Math 2600/5600 – Fall 2015 – Test 3 Name: Solutions

1 min

21 min

- Write answers in spaces provided. The backs of pages may be used for rough work.
- Marks are shown in brackets [].
- Please sign the Honor System pledge on the last page.
- You may use calculators (for basic calculations ONLY) and LA unless stated otherwise.
- No other devices (cell phones, etc.) may be used.
- On all questions you are expected to EXPLAIN YOUR WORKING and SHOW ALL CALCULATIONS unless explicitly stated otherwise.

40	1.

21 min

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1. [7] Complete the following definitions.

(a) If $T \in L(V)$ then a subspace W of V is *T-invariant* if ...

$T(W) \subseteq W$, i.e. $Tw \in W \ \forall w \in W$.

(b) A *permutation* of a set X is ...

a bijection from X to itself

(c) If $T \in L(V)$ then we say $v \in V$ is an *eigenvector* for *eigenvalue* λ if ...

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$v \neq 0$ and $Tv = \lambda v$

(d) We say a polynomial *splits* in a given field F if ...

it factors completely into linear factors

(e) The *cofactor* of an entry A_{ij} in $A \in F^{n\times n}$ is ... (note: also define any matrices you use here)

$(-1)^{i+j} \det \tilde{A}_{ij}$ where $\tilde{A}_{ij}$ is A with row i & column j deleted.

2

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5 min

2.

2. [9] (a) Compute the following **complex** determinant as efficiently as possible. Show your working and clearly indicate any results that you use.

$$|A| = \left|\begin{array}{ccc|cc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 7 & 4+i & 5 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ \hline 9 & 1 & 7 & -5i & 2 & 0 \\ 4 & 1 & 3 & 2 & 3i & 0 \\ \hline 9 & 8 & 7 & 6 & 3 & 2i \end{array}\right|$$

This is block lower triangular, so

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 7 & 4+i & 5 \\ 1 & 0 & 2 \end{vmatrix} \begin{vmatrix} -5i & 2 \\ 2 & 3i \end{vmatrix} \, |2i|$$

4

$$= [2(4+i) - (4+i)]\,[-15i^2 - 4]\; 2i$$

$$= (4+i)(15-4)(2i) = (4+i)(11)(2i)$$

$$= 22i(4+i) = 88i + 22i^2 = -22 + 88i.$$

(b) Compute the following real determinant by using elementary row operations to reduce to upper triangular form. Describe all elementary row operations and show intermediate determinants.

$$\begin{vmatrix} 4 & 6 & 2 \\ 7 & -6 & 3 \\ 9 & -2 & 8 \end{vmatrix} = 4 \begin{vmatrix} 1 & 3/2 & 1/2 \\ 7 & -6 & 3 \\ 9 & -2 & 8 \end{vmatrix} \; R_1' = R_1/4 \quad = 4 \begin{vmatrix} 1 & 3/2 & 1/2 \\ 0 & -33/2 & -1/2 \\ 0 & -31/2 & 7/2 \end{vmatrix} \; \begin{array}{l} R_2' = R_2 - 7R_1 \\ R_3' = R_3 - 9R_1 \end{array}$$

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$$= 4\left(\frac{-33}{2}\right) \begin{vmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1/33 \\ 0 & -31/2 & 7/2 \end{vmatrix} \; R_2' = -\frac{2}{33}R_2 \quad = -66 \begin{vmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1/33 \\ 0 & 0 & 131/33 \end{vmatrix} \; R_3' = R_3 + \frac{31}{2}R_2$$

$$= -\cancel{66}^{\,2}\left(\frac{131}{\cancel{33}}\right) = -2 \times 131 = -262.$$

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7 min

3. [8] Working in the field ~~$\mathbb{C}$~~ $\mathbb{R}$, the matrix A below is known to have three distinct eigenvalues λ:

$\lambda = 1$ with $E_1(A) = \text{span}\,\{(1, 0, -1, 2)\}$,
$\lambda = 2$ with $E_2(A) = \text{span}\,\{(2, 1, -2, 6)\}$, and
$\lambda = 4$ with algebraic multiplicity 2.

$$A = \begin{bmatrix} 7 & -4 & 12 & 3 \\ 6 & 2 & 6 & 0 \\ -3 & 4 & -8 & -3 \\ 18 & -12 & 36 & 10 \end{bmatrix}$$

3.

(a) Find a basis for $E_4(A)$.

$$E_4(A) = N(4I - A) = N\left(\underbrace{\begin{bmatrix} -3 & 4 & -12 & -3 \\ -6 & 2 & -6 & 0 \\ 3 & -4 & 12 & 3 \\ -18 & 12 & -36 & -6 \end{bmatrix}}_{B}\right) : \text{ system } Bv = 0, \text{ aug. matrix } [B|0]$$

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$$[B|0] = \left[\begin{array}{cccc|c} -3 & 4 & -12 & -3 & 0 \\ -6 & 2 & -6 & 0 & 0 \\ 3 & -4 & 12 & 3 & 0 \\ -18 & 12 & -36 & -6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow$ system
$$x_1 - \tfrac{1}{3}x_4 = 0 \qquad x_1 = \tfrac{1}{3}x_4$$
$$x_2 - 3x_3 - x_4 = 0 \qquad x_2 = 3x_3 + x_4$$

$$E_4(A) = \{(\tfrac{1}{3}x_4, 3x_3 + x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{C}\}$$
$$= \{x_3(0, 3, 1, 0) + x_4(\tfrac{1}{3}, 1, 0, 1) \mid x_3, x_4 \in \mathbb{C}\}$$
$$= \text{span}\,\underbrace{\{(0,3,1,0), (\tfrac{1}{3}, 1, 0, 1)\}}_{\text{basis for } E_4(A)}$$

(b) Is A diagonalizable? Why or why not? If A is diagonalizable, provide a diagonal matrix D and invertible matrix Q such that $D = Q^{-1}AQ$.

Yes, because we get a basis for $\mathbb{R}^4$ by combining bases of $E_1(A)$, $E_2(A)$ & $E_4(A)$: total of $\underbrace{1+1+2=4}_{\text{sum of geometric multiplicities}}$ vectors.

Taking ord. basis $C = (\underbrace{(1,0,-1,2)}_{c_1}, \underbrace{(2,1,-2,6)}_{c_2}, \underbrace{(0,3,1,0)}_{c_3}, \underbrace{(\tfrac{1}{3},1,0,1)}_{c_4})$

we have

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$$D = [L_A]_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad Q = [c_1|c_2|c_3|c_4] = [I]_C^{\text{std basis}}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 1 & 3 & 1 \\ -1 & -2 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{bmatrix}$$

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3 min

4.

4. [3] Suppose $A, B \in F^{n\times n}$ and $v \in F^n$. Prove that if v is an eigenvector for A for eigenvalue λ, and v is also an eigenvector of B for eigenvalue μ, then v is an eigenvector of AB. Justify each step. What is the corresponding eigenvalue of AB?

We have $v \neq 0$, $Av = \lambda v$, $Bv = \mu v$. So

$(AB)v = A(Bv) = A(\mu v)$ since $Bv = \mu v$

$= \mu(Av)$ by prop. of matrix mult

$= \mu(\lambda v)$ since $Av = \lambda v$

$= (\lambda\mu) v$

Since $v \neq 0$ & $(AB)v = (\lambda\mu)v$, v is eigenvector for eigenvalue $\lambda\mu$.

5. [4] Prove that if $A, B \in F^{n\times n}$ are similar matrices, then $\chi_A(t) = \chi_B(t)$.

Since A, B are similar there is invertible Q with $B = Q^{-1}AQ$. Now

$\chi_B(t) = \det(tI - B) = \det(tI - Q^{-1}AQ)$

$= \det(tQ^{-1}IQ - Q^{-1}AQ)$

$= \det(Q^{-1}(tI - A)Q)$

$= \det Q^{-1} \det(tI - A) \det Q$

$= \det(tI - A) \det Q^{-1} \det Q$

$= \chi_A(t) \cdot \det Q^{-1}Q = \chi_A(t) \det I$

$= \chi_A(t) \cdot 1 = \chi_A(t)$.

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5 min

5.

6. [3] Suppose $T \in L(P_2(\mathbf{R}))$ has matrix $A = [T]_B$ with respect to the standard basis $B = (1, x, x^2)$. Suppose that A has eigenvalues -1 with $E_{-1}(A) = \operatorname{span}\{(1,-1,0)\}$ and 7 with $E_7(A) = \operatorname{span}\{(0,1,4),(1,-2,3)\}$. Give an ordered basis C and a diagonal matrix D so that $D = [T]_C$.

Take C = vectors in $P_2(\mathbb{R})$ whose B coordinate vectors are the basis vectors for the eigenspaces of A:

$$(1,-1,0) \qquad (0,1,4) \qquad (1,-2,3)$$

$$C = (1-x, \quad x+4x^2, \quad 1-2x+3x^2)$$

$$\lambda=-1 \qquad \lambda=7 \qquad \lambda=7.$$

Then $D = [T]_C$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

2 1

7. [3] What term in the determinant of $A \in F^{7\times 7}$ comes from $\pi = [7513264] \in S_7$ (including its sign)? Show your working.

Need to work out sign:

$[7513264] \to [1573264] \to [1273564] \to [1237564]$
$\to [1234567]$ 4 transpositions $\therefore$ $\operatorname{sgn}\pi = +1$. (even) 2

So term is

$+A_{1,\pi(1)}A_{2,\pi(2)}A_{3,\pi(3)}\cdots A_{7,\pi(7)} = +A_{17}A_{25}A_{31}A_{43}A_{52}A_{66}A_{74}.$ 1

8. [3] Short answer questions. Working is not required but may be considered for partial credit if given.

For each of the following two situations answer Y, N or P to indicate which of the following is true.

Y. Definitely diagonalizable.
N. Definitely not diagonalizable.
P. Perhaps diagonalizable, perhaps not.

(a) $T \in L(\mathbf{C}^{17})$ has all of its eigenvalues of algebraic multiplicity 1. ___Y___

Since in $\mathbb{C}$, χ_T splits $\therefore$ 17 eigenvalues, all have geom. mult ≥ 1.

(b) $A \in \mathbf{R}^{21\times 21}$ has all of its eigenvalues of geometric multiplicity 1. ___P___

Don't know if χ_A splits.

(c) Suppose $T \in L(\mathbf{R}^4)$ has characteristic polynomial $\chi_T(t) = t^4 - 4t^3 + 2t^2 + 4t - 3 = (t-1)^2(t+1)(t-3)$. If W is a T-invariant subspace, and T_W is the restriction of T with domain and codomain W, then which of the following is **not** possible? ___C___

A. $\chi_{T_W}(t) = t^2 - 1$. ✓ C. $\chi_{T_W}(t)$ cannot be factored into linear factors. ✗ — uses some of factors of χ_T.
B. T_W has eigenvalues 1, -1 and 3. ✓ D. T_W has an eigenvalue of geometric multiplicity 2. ✓

7:57

I pledge on my honor that I have neither given nor received improper aid on this test or quiz.

Signed: ______________________