Math 2600/5600 - Linear Algebra - Fall 2015

Review Sheet for Test 3

Test 3 covers material on determinants and eigenvalues, but not including the Cayley-Hamilton Theorem (except that some results from Section 5.4 that we proved earlier are included). Earlier material may be included if it is necessary as background.

Below I try to give an indication of various things from the course that you are expected to know, and the level at which you need to know them. This list DOES NOT include everything that you are supposed to know.

Remember that anything that is on a homework or practice problem is fair game for the test. One of the best ways to study for the test is to make sure you can do all the practice problems.

- You should know and be able to state definitions of the following **from the 'Lecture notes on determinants' posted on the class web site**. Do not use the book's definitions.

a *permutation* of a set X;
transpositions acting on a permutation;
identity permutation;
even and *odd* permutations;
sign sgn (π) of a permutation;
determinant of a matrix (using permutations);
upper triangular matrix;
lower triangular matrix;
multilinear function;
cofactor of A_{ij};
cofactor expansion along row i or column j;
adjugate or *classical adjoint* matrix.

- There are certain things that you should be able to do involving determinants:

Determine whether a permutation is even or odd, and compute its sign.

Construct inverse of a permutation.

Identify term in a determinant that comes from a given permutation.

Compute following determinants quickly: 2×2 matrix, 3×3 matrix (forwards/backwards diagonals), upper or lower triangular matrix (including diagonal matrices).

Calculate determinant by using elementary row operations to reduce to upper triangular form.

More generally, be able to use both elementary row and column operations to put matrix in a simpler form as part of computing determinant.

Expand a determinant using a cofactor expansion along any row or any column.

Know how to compute the determinant of a block upper triangular matrix (as in A5.3). Also in situation where there are several diagonal blocks (not just two). Also for block lower triangular matrices.

- There are certain properties of determinants you should know:

$\det A^{\mathrm{T}} = \det A$; you should know that this can be proved using the fact that sgn $(\pi) =$ sgn (π^{-1}).

det is multilinear in rows (also in columns).

$\det A$ is zero if A has equal rows, a row of zeroes, or more generally, linearly dependent rows. Similarly for columns.

How $\det A$ is affected by elementary row operations.

$\det AB = \det A \det B$; you should have a general idea of how this is proved (using elementary matrices).

Many things equivalent to $\det A \neq 0$ for $A \in F^{n\times n}$: A is invertible; L_A (or any other $T \in L(V,W)$ with $[T]_B^C = A$) is an isomorphism; rank $A = n$; rows of A are linearly independent; rows of A are a basis of F^n; rows of A span F^n; columns of A are linearly independent, etc.

Relationship between adjugate matrix and inverse.

- There are certain things involving determinants you do **NOT** have to know:

Extended form of definition of determinant using row labels R and column labels C.

Proof of cofactor expansion.

- You should know and be able to state definitions of the following. Definitions from the book are acceptable except as noted below.

eigenvector and *eigenvalue* of $T \in L(V)$ or $A \in F^{n\times n}$;
$E_\lambda(T)$ or $E_\lambda(A)$;
eigenspace of T or A for λ;
geometric multiplicity of λ;
diagonalizable linear transformation;
diagonalizable matrix;
characteristic polynomial $\chi_A(t)$ of $A \in F^{n\times n}$;
characteristic polynomial $\chi_T(t)$ of $T \in L(V)$;
characteristic equation of A or T;
T-*invariant* subspace of V if $T \in L(V)$;
(algebraic) multiplicity of λ;
what it means for a polynomial to *split*;

Some of these definitions were given differently in class from the version in the book, or were not given in the book at all. You should know the version from class:

The *geometric multiplicity* of λ is dim $E_\lambda(T)$.

For $A \in F^{n\times n}$, $\chi_A(t) = \det(tI - A)$ is the *characteristic polynomial* of A.
For $T \in L(V)$, dim $V = n$ (finite), define $\chi_T(t)$ to be $\chi_A(t)$ for any $A = [T]_B$; exact ordered basis B does not matter.
The equation $\chi_A(t) = 0$ or $\chi_T(t) = 0$ is the *characteristic equation* of A or T.

- There are certain things that you should be able to do involving eigenvalues:

Find the characteristic polynomial of $A \in F^{n\times n}$ or $T \in L(V)$ and use it to find the eigenvalues and their algebraic multiplicities.

Find bases for the eigenspaces $E_\lambda(T) = N(\lambda I - T)$ and geometric multiplicities of the eigenvalues.

Determine whether A or T is diagonalizable. For diagonalizable T, give a basis B and diagonal D for which $[T]_B = D$. For diagonalizable A, give invertible Q and diagonal D for which $D = Q^{-1}AQ$.

Distinguish between what happens in $\mathbf{R}^n$ versus $\mathbf{C}^n$ when taking eigenvalues, trying to diagonalize, and so on.

- You should know the following results:

Lemma TI: (FIS Theorem 5.21, in 5.4) If W is T-invariant and T_W is the restriction of T to domain and codomain W, then $\chi_{T_W}(t)$ is a factor of $\chi_T(t)$.

Lemma GM: (FIS Theorem 5.7) If λ is eigenvalue of T or A with algebraic multiplicity m and geometric multiplicity k then $1 \le k \le m$.

Lemma: (FIS Theorem 5.5) Any collection of eigenvectors, each of which comes from a different eigenspace, is linearly independent.

Corollary: (FIS Theorem 5.8) If we take linearly independent sets S_1, S_2, ..., S_p, where each S_i is a subset of a different $E_{\lambda_i}(T)$, then $S_1 \cup S_2 \cup \ldots \cup S_p$ is linearly independent.

Theorem: (combines FIS Theorems 5.6, 5.9) For $T \in L(V)$, V finite-dimensional, the following are equivalent.
(a) T is diagonalizable.
(b) V has a basis B consisting of eigenvectors of T; then $[T]_B$ is diagonal.
(c) The geometric multiplicities of the eigenvalues add up to dim V.
(d) (i) $\chi_T(t)$ *splits*, i.e., factors completely into linear terms,
and (ii) for every eigenvalue λ, its geometric multiplicity equals its algebraic multiplicity.

A basis B for which $[T]_B$ is diagonal is exactly the union of bases of all the eigenspaces.