Math 2600/5600 – Fall 2015 – Test 2 Name: Solutions

- Write answers in spaces provided. The backs of pages may be used for rough work.
- Marks are shown in brackets [].
- Please sign the Honor System pledge on the last page.
- You may use calculators (for basic calculations ONLY) and LA unless stated otherwise.
- No other devices (cell phones, etc.) may be used.
- On all questions you are expected to EXPLAIN YOUR WORKING and SHOW ALL CALCULATIONS unless explicitly stated otherwise.

	40
1.	

18 min.

1. [9] Complete the following definitions.

(a) A matrix is in *reduced row echelon form* if ...

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(1) All zero rows are at the bottom;

(2) The leading (first nonzero) entry in each row is to the right of the leading entry in the previous row;

(3) Every leading entry is 1;

4 (4) The other entries in the column of a leading entry are 0.

Define matrix multiplication:

(b) If $A \in F^{m\times n}$ and $B \in F^{n\times p}$ then the product AB is ...

the matrix in $F^{m\times p}$ with $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$

$2\frac{1}{2}$ for all $i = 1,2,..,m$ and $j = 1,2,..,p$.

(c) A *vector space isomorphism* from a vector space V to a vector space W is ...

$1\frac{1}{2}$ an invertible linear transformation from V to W

(d) A system of linear equations is *consistent* if ...

1 it has at least one solution,

or solution set is nonempty.

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2. [8] Use Gauss-Jordan elimination to find the inverse, if any, of the matrix A below. Show working, including all elementary row operations used. If there is no inverse, explain how you know that.

2.

$$A = \begin{bmatrix} 0 & 3 & 8 \\ 1 & 5 & 1 \\ 3 & 11 & -7 \end{bmatrix} \qquad [A|I] = \left[\begin{array}{ccc|ccc} 0 & 3 & 8 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 & 1 & 0 \\ 3 & 11 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1' = R_2 \\ R_2' = R_1}} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 0 & 1 & 0 \\ 0 & 3 & 8 & 1 & 0 & 0 \\ 3 & 11 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3' = R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 0 & 1 & 0 \\ 0 & 3 & 8 & 1 & 0 & 0 \\ 0 & -4 & -10 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2' = R_2/3} \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 0 & 1 & 0 \\ 0 & 1 & 8/3 & 1/3 & 0 & 0 \\ 0 & -4 & -10 & 0 & -3 & 1 \end{array}\right] \xrightarrow{\substack{R_1' = R_1 - 5R_2 \\ R_3' = R_3 + 4R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & -37/3 & -5/3 & 1 & 0 \\ 0 & 1 & 8/3 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 4/3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_3' = \frac{3}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -37/3 & -5/3 & 1 & 0 \\ 0 & 1 & 8/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 2 & -\frac{9}{2} & \frac{3}{2} \end{array}\right] \xrightarrow{\substack{R_1' = R_1 + \frac{37}{3}R_3 \\ R_2' = R_2 - \frac{8}{3}R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23 & -\frac{109}{2} & \frac{37}{2} \\ 0 & 1 & 0 & -5 & 12 & -4 \\ 0 & 0 & 1 & 2 & -\frac{9}{2} & \frac{3}{2} \end{array}\right]$$

A^{-1}

So $A^{-1} = \begin{bmatrix} 23 & -\frac{109}{2} & \frac{37}{2} \\ -5 & 12 & -4 \\ 2 & -\frac{9}{2} & \frac{3}{2} \end{bmatrix}$ exists.

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3. [4] Suppose T is a linear operator on $P_2(\mathbf{R})$ whose action is given by $T(p(x)) = \frac{d}{dx}((2x+5)p(x))$. For example, $T(3x-2) = \frac{d}{dx}((2x+5)(3x-2)) = \frac{d}{dx}(6x^2+11x-10) = 12x+11$. Find $[T]_B$ where $B = (1, x, x^2)$ is the standard ordered basis of $P_2(\mathbf{R})$.

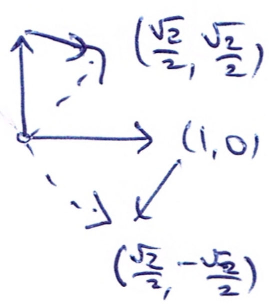
3.

$$T(1) = \frac{d}{dx}(2x+5) = 2$$
$$T(x) = \frac{d}{dx}(2x^2+5x) = 4x+5$$
$$T(x^2) = \frac{d}{dx}(2x^3+5x^2) = 6x^2+10x$$

$$[T]_B = [T]_B^B = \left[[T(1)]_B \; [T(x)]_B \; [T(x^2)]_B \right]$$

$$= \begin{bmatrix} 2 & 5 & 0 \\ 0 & 4 & 10 \\ 0 & 0 & 6 \end{bmatrix}$$

4. [5] (a) Find the standard matrix of the linear operator S on $\mathbf{R}^2$ which rotates points by 45° clockwise about the origin. (Give an **exact** answer in as simple a form as possible.)



$$[S] = [S(1,0) \mid S(0,1)] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(b) If R is the linear operator on $\mathbf{R}^2$ with standard matrix $[R] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $[R]_C$ where C is the ordered basis $(\underset{c_1}{(5,11)}, \underset{c_2}{(1,3)})$.

Let B = std ord basis.

$$[R]_C = [R]_C^C = [IRI]_C^C = \underbrace{[I]_B^C}_{Q^{-1}} \underbrace{[R]_B^B}_{A} \underbrace{[I]_C^B}_{Q}$$

$[R]_B^B = [R]$

Now $[I]_C^B = \left[[Ic_1]_B \mid [Ic_2]_B \right] = [c_1 \mid c_2] = \begin{bmatrix} 5 & 1 \\ 11 & 3 \end{bmatrix} = Q$

and $[I]_B^C = ([I]_C^B)^{-1}$ so we have

$$\underbrace{\begin{bmatrix} 5 & 1 \\ 11 & 3 \end{bmatrix}^{-1}}_{Q^{-1}} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 5 & 1 \\ 11 & 3 \end{bmatrix}}_{Q} = \begin{bmatrix} 3/4 & -1/4 \\ -11/4 & 5/4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 11 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{11}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{\text{by LA}}$$

5. [8] (a) State the Rank-Nullity Theorem.

If $T \in L(V, W)$ and V is finite-dimensional then

$$\text{rank } T + \text{null } T = \dim V$$

(b) Suppose that V and W are both finite-dimensional and $T \in L(V, W)$.

(i) State a condition in terms of rank T for T to be onto.

T onto $\Leftrightarrow$ rank $T = \dim W$

$-\frac{1}{2}$ for both if don't make clear it's necessary & sufficient.

(ii) State a condition in terms of null T for T to be one-to-one.

T 1-1 $\Leftrightarrow$ null $T = 0$

(c) With V, W, T as in (b), prove the following.

(i) If T is onto then $\dim V \geq \dim W$.

Suppose T is onto. Then

$$\begin{aligned} \dim V = \text{null } T + \text{rank } T &= \text{null } T + \dim W \quad \text{by (b)(i)} \\ &\geq \dim W \end{aligned}$$

because null $T \geq 0$

(ii) If T is one-to-one then $\dim V \leq \dim W$.

Suppose T is 1-1. Then

$$\begin{aligned} \dim V = \text{null } T + \text{rank } T &= 0 + \text{rank } T \quad \text{by (b)(ii)} \\ &= \text{rank } T \leq \dim W \end{aligned}$$

because $R(T) \subseteq W$, so that rank $T \leq \dim W$.

1:04

6. [6] Short answer questions. Working is not required but may be considered for partial credit if given.

5.

(a) Suppose S and T are linear operators on vector space V, and A and B are ordered bases of V. Then $[TS]_A^B$ is equal to what?

B

A. $[T]_B^B[S]_A^A$.

B. $[T]_B^B[S]_A^B$.

C. $[T]_A^B[S]_A^B$.

D. $[T]_B^A[S]_A^B$.

(b) If $A, B \in F^{m\times n}$, what do we know about rank $(A+B)$?

C

A. rank $(A+B) =$ rank $A +$ rank B.

B. rank $(A+B) \geq$ rank $A +$ rank B.

C. rank $(A+B) \leq$ rank $A +$ rank B.

D. rank $(A+B) \neq$ rank $A +$ rank B.

(c) Give the most accurate answer. The rank of a row echelon form matrix A is equal to:

D

A. The maximum number of linearly independent rows. ✓ always

B. The number of nonzero rows. ✓ since row ech form

C. The number of nonzero columns.

D. Both A and B.

E. All of A, B and C.

(d) If $A \in F^{m\times n}$, and $P \in F^{m\times m}$ is invertible, then which of the following is most accurate?

A

A. rowsp $PA =$ rowsp A and colsp $PA \cong$ colsp A.

B. rowsp $PA \cong$ rowsp A and colsp $PA \cong$ colsp A.

C. rowsp $PA =$ rowsp A and colsp $PA =$ colsp A.

D. rowsp $PA \cong$ rowsp A and colsp $PA =$ colsp A.

(e) We can interpret a matrix product AB in which of the following ways?

D

A. Each row of A says how to combine the rows of B to get a column of AB.

B. Each row of A says how to combine the columns of B to get a column of AB.

C. Each column of B says how to combine the rows of A to get a row of AB.

D. Each column of B says how to combine the columns of A to get a column of AB.

(f) Suppose $T \in L(V, W)$, $U \in L(W, X)$ and dim $V =$ dim $W =$ dim X. Suppose UT is one-to-one. Which of the following is **NOT** necessarily true?

A

A. $R(UT) = W$. $\underline{R(UT) = X!}$

B. U is onto.

C. $N(T) = \{0\}$.

D. UT is an isomorphism.

1:05 UT 1-1 $\Rightarrow$ UT invertible $\Rightarrow$ U, T both invertible (A2.3) so B, C, D all true.

I pledge on my honor that I have neither given nor received improper aid on this test or quiz.

Signed: