Math 2600/5600 - Linear Algebra - Fall 2015

Extra Problems and Answers/Solutions to Practice Problems for Chapter 6

Note: For some problems answers without full explanation are given, while for some problems complete solutions are given. On homework you are always expected to give complete solutions with full details.

6.1.1. See book for problem. Answers in book.

6.1.3. See book for problem. Answers in book.

6.1.9. See book for problem.

Answer: (a) Suppose $\langle x, z\rangle = 0 \ \forall\ z \in \beta$. Since β is a basis, we can write $x = \alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_n z_n$ for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $z_1, z_2, \ldots, z_n \in \beta$. Then
$\langle x, x\rangle = \langle x, \alpha_1 z_1 + \ldots + \alpha_n z_n\rangle = \alpha_1\langle x, z_1\rangle + \ldots + \alpha_n\langle x, z_n\rangle = 0$
because $\langle x, z\rangle = 0 \ \forall\ z \in \beta$. Since $\langle x, x\rangle = 0$, we must have $x = 0$.

(b) Suppose $\langle x, z\rangle = \langle y, z\rangle \ \forall\ z \in \beta$. Then $\langle x - y, z\rangle = \langle x, z\rangle - \langle y, z\rangle = 0 \ \forall\ z \in \beta$. Hence, by (a), $x - y = 0$, so $x = y$.

6.1.10. See book for problem.

Answer: Suppose x and y are orthogonal. We have

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle + \langle y, y\rangle \quad \text{(because } \langle x, y\rangle = 0) \quad = \|x\|^2 + \|y\|^2.\end{aligned}$$

In $\mathbf{R}^2$ (or $\mathbf{R}^n$ or in fact any real inner product space) if we take a triangle with vertices a, b and c and let $x = b - a$, $y = c - b$, then the third side is $x + y = c - a$, and there is a right angle at b precisely if $\langle x, y\rangle = 0$. Then $\|x + y\|$ represents the length of the hypotenuse of the right triangle whose other sides have lengths $\|x\|$ and $\|y\|$, so the formula above gives Pythagoras's Theorem.

6.1.17. See book for problem.

Answer: Suppose $\|Tx\| = \|x\|$ for all $x \in V$. Consider $x \in N(T)$, then $Tx = 0$ so $\|x\| = \|Tx\| == \|0\| = 0$, from which $x = 0$. Thus, $N(T) = \{0\}$ and so T is one-to-one.

6.2.1. See book for problem. Answers in book.

6.2.2. See book for problem. Answers in book.

X12. (Algebraic derivation of $\text{proj}_x y$) Prove algebraically that if $x \neq 0$ then $y - \alpha x \perp x$ if and only if $\alpha = \langle y, x\rangle / \langle x, x\rangle$.

Answer: Note that $\langle x, x\rangle \neq 0$ because $x \neq 0$. Hence $y - \alpha x \perp x \Leftrightarrow 0 = \langle y - \alpha x, x\rangle = \langle y, x\rangle - \alpha\langle x, x\rangle \Leftrightarrow \alpha = \langle y, x\rangle / \langle x, x\rangle$.

6.2.13. See book for problem.

Answer: (a) Suppose $S_0 \subseteq S$. To prove that $S^\perp \subseteq S_0^\perp$, we must show that every $v \in S^\perp$ also belongs to $S_0^\perp$. Suppose $v \in S^\perp$, then $\langle v, s\rangle = 0 \ \forall\ s \in S$. If $s_0 \in S_0$, then s_0 is also in S, so $\langle v, s_0\rangle = 0$, and this holds $\forall\ s_0 \in S_0$. Hence $v \in S_0^\perp$.

(b) To prove that $S \subseteq (S^\perp)^\perp$, we must show that every $s \in S$ also belongs to $(S^\perp)^\perp$. Suppose $s \in S$. Then for every $v \in S^\perp$ we have $0 = \langle v, s\rangle$ (by definition of $S^\perp$) $= \langle s, v\rangle$. Since $\langle s, v\rangle = 0 \ \forall\ v \in S^\perp$, that means that $s \in (S^\perp)^\perp$.

X13. Find a basis for $\{(1, 2, 3, 4), (-1, 0, 1, 0)\}^\perp$ in $\mathbf{R}^4$ (by solving a system of linear equations).

Answer: Let $S = \{s_1 = (1,2,3,4), s_2 = (-1,0,1,0)\}$. Then $x = (x_1,x_2,x_3,x_4) \in S^\perp$ if and only if $\langle x, s_1\rangle = \langle x, s_2\rangle = 0$. This gives a linear system of equations which we solve by transforming into an augmented matrix and reducing to reduced row echelon form:

$$\begin{array}{rcrcrcrcl} \langle x,s_1\rangle &=& x_1 &+& 2x_2 &+& 3x_3 &+& 4x_4 = 0 \\ \langle x,s_2\rangle &=& x_1 && &-& x_3 && = 0 \end{array} \rightarrow \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{rcrcrcl} x_1 && &-& x_3 && = 0 \\ && x_2 &+& 2x_3 &+& 2x_4 = 0 \end{array} \rightarrow \begin{array}{rcl} x_1 &=& x_3 \\ x_2 &=& -2x_3 - 2x_4 \end{array}$$

Hence,

$$\begin{aligned} S^\perp &= \{(x_3, -2x_3 - 2x_4, x_3, x_4) \mid x_3, x_4 \in \mathbf{R}\} \\ &= \{x_3(1,-2,1,0) + x_4(0,-2,0,1) \mid x_3, x_4 \in \mathbf{R}\} = \text{span}\,\{(1,-2,1,0),(0,-2,0,1)\} \end{aligned}$$

where $\{(1,-2,1,0),(0,-2,0,1)\}$ is a basis for $S^\perp$.

X14. Prove that for any subset S of V, $S^\perp = (\text{span}\, S)^\perp$.

Answer: We need to show that $S^\perp \subseteq \text{span}\, S^\perp$ and $(\text{span}\, S)^\perp \subseteq S^\perp$.

Suppose $v \in S^\perp$, so that $\langle v, s\rangle = 0\ \forall\ s \in S$. Suppose $y \in \text{span}\, S$, then $y = \alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_k s_k$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $s_1, s_2, \ldots, s_k \in S$. Then

$\langle v, y\rangle = \langle v, \alpha_1 s_1 + \ldots + \alpha_k s_k\rangle = \alpha_1\langle v, s_1\rangle + \ldots + \alpha_k\langle v, s_k\rangle = 0$

because $\langle v, s\rangle = 0\ \forall\ s \in S$. Thus, $\langle v, y\rangle = 0\ \forall\ y \in \text{span}\, S$. Hence, $v \in (\text{span}\, S)^\perp$.

Conversely, suppose $v \in (\text{span}\, S)^\perp$, so that $\langle v, y\rangle = 0\ \forall\ y \in \text{span}\, S$. But every $s \in S$ also belongs to span S, so $\langle v, s\rangle = 0\ \forall\ s \in S$. Hence $v \in S^\perp$.