{Z}_2 -well-covered but not well-covered if and only if G is one of*

- (i) R_{12} (shown in Figure III.3),
- (ii) $C_4 \times K_2 \cong X_8$,
- (iii) $C_m \times K_2$ for $m \geq 6$,
- (iv) M_{2m} for $m \geq 5$ (described in Section III), or
- (v) X_{4k} for $k \geq 3$ (described in Section III).

Proof. From Section III, the graphs in (i) to (v) are all \mathbf{Z}_2 -well-covered but not well-covered.

Conversely, suppose that G is \mathbf{Z}_2 -well-covered but not well-covered. Assume that G is not isomorphic to $C_4 \times K_2 \cong X_8$, M_{10} , or R_{12} . Since G is connected, by Theorem IV.1 all vertices of G are star vertices of the same type, (A) or (B). Define an edge e of G to be of *type i* (i a nonnegative integer) if e belongs to i 4-cycles of G .

Suppose all vertices of G are star vertices of type (A). Looking at Figure IV.9(b), we see that the edges of G are all type 1 or type 2. The edges of type 2 form a perfect matching. If we form a graph H whose vertices are the type 2 edges of G , with two type 2 edges adjacent if they belong to a common 4-cycle of G , then H is 2-regular, and hence is a cycle. It follows that G is isomorphic to $C_m \times K_2$ or M_{2m} for $m \geq 6$.

Now suppose that all vertices of G are star vertices of type (B). Looking at Figure IV.14(a), we see that the edges of G are all type 0 or type 1. The spanning subgraph formed by the edges of type 1 is a 2-factor whose components are all the 4-cycles of G . Each pair of opposite vertices of a 4-cycle is joined in G by a pair of type 0 edges to a pair of opposite vertices in another 4-cycle. Therefore, if we form a graph J whose vertices are the 4-cycles of G , with two 4-cycles adjacent if they are joined by a pair of edges, then J is 2-regular, and hence is a cycle. It follows that G is isomorphic to X_{4k} for $k \geq 3$. ■

Combining this with Theorem II.7, we obtain the following characterisation of cubic parity (\mathbf{Z}_2 -well-covered) graphs. Note that a graph is a parity graph if and only if each of its components is a parity graph, so we may restrict our attention to connected graphs.

Corollary IV.4. *Suppose G is a connected cubic graph. Then G is \mathbf{Z}_2 -well-covered, i.e., a parity graph, if and only if it is one of the graphs of either Theorem II.7 or Theorem IV.3.*

V. Concluding remarks

An important point of this paper is that minimal non-well-covered graphs are a highly useful tool. Here, they enabled us to characterise cubic parity graphs by extending the characterisation of cubic well-covered graphs, rather than by starting from scratch. The original case analysis for cubic well-covered graphs took roughly 26 pages (single-spaced, with diagrams) if given in full. The case analysis in this paper takes roughly 14 pages if given in full. One would guess that a characterisation of the cubic parity graphs from scratch, which would have the well-covered and non-well-covered cases tangled together, would probably cover essentially the same cases and so take about 40 pages. Minimal non-well-covered graphs allow us to separate the well-covered and non-well-covered cases, using Lemma II.9, and so we only had to do the 14 extra pages of the non-well-covered case, rather than the whole 40 pages.

As mentioned in Section I, it is interesting that cubic parity graphs divide sharply into two classes, those that are well-covered and those that are not. The structures of these two classes, as described in Theorems II.7 and IV.3, respectively, are quite different. This is surprising, because for graphs of girth 6 or more, as investigated in [5, 9], the structure of \mathbf{Z}_m -well-covered graphs seems to be a natural generalisation of the structure of the well-covered graphs.

In Theorem II.7, the connected cubic well-covered graphs are, with six exceptions, constructed from the three basic well-covered building blocks A , B and C . The decomposition into building blocks is usually unique. One might reasonably have imagined that cubic \mathbf{Z}_2 -well-covered graphs would include a few more building blocks that could be used together with A , B and C . However, the connected cubic \mathbf{Z}_2 -well-covered non-well-covered graphs, as described in Theorem II.7, do not seem to be made up of simple building blocks in the same way. One can decompose graphs in the families $C_m \times K_2$, M_{2m} , and X_{4k} into vertex-disjoint \mathbf{Z}_2 -well-covered subgraphs isomorphic to $K_{1,3}$, sometimes also using one copy of P_2 or $\overline{K_2}$, but the decomposition is far from unique. The building blocks A , B and C do not appear in these graphs at all.

For well-covered graphs of bounded degree, it seems reasonable to conjecture that there will always be a decomposition into a finite family of building blocks. This is supported by the characterisation of cubic well-covered graphs, by Ramey's characterisation of well-covered graphs of maximum degree 3 in his Ph.D. thesis [13], and also by Hartnell and Plummer's characterisation of 4-connected 4-regular claw-free well-covered graphs [10].

Conjecture V.1. *Fix $d \geq 3$. Then there is a collection $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ of well-covered graphs of maximum degree at most d , and a list of rules for connecting the elements of \mathcal{G} together via edges, such that with finitely many exceptions every well-covered graph of maximum degree at most d is obtained by joining together elements of \mathcal{G} according to these rules.*

One might also conjecture that the decomposition of a graph into elements of \mathcal{G} is, with finitely many exceptions, unique. For \mathbf{Z}_m -well-covered graphs, it is less clear whether something similar to Conjecture V.1 is likely to hold. We expect that graphs that are minimal non-well-covered but \mathbf{Z}_m -well-covered, such as $K_{p, km+p}$ for $p, k \geq 1$, will play an important role in the structure of graphs that are \mathbf{Z}_m -well-covered but not well-covered, and this is shown by Corollary II.6 and Theorem IV.1, and by the results for girth at least 6 by Finbow and Hartnell [9] and Caro and Hartnell [5].

It would be useful to have more evidence as to the structure of \mathbf{Z}_m -well-covered graphs with degree restrictions. Using the techniques of this paper, it would probably be possible to extend Ramey's characterisation of well-covered graphs of maximum degree at most 3 to \mathbf{Z}_2 -well-covered graphs of maximum degree at most 3. However, this would be a long and tedious exercise. One alternative approach is to use a computer to determine the possible structures close to a vertex. From [4] we know that A -well-coveredness is a local property, determined by the structure of the graph out to distance 4 around each vertex. Once the possible local structures are found, a computer could also be used to determine which pairs of structures could belong to adjacent vertices. In this way, we might obtain enough information to give a structural characterisation without a tedious manual case analysis. Such an approach might also allow us to extend what is known about 4-regular graphs, or even graphs of maximum degree 4, both for well-covered graphs and for \mathbf{Z}_m -well-covered graphs.

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