Math 2600/5600 – Fall 2015 – Test 1 Name: Solutions

2 min

1.

- Write answers in spaces provided. The backs of pages may be used for rough work.
- Marks are shown in brackets [].
- Please sign the Honor System pledge on the last page.
- You may use calculators (for basic calculations ONLY) unless stated otherwise.
- No other devices: computers, cell phones, etc.
- On all questions you are expected to EXPLAIN YOUR WORKING and SHOW ALL CALCULATIONS unless explicitly stated otherwise.

40

18 min.

3:13

1. [8] Complete the following definitions.

(a) A collection of vectors in a vector space is *linearly dependent* if ...

it contains finitely many vectors $v_1, v_2, \ldots, v_k$ $(k \geq 1)$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ not all zero so that

3

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0,$$

(b) For vectors spaces V and W, $L(V, W)$ (or $\mathcal{L}(V, W)$ if you prefer) is ...

the set of all linear transformations $T : V \to W$.

2

(c) The *transpose* A^{T} of a matrix A is obtained by ...

interchanging rows and columns,

1

so that $[A^T]_{ij} = A_{ji}$ for each suitable i & j.

(d) The *standard basis* for the vector space F^n is ...

$$\{ e_1 = (1, 0, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0),\ \ldots\ e_n = (0, 0, \ldots, 1) \}$$

2

where each e_i has a 1 as ith coordinate and all other coordinates equal to 0.

3:15

4 min

2.

2. [8] (a) Let $\mathbf{R}^{\mathbf{R}} = \mathcal{F}(\mathbf{R},\mathbf{R})$ be the vector space (over $\mathbf{R}$) of real-valued functions defined on the real numbers (with addition and scalar multiplication defined pointwise). Consider the subset of 2π-periodic functions:

$$P = \{f \in \mathbf{R}^{\mathbf{R}} \mid f(x+2\pi) = f(x) \text{ for every } x \in \mathbf{R}\}.$$

For example, if $f(x) = \sin x$ then $f \in P$. Prove that P is a subspace of $\mathbf{R}^{\mathbf{R}}$.

We check the conditions of the Subspace Theorem (or streamlined version)

(SS1) Let $z(x) = 0 \ \forall x \in \mathbb{R}$; z is the zero vector in $\mathbb{R}^{\mathbb{R}}$.
We have $z(x+2\pi) = 0 = z(x) \ \forall x \in \mathbb{R}$, so $z \in P$.

(SS2) Suppose $f, g \in P$. Then
$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$
$g(x+2\pi) = g(x) \quad \forall x \in \mathbb{R}$.
So
$(f+g)(x+2\pi) = f(x+2\pi) + g(x+2\pi)$
$= f(x) + g(x) = (f+g)(x) \ \forall x \in \mathbb{R}$
So $f+g \in P$.

(SS3) Suppose $f \in P$ and $\alpha \in \mathbb{R}$. Then
$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$.
So
$(\alpha f)(x+2\pi) = \alpha f(x+2\pi)$
$= \alpha f(x) = (\alpha f)(x) \ \forall x \in \mathbb{R}$
So $\alpha f \in P$

(SSP) Suppose $f, g \in P$ and $\alpha, \beta \in \mathbb{R}$. Then
$f(x+2\pi) = f(x) \ \forall x \in \mathbb{R}$
$g(x+2\pi) = g(x) \ \forall x \in \mathbb{R}$
So
$(\alpha f + \beta g)(x+2\pi)$
$= (\alpha f)(x+2\pi) + \beta g(x+2\pi)$
$= \alpha f(x+2\pi) + \beta g(x+2\pi)$
$= \alpha f(x) + \beta g(x)$
$= (\alpha f)(x) + (\beta g)(x)$
$= (\alpha f + \beta g)(x) \ \forall x \in \mathbb{R}$
So $\alpha f + \beta g \in P$.

Since (SS1)–(SS3) (or (SS1) & (SSP)) hold, P is a subspace.

(b) Prove that $Q_1 = \{(x,y) \in \mathbf{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$ (the set of vectors in the first quadrant) is not a subspace of $\mathbf{R}^2$.

$(1,1) \in Q_1$ but $-1(1,1) = (-1,-1) \notin Q_1$
So Q_1 is not closed under scalar mult'n, (SS3) fails. So Q_1 is not a subspace.

3:19

3. [4] If F is a field and $E \subseteq F$, one of the conditions for E to be a subfield of F is that E is closed under taking reciprocals of nonzero elements. Prove that $\mathbf{Q}(i) = \{a + bi \mid a, b \in \mathbf{Q}\}$, which is a subset of the field of complex numbers $\mathbf{C}$, has this property: if $\alpha \in \mathbf{Q}(i)$ with $\alpha \neq 0$, then $\alpha^{-1} \in \mathbf{Q}(i)$.

5 min.

3.

3:21

Suppose $\alpha = a+bi \in \mathbb{Q}(i)$, $a, b \in \mathbb{Q}$, $\alpha \neq 0$.

Then

$$\alpha^{-1} = \frac{1}{\alpha} = \frac{1}{a+bi}$$

Since $\alpha \neq 0$, at least one of $a, b \neq 0$, so $a-bi \neq 0$.

Hence

$$\alpha^{-1} = \frac{1}{a+bi}\,\frac{a-bi}{a-bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2-(bi)^2}$$

$$= \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{(-b)}{a^2+b^2}\,i$$

Since $a, b \in \mathbb{Q}$, $\frac{a}{a^2+b^2} \in \mathbb{Q}$ and $\frac{-b}{(a^2+b^2)} \in \mathbb{Q}$,

So $\alpha^{-1} \in \mathbb{Q}(i)$, as required.

3:24 4. [4] Determine whether or not $(-2, 0, 3)$ belongs to span $\{(1, 3, 0), (2, 4, -1)\}$ in $\mathbf{R}^3$.

We want to know if there are α_1, α_2 with

$$\alpha_1 (1,3,0) + \alpha_2 (2,4,-1) = (-2,0,3).$$

Looking at components:

$$\alpha_1 + 2\alpha_2 = -2 \quad (1)$$
$$3\alpha_1 + 4\alpha_2 = 0 \quad (2)$$
$$-\alpha_2 = 3 \quad (3)$$

From (3), $\alpha_2 = -3$.

Then from (1) we get

$$\alpha_1 - 6 = -2$$
$$\alpha_1 = -2+6 = 4$$

We verify that this satisfies (2) as well:

$$3\alpha_1 + 4\alpha_2 = 3(4) + 4(-3) = 12-12=0$$

We know (3) & (1) hold since we used them to get α_1, α_2.

So α_1, α_2 <u>do</u> exist, $(-2,0,3)$ <u>is</u> ~~contained~~ in the span.

3:26

V= W=

5. [8] Consider the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^4$ by

$$T(x_1, x_2, x_3, x_4) = (x_2 + 2x_3 + x_4, x_1 + x_2, x_1 + x_2, 3x_1 + 7x_2 + 8x_3 + 4x_4).$$

4 min.

4.

5:01 (a) By starting from a basis of $\mathbf{R}^4$, find a spanning set ~~of~~ for $R(T)$.

If $V = \text{span } S$ then $R(T) = \text{span } T(S)$: take S = std basis.

$R(T) = \text{span}\{Te_1, Te_2, Te_3, Te_4\}$

$= \text{span}\{T(1,0,0,0), T(0,1,0,0), T(0,0,1,0), T(0,0,0,1)\}$

3 $= \text{span}\underbrace{\{(0,1,1,3), (1,1,1,7), (2,0,0,8), (1,0,0,4)\}}_{\text{spanning set}}$.

(b) Reduce your spanning set from (a) to a basis of $R(T)$. Each time you discard an element, explain why you can discard it.

$R(T) = \text{span}\{(0,1,1,3), (1,1,1,7), (1,0,0,4)\}$

because $(2,0,0,8) = 2(1,0,0,4)$

$2\frac{1}{2}$

$= \text{span}\{(0,1,1,3), (1,0,0,4)\}$

because $(1,1,1,7) = (0,1,1,3) + (1,0,0,4)$

So basis is $\{\underset{v_1}{(0,1,1,3)}, \underset{v_2}{(1,0,0,4)}\}$. (not multiples of each other)

(c) Verify that your answer to (b) is linearly independent.

Suppose $\alpha_1(0,1,1,3) + \alpha_2(1,0,0,4) = 0$

$2\frac{1}{2}$ then $(0, \alpha_1, \alpha_1, 3\alpha_1) + (\alpha_2, 0, 0, 4\alpha_2) = 0$

$(\alpha_2, \alpha_1, \alpha_1, 3\alpha_1 + 4\alpha_2) = 0$

From first coord. $\alpha_2 = 0$, from second coord. $\alpha_1 = 0$

so only trivial lin. comb is 0: v_1, v_2 are lin. ind.

5:05

6. [8] Short answer questions. Working is not required but may be considered for partial credit if given.

3 min

5.

(a) Suppose $\{u, v\}$ is a linearly dependent set of vectors. What is the most accurate statement you can make?

D

A. u is a scalar multiple of v.

B. v is a scalar multiple of u.

C. Both A and B are true.

D. At least one of A or B is true.

E. Exactly one of A or B is true.

C not true, e.g. $\{u,v\} = \{0, v \neq 0\}$.

(b) Suppose S_0 is a spanning set for vector space V and $|S_0| = 27$. Suppose we discard 12 elements from S_0 to form a basis B_1 of V. Then we use elements of B_1 to build up a linearly independent set I_2 with $|I_2| = 9$ to get a new basis B_3 of V. How many elements of B_1 do we add to I_2?

6

dim $V = 27 - 12 = 15$

$15 - 9 = 6$

(c) Suppose V is a vector space over a field F, and E is a subfield of F. When is V a vector space over E (with the same addition, and the restriction of the same scalar multiplication to elements of E)?

A

A. Always.

B. Never.

C. Sometimes; it depends on both E & F.

D. Provided F has characteristic 0.

(d) Suppose vector space V contains a spanning set of size 10, and a linearly independent set of size 7. Let $A \subseteq V$. Which of the following is **NOT** true?

C

A. If $|A| = 6$ then A does not span V.

B. If $|A| = 11$ then A is linearly dependent.

C. If $|A| = 10$ then A cannot be a basis for V.

D. If $|A| = 7$ then A might be a basis for V.

- In the following true/false questions answer 'True' only if the assertion is **ALWAYS** true.

(e) True or false: The empty set is a subspace of every vector space.

F

Empty set is not a vector space: no zero vector.

(f) True or false: If S is linearly dependent then every element of S is a linear combination of other elements of S.

F

The "every" is false. "Some" would be true.

(g) True or false: If $T : V \to W$ is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.

F

False, e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1,x_2) = (0,0)$ for all (x_1,x_2)!
lin. ind., $T(\{(1,1)\}) = \{(0,0)\}$ — lin. dep.

(h) True or false: The zero vector belongs to the span of any set of vectors.

T

True, even the span of the empty set.

I pledge on my honor that I have neither given nor received improper aid on this test or quiz.

Signed: