Math 2600/5600 - Linear Algebra - Fall 2015

Extra Problems and Answers/Solutions to Practice Problems for Chapter 1

Note: For some problems answers without full explanation are given, while for some problems complete solutions are given. On homework you are always expected to give complete solutions with full details.

1.2.1. See book for problem. Answers in book.

1.2.8. See book for problem.

Solution: $(a+b)(x+y) = a(x+y) + b(x+y)$ by (V9)
$= (ax+ay) + (bx+by)$ by (V10), twice
$= ax+ay+bx+by$ by a consequence of (V3).

1.2.11. See book for problem.

Solution: To show that $V = \{0\}$ is a vector space we check all the axioms.
(V1) $u+v$ exists in V for all $u, v \in V$ (closed).
The only possible values of u and v are $u = v = 0$ and $u + v = 0 + 0 = 0 \in V$.
(V2) $u+v = v+u$ for all $u, v \in V$ (commutative).
The only possible values of u and v are $u = v = 0$ and $u + v = 0 + 0 = 0 = 0 + 0 = v + u$.
(V3) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$ (associative).
The only possible values of u, v, w are $u = v = w = 0$ and $u + (v + w + = 0 + (0 + 0) = (0 + 0) + 0 = (u + v) + w$.
(V4) There exists $0 \in V$ so that $v + 0 = 0 + v = v$ for all $v \in V$ (identity).
Certainly $0 \in V$. The only possible value of v is $v = 0$ and $v + 0 = 0 + 0 = 0$ and $0 + v = 0 + 0 = 0$, as required.
(V5) For every $v \in V$ there exists $-v \in V$ such that $v + (-v) = (-v) + v = 0$ (inverses).
The only possible value of v is $v = 0$ and if we take $-v = 0$ then $v + (-v) = 0 + 0 = 0$ and $(-v) + v = 0 + 0 = 0$, as required.
(V6) αv exists in V for all $\alpha \in F, v \in V$.
The only possible value of v is $v = 0$ and for any α, $\alpha 0 = 0$ exists in V.
(V7) $1v = v$ for all $v \in V$.
The only possible value of v is $v = 0$ and $1v = 1(0) = 0 = v$.
(V8) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F, v \in V$.
The only possible value of v is $v = 0$ and for any α and β, $\alpha(\beta v) = \alpha(\beta 0) = \alpha 0 = 0 = (\alpha\beta)0 = (\alpha\beta)v$.
(V9) $(\alpha + \beta)v = \alpha v + \beta v$ for all $\alpha, \beta \in F, v \in V$.
The only possible value of v is $v = 0$ and for any α and β, $(\alpha + \beta)v = (\alpha + \beta)0 = 0 = 0 + 0 = \alpha 0 + \beta 0 = \alpha v + \beta v$.
(V10) $\alpha(u + v) = \alpha u + \alpha v$ for all $\alpha \in F, u, v \in V$.
The only possible values of u and v are $u = v = 0$, and for any al we have $\alpha(u + v) = \alpha(0 + 0) = \alpha 0 = 0 = 0 + 0 = \alpha 0 + \alpha 0 = \alpha u + \alpha v$.

[This is very tedious. It is much easier to prove this using the Subspace Theorem from 1.3, taking the set containing the 0 vector inside something larger we already know is a vector space.]

1.2.13. See book for problem. Answer in book.

1.2.16. See book for problem.

Answer: Yes, $\mathbf{R}^{m\times n} = M_{m\times n}(\mathbf{R})$ is a vector space over $\mathbf{Q}$. The addition axioms (V1)–(V5) do not depend on the field, so they still hold. All the axioms involving scalar multiplication, (V6)–(V10), still hold because any scalar in $\mathbf{Q}$ is also in $\mathbf{R}$.

As a general principle, if V is a vector space over F, and K is a subfield of F, then V is also a vector space over K.

1.2.18. See book for problem.

Answer: No; for one thing, addition is not commutative.

X1. Is the empty set a (real) vector space?

Solution: No. Axiom (V4) (book's (VS3)) says that a vector space contains an additive identity, so it must have at least one element.

1.3.1. See book for problem. Answers in book.

1.3.5. See book for problem.

Solution: A matrix M is symmetric if $M_{ij} = M_{ji}$. Suppose A is a square matrix, say $n \times n$. Then A^{T} is also $n \times n$, so $B = A + A^{\mathrm{T}}$ exists. Take i, j with $1 \leq i, j \leq n$. We have $B_{ij} = A_{ij} + (A^{\mathrm{T}})_{ij} = A_{ij} + A_{ji}$. We have $(B^{\mathrm{T}})_{ij} = B_{ji} = A_{ji} + A_{ij}$ (swapping the roles of i and j in the previous equation). We see that $B_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = (B^{\mathrm{T}})_{ij}$ for every i and j, so $B = B^{\mathrm{T}}$, and B is symmetric.

1.3.8. See book for problem.

Answer: Answers to (a), (c), (e) are in book. (f) No. (Contains 0 and closed under scalar multiplication, but not closed under addition. Geometrically this is a (two-sided) elliptical cone.)

1.3.11. See book for problem. Answer in book.

1.3.17. See book for problem.

Solution: This problem asks us to show that W is a subspace of V if and only if three conditions hold: (a) $W \neq \emptyset$, (b) $ax \in W$ whenever $a \in F$ and $x \in W$, and (c) $x + y \in W$ whenever $x, y \in W$. Conditions (b) and (c) are just (SS3) (closure under scalar multiplication) and (SS2) (closure under addition) of the Subspace Theorem, so what we are being asked to show is that (SS1), $0 \in W$, can be replaced by (a) in the Subspace Theorem.

Suppose conditions (a)–(c) hold. Then (SS2)=(c) and (SS3)=(b) hold. SInce $W \neq \emptyset$ there is some $w \in W$. Then by (b), $0w = 0 \in W$, so (SS1) holds. Since (SS1)–(SS3) hold, W is a subspace of V by the Subspace Theorem.

Now suppose that W is a subspace of V. By the Subspace Theorem, (b)=(SS3) and (c)=(SS2) hold. Also, (SS1) holds, so $0 \in W$, which means that $W \neq \emptyset$, so (a) holds. Hence (a)–(c) hold.

1.3.20. See book for problem.

Solution: We prove by induction on n that if $a_1, a_2, \ldots, a_n$ are scalars, and $w_1, w_2, \ldots, w_n \in W$ (a subspace of V), then $a_1w_1 + a_2w_2 + \ldots + a_nw_n \in W$.
Basis: If $n = 1$ then $a_1w_1 \in W$ since W is closed under scalar multiplication by (SS3).
Induction step: Suppose that $n = k \geq 2$, and assume the induction hypothesis, that the result holds when $n = k - 1$. We have

$$a_1w_1 + a_2w_2 + \ldots + a_kw_k = (a_1w_1 + a_2w_2 + \ldots + a_{k-1}w_{k-1}) + a_kw_k.$$

Now $a_1w_1 + a_2w_2 + \ldots + a_{k-1}w_{k-1} \in W$ by the induction hypothesis, and $a_kw_k \in W$ by (SS3) (closure under scalar multiplication). Since (SS2) (closure under addition) holds for W, we therefore conclude that $a_1w_1 + a_2w_2 + \ldots + a_kw_k = (a_1w_1 + a_2w_2 + \ldots + a_{k-1}w_{k-1}) + a_kw_k \in W$, as required. So the result holds when $n = k$.
Conclusion: By the Principle of Mathematical Induction, the result holds for all integers $n \geq 1$.

[Note: We can also say that the result holds for $n = 0$, since a linear combination of no vectors is equal to the zero vector, which belongs to W by (SS1).]

X2. Use the Subfield Theorem to show that $\mathbf{Q}(\sqrt{2})$ is a subfield of $\mathbf{R}$. (Hint: you should know how to rationalize a denominator.)

Solution: $\mathbf{Q}(\sqrt{2})$ is defined as $\{a + b\sqrt{2} \mid a, b \in \mathbf{Q}\}$. We check the conditions of the Subfield Theorem in the notes.
(SF1) $0 = 0 + 0\sqrt{2} \in \mathbf{Q}(\sqrt{2})$.
(SF2) $1 = 1 + 0\sqrt{2} \in \mathbf{Q}(\sqrt{2})$ and $-1 = -1 + 0\sqrt{2} \in \mathbf{Q}(\sqrt{2})$.
(SF3) If $x_1 = a_1 + b_1\sqrt{2}$ and $x_2 = a_2 + b_2\sqrt{2}$ are in $\mathbf{Q}(\sqrt{2})$, where $a_1, b_1, a_2, b_2 \in \mathbf{Q}$ then we have $x_1 + x_2 = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$ which belongs to $\mathbf{Q}(\sqrt{2})$ because $a_1 + a_2, b_1 + b_2 \in \mathbf{Q}$. Thus, $\mathbf{Q}(\sqrt{2})$ is closed under addition.
(SF4) If $x_1 = a_1 + b_1\sqrt{2}$ and $x_2 = a_2 + b_2\sqrt{2}$ are in $\mathbf{Q}(\sqrt{2})$, where $a_1, b_1, a_2, b_2 \in \mathbf{Q}$ then we have $x_1x_2 = (a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = a_1a_2 + (a_1b_2 + b_1a_2)\sqrt{2} + b_1b_2\sqrt{2}^2) = (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}$ which belongs to $\mathbf{Q}(\sqrt{2})$ because $a_1a_2 + 2b_1b_2, a_1b_2 + b_1a_2 \in \mathbf{Q}$. Thus, $\mathbf{Q}(\sqrt{2})$ is closed under multiplication.
(SF5) If $x = a + b\sqrt{2} \in \mathbf{Q}(\sqrt{2}) - \{0\}$ then we know $x^{-1} = 1/x$ exists in $\mathbf{R}$; we just need to prove that it belongs to $\mathbf{Q}(\sqrt{2})$. We know that $a - b\sqrt{2} \neq 0$ (if $a - b\sqrt{2} = 0$ then we would have $a = b\sqrt{2}$, which is impossible for rational numbers a, b unless both are zero). So

$$x^{-1} = \frac{1}{x} = \frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} + \frac{-b}{a^2 - 2b^2}\sqrt{2}$$

which belongs to $\mathbf{Q}(\sqrt{2})$ because $\frac{a}{a^2 - 2b^2}, \frac{-b}{a^2 - 2b^2} \in \mathbf{Q}$. Thus, $\mathbf{Q}(\sqrt{2})$ is closed under taking reciprocals.

Since (SF1)–(SF5) hold, $\mathbf{Q}(\sqrt{2})$ is a subfield of $\mathbf{R}$.

X3. Show that $\mathbf{Q}(i)$ is a subfield of $\mathbf{C}$.

Answer: $\mathbf{Q}(i)$ is defined as $\{a + bi \mid a, b \in \mathbf{Q}\}$, where $i^2 = -1$. As in X2, we check the conditions of the Subfield Theorem. Conditions (SF1)–(SF3) are very similar to X2. For (SF4) we get $x_1x_2 = (a_1 + b_1i)(a_2 + b_2i) = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i \in \mathbf{Q}(i)$. For (SF5) we get $x^{-1} = 1/x = 1/(a + bi) = (a - bi)/(a^2 + b^2) = (a/(a^2 + b^2)) + (-b/(a^2 + b^2))i \in \mathbf{Q}(i)$.

X4. (a) Construct the addition and multiplication tables for $\mathbf{Z}_7$.
(b) Using these tables, write down a two-column table with α and $-\alpha$ for each $\alpha \in \mathbf{Z}_7$, and another two-column table with α and α^{-1} for $\alpha \in \mathbf{Z}_7 - \{0\}$.
(c) What are $4 - 6$ and $4 \div 3$ in $\mathbf{Z}_7$?

Solution: (a)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

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×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

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(b) To save room I give two-row tables rather than two-column tables. For $-\alpha$, find entry β with $\alpha + \beta = 0$. For α^{-1}, find entry β with $\alpha\beta = 1$.

α	0	1	2	3	4	5	6
$-\alpha$	0	6	5	4	3	2	1

,

α	1	2	3	4	5	6
α^{-1}	1	4	5	2	3	6

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(c) $4 - 6 = 4 + (-6) = 4 + 1 = 5$. $4 \div 3 = 4 \times (3^{-1}) = 4 \times 5 = 6$.

X5. We can define $\mathbf{Z}_n = \{0, 1, 2, \ldots, n - 1\}$ with addition and multiplication modulo n even if n is not a prime. But when n is not a prime, this is not a field.
(a) Construct the multiplication table for $\mathbf{Z}_6$, and use it to explain why $\mathbf{Z}_6$ is not a field.
(b) Generalize your answer to (a) to explain why $\mathbf{Z}_n$ is not a field when $n \geq 4$ is not prime.

Solution: (a)

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

We see that there is no 1 in several of the rows for nonzero elements: in particular, for rows 2, 3 and 4. Therefore, 2, 3 and 4 have no multiplicative inverse in $\mathbf{Z}_6$.

(b) In general, suppose $n \geq 4$ is not prime, then $n = ab$ where $a, b \geq 2$ are integers. We claim that a (or similarly b) does not have a multiplicative inverse in $\mathbf{Z}_n$. Informally, the a row in the multiplication table for $\mathbf{Z}_n$ contains only multiples of a, and so cannot contain 1. We can make this more formal as follows.

Consider any $a \times s$ in $\mathbf{Z}_n$. This equals the remainder r after dividing $a \times s$ (in $\mathbf{R}$) by n, so that (in $\mathbf{R}$) $as = qn + r$ for some integer q. Then $r = as - qn = as - qab = a(s - qb)$, which is divisible by a. Hence r cannot equal 1, and so there is no s with $a \times s = 1$ in $\mathbf{Z}_n$.

1.4.1. See book for problem. Answers in book.

1.4.3. See book for problem. Answers in book.

1.4.4. See book for problem. Answers in book.

1.4.5. See book for problem. Answers in book.

1.4.10. See book for problem.

Solution: An $n \times n$ matrix is symmetric provided $A_{ij} = A_{ji}$ for every i, j with $1 \leq i, j \leq n$. We only need to check this when $i < j$: when $i = j$ it is always true, and for $i > j$ we can swap i and j. So a 2×2 matrix A is symmetric precisely when $A_{12} = A_{21}$.

Assuming our scalars are from a general field F, we have

$$\begin{aligned}
\text{span}\,\{M_1, M_2, M_3\} &= \text{span}\,\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\} \\
&= \{\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in F\} \\
&= \{\begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{bmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in F\} = S \text{ (say)}.
\end{aligned}$$

Certainly every matrix A in S is symmetric because $A_{12} = \alpha_3 = A_{21}$. On the other hand, given any symmetric matrix A, we see that $A \in S$ by taking $\alpha_1 = A_{11}$, $\alpha_2 = A_{22}$, and $\alpha_3 = A_{12} = A_{21}$. Therefore, S is precisely the set of symmetric 2×2 matrices (with entries from F).

1.4.11. See book for problem.

Answer: The linear combinations of a single vector x are just the scalar multiples of x. Geometrically in $\mathbf{R}^3$, span $\{x\}$ is just a line through the origin in the direction of the vector x, unless $x = 0$, when it is just the subspace $\{0\}$.

1.4.13. See book for problem.

Solution: Because $S_1 \subseteq S_2$, every linear combination of elements of S_1 is also a linear combination of elements of S_2. In other words, span $S_1 \subseteq$ span S_2.

If $S_1 \subseteq S_2 \subseteq V$ then applying the above twice, we have span $S_1 \subseteq$ span $S_2 \subseteq$ span V. But span $V = V$, so if span $S_1 = V$ we have

$$V = \text{span}\, S_1 \subseteq \text{span}\, S_2 \subseteq \text{span}\, V = V.$$

Since $V \subseteq$ span S_2 and span $S_2 \subseteq V$, span $S_2 = V$, as required.

1.5.1. See book for problem. Answers in book.

1.5.2. See book for problem. Answers in book.

1.5.8. See book for problem.

Solution: From $\alpha_1(1,1,0)+\alpha_2(1,0,1)+\alpha_3(0,1,1)=0$ we get a linear system

$$\begin{array}{ccccccl} \alpha_1 & + & \alpha_2 & & & = & 0 \quad (1) \\ \alpha_1 & & & + & \alpha_3 & = & 0 \quad (2)\,. \\ & & \alpha_2 & + & \alpha_3 & = & 0 \quad (3) \end{array}$$

(a) In $\mathbf{R}$, (1)+(2)−2(3) gives $2\alpha_1 = 0$, so $\alpha_1 = 0$, and then substituting in (1) and (2) gives $\alpha_2 = \alpha_3 = 0$, so the vectors are linearly independent.

(b) However, if F has characteristic 2 then $1+1=0$ in F, so we see that $\alpha_1 = \alpha_2 = \alpha_3 = 1$ is a solution of the system, so there is a nontrivial linear combination that is 0, and the vectors are linearly dependent.

1.5.9. See book for problem.

Answer: You can prove this directly, or use Lemma LI1 from class (problem 1.5.14 from book).

1.5.18. See book for problem.

Answer: Consider a nontrivial linear combination of polynomials from S. Look at the polynomial with nonzero coefficient and subject to that, highest degree, say n. Consider the coefficient of x^n.

1.5.20. See book for problem.

Answer: Suppose that $\alpha_1 f + \alpha_2 g = 0$; that means that $\alpha_1 f(t) + \alpha_2 g(t) = 0$ for every real number t. Plug in two different values of t, say $t = 0$ and $t = 1$, and show that you get a system of linear equations whose only solution is $\alpha_1 = \alpha_2 = 0$.

1.6.1. See book for problem. Answers in book.

1.6.2. See book for problem. Answers in book.

1.6.3. See book for problem. Answers in book.

1.6.4. See book for problem. Answer in book.

1.6.5. See book for problem. Answer in book.

1.6.11. See book for problem.

Answer: Since V has dimension 2 (because $\{u, v\}$ is a basis), and the given sets have size 2, you just need to show that they are spanning sets.

1.6.14. See book for problem.

Answer: One basis for W_1 is $\{(0,1,0,0,0),(1,0,1,0,0),(1,0,0,1,0),(0,0,0,0,1)\}$; $\dim W_1 = 4$. One basis for W_2 is $\{(1,0,0,0,-1),(0,1,1,1,0)\}$; $\dim W_2 = 2$.

1.6.17. See book for problem. Answer in book.

1.6.26. See book for problem. Answer in book.

Answer: Think about multiples of the polynomial $(x-a)$.