Products of generating functions

Example: (1) Subsets of $\mathbb{N}_0$, with weight = value: $A = \{3, 4\}, A(x) = x^3 + x^4$; $B = \{2, 7, 9\}, B(x) = x^2 + x^7 + x^9$. Then $A \times B = \{(3, 2), (3, 7), (3, 9), (4, 2), (4, 7), (4, 9)\}$.

Weight elements of $A \times B$ by $w(a, b) = a + b$. Then

$$(A \times B)(x) = x^5 + x^{10} + x^{12} + x^6 + x^{11} + x^{13}$$

$$= x^{3+2} + x^{3+7} + x^{3+9} + x^{4+2} + x^{4+7} + x^{4+9}$$

$$= (x^3 + x^4)(x^2 + x^7 + x^9) = A(x)B(x)$$

Product Lemma: Suppose have $A$ weighted by $u$, $B$ weighted by $v$, and we weight $A \times B$ by $w(\alpha, \beta) = u(\alpha) + v(\beta)$. Then

$$(A \times B)_w(x) = \sum_{(\alpha, \beta) \in A \times B} x^{w(\gamma)} = \sum_{(\alpha, \beta) \in A \times B} x^{u(\alpha) + v(\beta)} = \sum_{(\alpha, \beta) \in A \times B} x^{u(\alpha) + v(\beta)} = \sum_{\alpha \in A} \sum_{\beta \in B} x^{u(\alpha)} x^{v(\beta)} = \sum_{\alpha \in A} x^{u(\alpha)} \sum_{\beta \in B} x^{v(\beta)} = A_u(x)B_v(x)$$

Note: Extends to arbitrary cartesian products $A_1 \times A_2 \times \ldots \times A_k$.

Bijection Rule: Suppose have $S$ weighted by $w$, $S'$ weighted by $w'$ and bijection $f : S \rightarrow S'$ with $w'(f(\sigma)) = w(\sigma)$ (weight-preserving) for all $\sigma \in S$. Then $S_w(x) = S_{w'}(x)$.

Combine to get:

General Product Lemma: Suppose have $A$ weighted by $u$, $B$ weighted by $v$, injective $(1 - 1)$ operation $*: A \times B \rightarrow C$ and $R = \{\alpha * \beta | \alpha \in A, \beta \in B\}$ (range of $*$) weighted by $w(\alpha * \beta) = u(\alpha) + v(\beta)$. Then $R_w(x) = A_u(x)B_v(x)$. Since $*$ is injective, provides weight-preserving bijection from $A \times B$ to $R$.

Example: (1) 01-strings, weight=length, $* = \text{concatenation}$.

$A = \{0, 00, 000\}$

$B = \{11, 1111\}$

$R = A * B = \{\alpha * \beta | \alpha \in A, \beta \in B\} = \{\alpha \beta | \alpha \in A, \beta \in B\}$

$$= \{011, 01111, 00111, 001111, 000111, 0001111\}.$$ 

Injective: $\alpha_1 \beta_1 = \alpha_2 \beta_2 \Rightarrow \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ (separate into 0s and 1s). So

$$R(x) = A(x)B(x) = (x + x^2 + x^3)(x^2 + x^4) = x^3 + x^4 + 2x^5 + x^6 + x^7.$$ 

Have to be careful, though.

(2) 01-strings, weight=length, $* = \text{concatenation}$.

$A = \{0, 00\}$

$B = \{000, 0000\}$

Then $R = A * B = \{0000, 00000, 00000, 000000\} = \{000, 00000, 000000\}$. We have

$$R(x) = x^4 + x^5 + x^6,$$

$$A(x)B(x) = (x + x^2)(x^3 + x^4) = x^4 + 2x^5 + x^6 \neq R(x)$$

because $*$ not injective: $0 * 0000 = 00 * 000$ (non-unique decomposition in terms of $*$).

Notes: (1) $*$ must be injective.

(2) Extends to operations where we combine more than two things, provided

(a) weight of result = sum of weights of things you combine, and

(b) injective: can only get each result in one way.
Example: 012-strings, weight=length.
\[ A = \{2,02,12,012,0012\} \quad \text{and} \quad A(x) = x + 2x^2 + x^3 + x^4 \]
\[ B = \{22,11,00,111,222,000\} \quad \text{and} \quad B(x) = 3x^2 + 3x^3 \]
Let * = concatenation of strings, then injective on A x B (can break after first 2, e.g. 01222 = 012 * 22). So without working out C = \{\alpha\beta \mid \alpha \in A, \beta \in B\}; we can calculate
\[ C(x) = A(x)B(x) = (x + 2x^2 + x^3 + x^4)(3x^2 + 3x^3) \]
yields
\[ = 3x^3 + 9x^4 + 9x^5 + 6x^6 + 3x^7. \]
So, for example, by concatenating things in A and B we get 9 strings of length 5.

G.f.s of some subsets of Z: Consider following subsets of Z, weighted by value:
\[ N = \{1,2,3,\ldots\} \quad \text{g.f. } N(x) = x + x^2 + x^3 + \ldots \]
\[ = x(1 + x + x^2 + \ldots) = x/(1-x) \]
\[ N_0 = \{0,1,2,3,\ldots\} \quad \text{g.f. } N_0(x) = 1 + x + x^2 + x^3 + \ldots = 1/(1-x) \]
\[ N_n = \{1,2,\ldots,n\} \quad \text{g.f. } N_n(x) = x + x^2 + \ldots + x^n \]
\[ = x(1 + x + x^2 + \ldots + x^{n-1}) = x(1-x^n)/(1-x) \]
(\text{note that } (1-x)(1+x+\ldots+x^{n-1}) = 1-x^n)
\[ 2N = \{2,4,6,\ldots\} \quad \text{g.f. } (2N)(x) = x^2 + x^4 + x^6 + \ldots = x^2/(1-x^2) \]

Equation solutions or compositions

Examples: (1) How many nonnegative integral solutions are there to
\[ x_1 + x_2 + x_3 + x_4 = 27 \quad ? \]
(Already know answer is \( \binom{4}{27} \) = \( \binom{30}{3} \) from ‘order form’ stuff.)
Solution: (using g.f.s) Think of problem as counting number of vectors in \( N_0^4 = \{(x_1,x_2,x_3,x_4) \mid x_1,x_2,x_3,x_4 \in N_0\} \) with weight 27, where to get weight we add the \( x_i \)'s:
\[ w(x_1,x_2,x_3,x_4) = x_1 + x_2 + x_3 + x_4 \]
Thus, we can use the Product Lemma: we want
\[ [x^{27}] (N_0^4)(x) = [x^{27}] \quad \underbrace{N_0(x)}_{\text{g.f. for } x_1} \quad \underbrace{N_0(x)}_{\text{g.f. for } x_2} \quad \underbrace{N_0(x)}_{\text{g.f. for } x_3} \quad \underbrace{N_0(x)}_{\text{g.f. for } x_4} \]
\[ = [x^{27}] (1-x)^{-1}(1-x)^{-1}(1-x)^{-1}(1-x)^{-1} \]
\[ = [x^{27}] (1-x)^{-4} = \left(\frac{\binom{4}{27}}{3}\right) = \binom{30}{3} \]
(which is the answer we already knew).

In general: Number of nonnegative integral solutions to \( x_1 + x_2 + x_3 + \ldots + x_n = k \) is
\[ [x^k] (N_0^n)(x) = [x^k] \quad \underbrace{N_0(x)}_{\text{g.f. for } x_1} \quad \underbrace{N_0(x)}_{\text{g.f. for } x_2} \quad \ldots \quad \underbrace{N_0(x)}_{\text{g.f. for } x_n} \]
\[ = [x^k] (1-x)^{-n} = \binom{n+k-1}{n} = \binom{n+k-1}{k}. \]

Replacing \( N_0 \) by \( N \), get number of positive solutions (\( n \)-part compositions of \( k \)), namely
\[ [x^k]x^n(1-x)^{-n} = [x^{k-n}](1-x)^{-n} = \binom{n}{k-n} = \binom{k-1}{k-n} \]
(2) Find the number of integral solutions to \( x_1 + x_2 + x_3 = 20 \) where \( 0 \leq x_1 \leq 10, 0 \leq x_2 \leq 12\) and \( 3 \leq x_3 \leq 7 \).
Solution: We are trying to count vectors of weight 20 in the set \( A = \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq 10, 0 \leq x_2 \leq 12, 3 \leq x_3 \leq 7\} \). Can use Product Lemma again: multiply g.f.s for \( x_1, x_2 \) and \( x_3 \):

\[
A(x) = (1 + x + x^2 + \ldots + x^{10})(1 + x + x^2 + \ldots + x^{12})(x^3 + x^4 + x^5 + x^6 + x^7)
\]

\[
= \frac{1 - x^{11}}{1 - x} \frac{1 - x^{13}}{1 - x} x^3(1 + x + x^2 + x^3 + x^4)
\]

\[
= \frac{1 - x^{11}}{1 - x} \frac{1 - x^{13}}{1 - x} x^3 \left(1 - x^5\right)
\]

\[
x^3(1 - x^{11})(1 - x^{13})(1 - x^5)(1 - x)^{-3}
\]

and we want the coefficient of \( x^{20} \), so

\[
\text{answer} = \left[ x^{20} \right] x^3(1 - x^{11})(1 - x^{13})(1 - x^5)(1 - x)^{-3}
\]

\[
= \left[ x^{17} \right] (1 - x^5 - x^{11} - x^{13} + x^{16} + x^{18} + x^{24} - x^{29})(1 - x)^{-3}
\]

\[
= \left[ x^{17} \right] (1 - x)^{-3} - \left[ x^{17} \right] x^5(1 - x)^{-3} - \left[ x^{17} \right] x^{11}(1 - x)^{-3} - \left[ x^{17} \right] x^{13}(1 - x)^{-3}
\]

\[
+ \left[ x^{17} \right] x^{16}(1 - x)^{-3} + \left[ x^{17} \right] x^{18}(1 - x)^{-3} + \left[ x^{17} \right] x^{24}(1 - x)^{-3} - \left[ x^{17} \right] x^{29}(1 - x)^{-3}
\]

\[
= \left( \binom{3}{17} - \binom{3}{12} - \binom{3}{6} - \binom{3}{4} + \binom{3}{1} \right)
\]

\[
= \binom{19}{2} - \binom{14}{2} - \binom{8}{2} - \binom{6}{2} + \binom{3}{2}
\]

\[
= 171 - 91 - 28 - 15 + 3 = 40
\]

Notice that in this example all the hard thinking about distributing balls first and subtracting off ones where we had too many balls in a box was done for us by the g.f.

(3) Find the number of ways to distribute 20 balls into 4 boxes if each box has at least one ball and box 4 has at most 10 balls.

Solution: Consider the set

\[
A = \{o, oo, ooo, \ldots\}^3 \times \{o, oo, \ldots, ooooo, \ldots\}
\]

We get g.f.

\[
A(x) = (x + x^2 + x^3 + \ldots)^3(x + x^2 + x^3 + \ldots + x^{10})
\]

\[
= x(1 + x + x^2 + \ldots)^3 x(1 + x + \ldots + x^9)
\]

\[
= [x(1 - x)^{-1}]^3 x(1 - x^{10})/(1 - x)
\]

\[
x^4(1 - x^{10})(1 - x)^{-4}
\]

and we want the coefficient of \( x^{20} \) in this:

\[
[x^{20}] A(x) = [x^{20}] x^4(1 - x^{10})(1 - x)^{-4}
\]

\[
= \left[ x^{16} \right] (1 - x^{10})(1 - x)^{-4}
\]

\[
= \left( [x^{16}] - [x^6] \right)(1 - x)^{-4}
\]

\[
= \left( \binom{4}{16} \right) - \binom{4}{6} = \binom{19}{3} - \binom{9}{3} = 969 - 84 = 885
\]

(4) Find the number of integral solutions to \( x_1 + x_2 + \ldots + x_{2k} = n \) with \( x_i \geq 1 \) and \( x_i \equiv i \pmod{2} \) for all \( i = 1, 2, \ldots, 2k \). (I.e. want \( x_1, x_3, \) etc. to be odd, \( x_2, x_4, \) etc. to be even.)
Solution: The $2k$-tuples $(x_1, x_2, \ldots, x_{2k})$ belong to the set
\[
\{1, 3, 5, \ldots\} \times \{2, 4, 6, \ldots\} \times \{1, 3, 5, \ldots\} \times \ldots \times \{2, 4, 6, \ldots\}
\]
so g.f. is
\[
A(x) = (x + x^3 + x^5 + \ldots)^k(x^2 + x^4 + x^6 + \ldots)^k
= [x(1 + x^2 + x^4 + \ldots)]^k[x^2(1 + x^2 + x^4 + \ldots)]^k
= x^k x^{2k}(1 + x^2 + x^4 + \ldots)^{2k}
= x^{3k}((1 - x^2)^{-1})^{2k} = x^{3k}(1 - x^2)^{-2k}
\]
So we want the coefficient of $x^n$ in this:
\[
[x^n] A(x) = [x^n] x^{3k}(1 - x^2)^{-2k}
\]
which is 0 if $n < 3k$, and if $n \geq 3k$ is:
\[
= [x^{n-3k}] (1 - x^2)^{-2k}
\]
which is 0 if $n - 3k$ is odd; if $n - 3k = 2m$ is even, write $y = x^2$ and we want:
\[
= [x^{2m}](1 - x^2)^{-2k} = [y^m](1 - y)^{-2k} = \binom{2k}{m} = \binom{2k}{\frac{n-3k}{2}}
\]
\[
= \binom{2k + \frac{n-3k}{2} - 1}{2k - 1} = \frac{n+k}{2} - 1
\]
So final answer is
\[
\begin{cases}
0 & \text{if } n < 3k \text{ or } n - 3k \text{ is odd} \\
\frac{n+k}{2k - 1} & \text{if } n \geq 3k \text{ and } n - 3k \text{ is even}
\end{cases}
\]
The condition ‘$n - 3k$ odd (even)’ is same as $n - k$ being odd (even), i.e. $n$ and $k$ have different (same) parity.