Theorem: If $A, B \in F^{n\times n}$ then $\det AB = \det A \det B$.

Proof: Suppose $\det A = 0$. Then the columns of A do not span F^n, so the columns of AB, which are linear combinations of the columns of A, do not span F^n, so $\det AB = 0$.

So suppose $\det A \neq 0$. Then A is invertible so $A = E_1^{-1}E_2^{-1}\dots E_k^{-1}I$ as above, and by Lemma DE (ii) $\det A = \det(E_1^{-1})\det(E_2^{-1})\dots\det(E_k^{-1})\det I = \det(E_1^{-1})\dots\det(E_k^{-1})$. But $AB = E_1^{-1}\dots E_k^{-1}IB = E_1^{-1}\dots E_k^{-1}B$ and hence by Lemma DE (ii) again, $\det AB = \det E_1^{-1}\dots\det E_k^{-1}\det B = \det A \det B$. ∎

Corollary: (i) $\det A^k = (\det A)^k$. (ii) If A is invertible then $\det A^{-1} = 1/\det A$.

Extension: (useful for submatrices) Suppose rows labelled by set R in order, columns by set C in order (for us R and C contain positive integers, but could be more general). Call bijection $\pi : R \to C$ even or odd according to the number of transpositions needed to change π into the *order-preserving* bijection $e_{RC} : R \to C$, then define sgn (π) as $+$ for even, $-$ for odd.

Example: $R = \{1,4,7,9\}$, $C = \{1,2,5,9\}$. Take

$\pi = \begin{vmatrix} 1 & 4 & 7 & 9 \\ 5 & 1 & 9 & 2 \end{bmatrix}$ (so $\pi(1) = 5$, $\pi(4) = 2$, etc.) $= [5192]$ (assume R given in order)

$\to [1592] \to [1295] \to [1259] = \begin{bmatrix} 1 & 4 & 7 & 9 \\ 1 & 2 & 5 & 9 \end{bmatrix} = e_{RC}$, now preserves order.

So π is odd, sgn $(\pi) = -1$.

Lemma: If $A \in F^{n\times n}$ has rows, columns labelled by R, C in order and $S(R,C)$ is set of bijections from R to C, then

$$\det A = |A| = \sum_{\pi\in S(R,C)} \text{sgn}\,(\pi) \prod_{r\in R} A_{r,\pi(r)}. \qquad (**)$$

Proof: this just becomes usual definition when relabel rows, columns in order with $\{1, 2, \dots, n\}$.

Example:

$$\begin{array}{cc|cc|} & C: & 3 & 7 \\ R: & 1 & A_{13} & A_{17} \\ & 4 & A_{43} & A_{47} \end{array} = A_{13}A_{47} - A_{17}A_{43} \quad \text{from} \quad \text{sgn}\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} = 1, \quad \text{sgn}\begin{bmatrix} 1 & 4 \\ 7 & 3 \end{bmatrix} = -1.$$

Useful case: Let $N_i = \{1, 2, \dots, n\} - \{i\}$. If $\pi \in S_n$ with $\pi(i) = j$ then get $\pi^- \in S(N_i, N_j)$ by restricting π. Want to compare sgn (π) and sgn (π^-).

Example: $\pi = [42153]$, $i = 3$ and $j = 1$.

$\pi^- = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 4 & 2 & 5 & 3 \end{bmatrix} = [4253] \in S(N_3, N_1) \to [2453] \to [2354] \to [2345] = e_{N_3,N_1}$;

$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{bmatrix} = [42153] \to [24153] \to [23154] \to [23145] \to [21345] \to [12345] = e$.

In general: can reduce π to e by same transpositions that reduce π^- to e_{N_i,N_j}, then bubble j to correct position with $|j - i|$ transpositions. Hence sgn $(\pi) = (-1)^{|j-i|}$sgn $(\pi^-) = (-1)^{i+j}$sgn (π^-).

Can use this to get recursive formula for determinant. Book uses this as definition.

Cofactor expansion: Suppose $A \in F^{n\times n}$. Fix $i \in \{1, 2, \dots, n\}$. Let $\widetilde{A}_{ij} = A$ with row i and column j deleted. Then

$$\begin{aligned}
\det A &= \sum_{\pi\in S_n} \text{sgn}\,(\pi)\ A_{1,\pi(1)}A_{2,\pi(2)}A_{3,\pi(3)}\dots A_{n,\pi(n)} \\
&= \sum_{j=1}^{n} \sum_{\pi\in S_n, \pi(i)=j} \text{sgn}\,(\pi)\ A_{1,\pi(1)}\dots A_{i,\pi(i)=j}\dots A_{n,\pi(n)} \\
&= \sum_{j=1}^{n} \sum_{\pi^-\in S(N_i,N_j)} (-1)^{i+j}\text{sgn}\,(\pi^-)\ A_{ij}\ A_{1,\pi^-(1)}\dots A_{i-1,\pi^-(i-1)}A_{i+1,\pi^-(i+1)}\dots A_{n,\pi^-(n)} \\
&= \sum_{j=1}^{n} (-1)^{i+j}A_{ij} \sum_{\pi^-\in S(N_i,N_j)} \text{sgn}\,(\pi^-)\ A_{1,\pi^-(1)}\dots A_{i-1,\pi^-(i-1)}A_{i+1,\pi^-(i+1)}\dots A_{n,\pi^-(n)}
\end{aligned}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det \widetilde{A}_{ij} = \sum_{j=1}^{n} c_{ij} A_{ij}$$

where $c_{ij} = (-1)^{i+j} \det \widetilde{A}_{ij}$ is the *cofactor* of A_{ij}. This is the *cofactor expansion along row* i. Shows explicitly that det depends linearly on row i, giving actual coefficients c_{ij}.

By (D1) can also do *cofactor expansion down column* j:

$$\det A = \sum_{i=1}^{n} c_{ij} A_{ij} \quad \text{where } c_{ij} = (-1)^{i+j} \det \widetilde{A}_{ij} \text{ again.}$$

Example: Expanding down the second column,

$$\begin{vmatrix} 2 & 0 & 3 & 1 \\ 4 & 0 & 0 & 1 \\ 1 & 3 & 6 & 9 \\ 6 & 0 & 4 & 7 \end{vmatrix} = 0 + 0 + (-1)^{2+3} 3 \begin{vmatrix} 2 & 3 & 1 \\ 4 & 0 & 1 \\ 6 & 4 & 7 \end{vmatrix} + 0$$

and expanding across the second row

$$= -3 \left((-1)^{2+1} 4 \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + 0 + (-1)^{2+3} 1 \begin{vmatrix} 2 & 3 \\ 6 & 4 \end{vmatrix} \right) = -3(-4(17) - 1(-10)) = -3(-58) = 174.$$

Use cofactor expansion when have row or column that is mostly zeroes, and also for computing determinants in cases where have symbolic expressions, not numbers, in matrix. Can mix cofactor and elimination methods.

Adjugate matrix: The *adjugate* or *classical adjoint* of $A \in F^{n \times n}$ is adj $A \in F^{n \times n}$ that is the transpose of the matrix of cofactors: $(\text{adj } A)_{ij} = c_{ji} = (-1)^{i+j} \det \widetilde{A}_{ji}$.

Adjugate and inverse: For $A \in F^{n \times n}$, $A(\text{adj } A) = (\text{adj } A)A = (\det A)I$.
Diagonal terms here are just cofactor expansions. Off-diagonal terms are cofactor expansions for matrix obtained from A by replacing one row by another - then have duplicate rows, so get 0.

Thus, if $\det A \neq 0$, $A \left(\dfrac{1}{\det A} \text{adj } A \right) = I$ and so $A^{-1} = \dfrac{1}{\det A} \text{adj } A)$.

Example: For 2×2 matrices:

$$\text{adj } A = \text{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (\det A)I.$$

By similar arguments can also get method called Cramer's Rule for giving solutions of square linear system in terms of determinants. Usually highly inefficient compared to elimination methods, so won't talk about it.