Math 2600/5600 - Linear Algebra - Fall 2015

Extra Problems and Answers/Solutions to Practice Problems for Chapter 5

Note: For some problems answers without full explanation are given, while for some problems complete solutions are given. On homework you are always expected to give complete solutions with full details.

5.1.1. See book for problem. Answers in book.

5.1.2. See book for problem. Answers in book.

5.1.3. See book for problem. Answers in book.

5.1.6. See book for problem.

Answer: Since taking coordinates is an isomorphism, $Tv = \lambda v \Leftrightarrow [Tv]_B = [\lambda v]_B$. But this is equivalent to $[T]_B[v]_B = \lambda[v]_B$.

5.1.8. See book for problem.

Answer: (a) 0 not an eigenvalue $\Leftrightarrow$ null $T = 0 \Leftrightarrow T$ is invertible, because its domain and codomain have the same dimension.
(b) If T is invertible then by (a) 0 is not an eigenvalue (and it is also not an eigenvalue of T^{-1}). So $Tv = \lambda v \Leftrightarrow v = T^{-1}(\lambda v) = \lambda T^{-1}v \Leftrightarrow T^{-1}v = \lambda^{-1}v$.

5.1.12. See book for problem.

Answer: (a) If B is similar to A, $B = Q^{-1}AQ$ for some invertible Q. Then $\chi_B(t) = \det(tI - B) = \det(tI - Q^{-1}AQ) = \det(Q^{-1}(tI)Q - Q^{-1}AQ) = \det Q^{-1}(tI - A)Q = \det Q^{-1}\det(tI - A)\det Q = \det(tI - A) = \chi_A(t)$ because $\det Q^{-1}\det Q = 1$.
(b) Different bases give similar matrices, but from (a) similar matrices have the same characteristic polynomial.

5.1.14. See book for problem.

Answer: $\chi_{A^{\mathrm{T}}}(t) = \det(tI - A^{\mathrm{T}}) = \det(tI^{\mathrm{T}} - A^{\mathrm{T}}) = \det(tI - A)^{\mathrm{T}} = \det(tI - A) = \chi_A(t)$, using $\det B^{\mathrm{T}} = \det B$.

5.1.15. See book for problem.

Answer: (a) A formal proof uses induction on m. The result is true for $m = 1$ because then the hypothesis and conclusion are identical. Suppose the result is true for $m = k-1 \geq 1$. Then $T^k x = T(T^{k-1}x) = T(\lambda^{k-1}x)$ (by the induction hypothesis) $= \lambda^{k-1}Tx = \lambda^{k-1}\lambda x = \lambda^k x$ and so the result also holds for $m = k$. Hence the result holds for all m by induction.

5.1.9. See book for problem.

Answer: If A is upper triangular, then so is $tI - A$, with diagonal entries $t - A_{11}$, $t - A_{22}$, ..., $t - A_{nn}$, so the characteristic polynomial is $\chi_A(t) = (t - A_{11})(t - A_{22})\dots(t - A_{nn})$, whose roots are exactly A_{11}, A_{22}, ..., A_{nn}, the diagonal entries of A.

5.1.16. See book for problem.

Answer: (a) Use the known fact tr AB = tr BA. If A and B are similar then $B = Q^{-1}AQ$, so tr B = tr $Q^{-1}AQ$ = tr $Q^{-1}(AQ)$ = tr $(AQ)Q^{-1}$ = tr AQQ^{-1} = tr AI = tr A.
(b) tr T = tr $[T]_B$ for any ordered basis B; doesn't matter which, because all such matrices $[T]_B$ are similar and so have the same trace by (a).

5.2.1. See book for problem. Answers in book.

5.2.2. See book for problem. Answers in book.

5.2.3. See book for problem. Answers in book.

5.2.8. See book for problem.

Answer: We know that the geometric multiplicities of λ_1 and λ_2 cannot add up to more than n, and the geometric multiplicity of λ_1 is at least 1, so they must add up to n, which means A is diagonalizable.

5.2.11. See book for problem.

Answer: Suppose A is similar to upper triangular U. Since U has the same characteristic polynomial as A, it has the same eigenvalues and (algebraic) multiplicities. But by 5.1.9, the eigenvalues with multiplicities are just the diagonal elements of U, so U has m_1 copies of λ_1, m_2 copies of λ_2, etc. on its diagonal.
(a) By 5.1.16(a) tr $A =$ tr $U = \sum_{i=1}^{k} m_i \lambda_i$.
(b) Since similar matrices have the same determinant, $\det A = \det U = (\lambda_1)^{m_1}(\lambda_2)^{m_2} \ldots (\lambda_k)^{m_k}$.

5.4.1. See book for problem. Answers in book.

5.4.2. See book for problem. Answers in book.

5.4.6. See book for problem. Answers in book.

5.4.11. See book for problem.

Answer: (a) Let $W = \text{span}\,\{v, Tv, T^2v, \ldots\}$. If $w \in W$ then w is a linear combination of finitely many $T^i v$, so for some finite p, $w = a_0 v + a_1 Tv + a_2 T^2 v + \ldots + a_p T^p v$. Then $Tw = a_0 Tv + a_1 T^2 v + \ldots + a_p T^{p+1} v$ which also belongs to W. So $T(W) \subseteq W$, as required.
(b) Suppose X is a T-invariant subspace containing v. Since $v \in X$ and X is T-invariant, $Tv \in X$. Since $Tv \in X$ and X is T-invariant, $T^2 v \in X$. Since $T^2 v \in X$ and X is T-invariant, $T^3 v \in X$. And so on (formally we could use a proof by induction), so X contains $\{v, Tv, T^2v, \ldots\}$. Since X is closed under taking linear combinations, X therefore contains span $\{v, Tv, T^2v, \ldots\} = W$.

5.4.17. See book for problem.

Answer: Since A satisfies its characteristic polynomial $\chi_A(t)$, which is monic of degree n, we know that A^n is a linear combination of $I, A, A^2, \ldots, A^{n-1}$. Then for any $k \geq 0$, we know that $A^{n+k} = A^k A^n$ is a linear combination of $A^k, A^{k+1}, \ldots, A^{k+n-1}$. In other words, any power A^j with $j \geq n$ can be reduced to a linear combination of smaller powers of A, and we can repeat this until we have reduced it to a linear combination of $I, A, A^2, \ldots, A^{n-1}$. Hence any linear combination of nonnegative powers of A can be reduced to a linear combination of $I, A, A^2, \ldots, A^{n-1}$. In other words, span $\{I, A, A^2, \ldots\}$ = span $\{I, A, A^2, \ldots, A^{n-1}\}$ which has dimension at most n because we have a spanning set of size n.