Math 2600/5600 - Linear Algebra - Fall 2015

Extra Problems and Answers/Solutions to Practice Problems for Chapter 3

Note: For some problems answers without full explanation are given, while for some problems complete solutions are given. On homework you are always expected to give complete solutions with full details.

3.1.1. See book for problem. Answers in book.

3.1.3. See book for problem. Incomplete answers in book.

3.1.4. See book for problem.

Answer: We provide a $1-1$ correspondence between elementary row operations and elementary column operations to create a matrix E from I, based on how E differs from an identity matrix.

(1) Exchanging R_i and R_j in I is equivalent to exchanging C_i and C_j: these both make $E_{ii} = E_{jj} = 0$ and $E_{ij} = E_{ji} = 1$.

(2) Multiplying R_i by α in I is equivalent to multiplying C_i by α: these both change E_{ii} from 1 to α.

(3) Adding βR_i to R_j creates an off-diagonal entry $E_{ji} = \beta$. The same effect is obtained by adding βC_j to C_i.

3.1.5. See book for problem.

Answer: Suppose E is an elementary matrix. If it is type (1) or (2), then $E^{\mathrm{T}} = E$ is also elementary. If it is type (3), obtained by adding βR_i to R_j, then it differs from I in the off-diagonal entry $E_{ji} = \beta$. Thus, E^{T} differs from I in the off-diagonal entry $(E^{\mathrm{T}})_{ij} = \beta$, so E^{T} can be obtained from I by adding βR_j to R_i, so E^{T} is also elementary.

(Similar arguments apply if we include matrices formed from I using elementary column operations as elementary matrices. Or we can just use problem 3.1.4 to say that we only need to consider elementary row operations.)

3.2.1. See book for problem. Answers in book.

3.2.2. See book for problem. Answers in book.

3.2.8. See book for problem.

Answer: Let $P = cI_m$, which is invertible with inverse $(1/c)I_m$, since $c \neq 0$. Hence rank $cA =$ rank $cI_mA =$ rank $PA =$ dim colsp $PA =$ dim colsp $A =$ rank A. We used the fact that colsp $PA \cong$ colsp A, so that dim colsp $PA =$ dim colsp A. (There are other ways to do this.)

3.2.14. See book for problem.

Answer: (a) By $X + Y$ we mean the set $\{x + y \mid x \in X, y \in Y\}$.

We want to show that $R(T + U) \subseteq R(T) + R(U)$. Let $w \in R(T + U)$, then $w = (T + U)v = Tv + Uv$ for some $v \in V$. But $Tv \in R(T)$ and $Uv \in R(U)$, so $w = Tv + Uv \in R(T) + R(U)$. Hence, $R(T + U) \subseteq R(T) + R(U)$.

(b) Note that for any sets X, Y we have span X + span $Y \subseteq$ span $(X \cup Y)$: a linear combination of things in X plus a linear combination of things in Y is just a linear combination of things in X and Y together.

If W is finite-dimensional then so are its subspaces $R(T), R(U), R(T + U)$, and so $T, U, U + T$ all have finite rank. Let X be a basis for $R(T)$ and Y be a basis for $R(U)$. Then $R(T + U) \subseteq R(U) + R(T) =$ span X + span $Y \subseteq$ span $(X \cup Y)$ and so rank $(T + U) =$ dim $R(T + U) \leq$ dim span $(X \cup Y) \leq |X \cup Y| \leq |X| + |Y| =$ rank T + rank U.

(c) rank $(A + B) =$ rank $L_{A+B} =$ rank $(L_A + L_B) \leq$ rank L_A + rank $L_B =$ rank A + rank B.

3.2.17. See book for problem.

Answer: We see that rank $A = \dim \operatorname{colsp} A$ is the maximum number of linearly independent columns of A. We have

$$BC = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} [\, c_1 \quad c_2 \quad c_3 \,] = \begin{bmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ b_3c_1 & b_3c_2 & b_3c_3 \end{bmatrix} = [\, c_1B \quad c_2B \quad c_3B \,]$$

Since the columns of BC are all multiples of B, at most one is linearly independent (maybe none, if they are all 0). So rank $BC \leq 1$.

Conversely, suppose A is 3×3 and rank $A = 1$. That means there is at most one linearly independent column. So either A is 0, or all columns of A are multiples of one nonzero column. In either case, there is a column vector B so that the columns of A can be regarded as c_1B, c_2B, c_3B for some scalars c_1, c_2, c_3 (take $c_i = 1$ if the ith column is the one that the other columns are multiples of). But from above this means $A = BC$ where $C = [c_1 \quad c_2 \quad c_3]$.

3.2.5. See book for problem. Answers in book.

3.2.6. See book for problem. Answers in book.

3.2.21. See book for problem.

Answer: Suppose A is $m \times n$ with rank m. Then $\dim \operatorname{colsp} A = m$, so the columns of A span $\mathbf{R}^m$. So we can set up an $n \times m$ matrix B whose j-th column tells how to combine the columns of A to give the j-th standard basis vector $e_j \in \mathbf{R}^m$. Then $AB = [e_1 \mid e_2 \mid \ \ldots \ \mid e_m] = I_m$.

3.3.1. See book for problem. Answers in book.

3.3.2. See book for problem. Answers in book.

3.3.3. See book for problem. Answers in book. (Note that answers can also be written in other ways.)

3.3.6. See book for problem. Answer in book. (Note that answer can also be written in other ways.)

3.3.10. See book for problem.

Answer: This is true. If rank $A = m$ then $m = \text{rank } A \leq \text{rank } [A \mid b] \leq m$, so rank $[A \mid b] = m = \text{rank } A$ and the system is consistent.

Another way to say this is that if rank $A = m$, then the columns of A form a spanning set for F^m, so there is certainly some linear combination of columns of A giving b, or in other words, there is some x so that $Ax = b$.

3.4.1. See book for problem. Answers in the book.

3.4.2. See book for problem. Answers in the book.

3.4.4. See book for problem. Answers in the book.

3.4.5. See book for problem. Answer in the book.

3.4.7. See book for problem. Answer in the book.

3.4.9. See book for problem.

Answer: Represent S in terms of the standard ordered basis of $\mathbf{R}^{2\times 2}$, $B = (E_{11}, E_{12}, E_{21}, E_{22})$ where E_{ij} is 0 except for a 1 in row i, column j. Put the coordinate vectors as the columns of a matrix A and reduce A to reduced row echelon form R:

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix} \quad \to \quad R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have an isomorphism from W to colsp A by taking coordinates, and an isomorphism from colsp A to colsp R from the elementary row operations that reduced A to R, so their composition is an isomorphism from W to colsp R, in which the ith element of S maps to the ith column of R.

The columns corresponding to the leading entries, namely columns $1, 2, 4$ form a basis for colsp R. Hence the first, second and fourth elements of S form a basis for W, namely $\left\{ \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \right\}$.

3.4.13. See book for problem. Answer to (b) in the book.

Answer: (a) Substitute vectors into equations to show they are in V; they are not multiples of each other so they are linearly independent.