4. DETERMINANTS

Determinant is number associated with every square matrix. Applications: volume of parallelopiped in n dimensions, general inverse of $n \times n$ matrix, scale factor for transformations specified by matrices (with $-$ sign if reverses orientation), Jacobian for equivalent of substitution in multivariable integrals.

Permutations

Definition: A *permutation* of a set X is a bijection (1 − 1 correspondence) $\pi : X \to X$. Set of permutations of $\{1, 2, \ldots, n\}$ is denoted S_n.

Example: S_5 = permutations of $\{1, 2, 3, 4, 5\}$:
$\pi = [13452]$ denotes bijection with $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 4$, $\pi(4) = 5$ and $\pi(5) = 2$.

Definition: Given a permutation $\pi = [\pi(1)\ \pi(2)\ \ldots \pi(n)] \in S_n$, it turns out that any sequence of *transpositions* (two-element swaps) reducing π to the *identity permutation* $e = [1\ 2\ \ldots\ n]$ always has length of the same parity. Proof not too hard: define a measure of how far permutation is from identity (inversion number), and show that it changes by odd amount for each transposition. So call a permutation *even* if length is always even, and *odd* if length is always odd.

Example: $[154326] \to [124356] \to [123456]$, 2 transpositions (move 1 into place, then 2, then 3, ...);
$[154326] \to [145326] \to [143526] \to [134526] \to [134256] \to [132456] \to [123456]$, 6 transpositions (each time fix earliest pair out of order).
Either way, $[154326]$ is even in S_6.

Definition: The *sign* $\text{sgn}\,(\pi)$ of a permutation π is 1 for an even permutation, -1 for an odd permutation. Often write '+' for 1 and '−' for -1.

Examples: For S_2, $\text{sgn}\,[12] = 1$, $\text{sgn}\,[21] = -1$.
For S_3, $\text{sgn}\,[123] = 1$, $\text{sgn}\,[132] = \text{sgn}\,[213] = \text{sgn}\,[321] = -1$, $\text{sgn}\,[231] = \text{sgn}\,[312] = 1$.

Determinants and permutations

Have probably seen 2×2, 3×3 determinants before. Can explain them using permutations and signs.

2×2 determinants:
$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

Each term has one entry from each row and one entry from each column: mapping rows → columns is a permutation.

Perm.	Sign	Term
[12]	+	$+A_{11}A_{22}$
[21]	−	$-A_{12}A_{21}$

– so $\det A = \sum_{\pi \in S_2} \text{sgn}\,(\pi)\, A_{1,\pi(1)} A_{2,\pi(2)}$.

3×3 determinants: (plus forward cyclic diagonals minus backward cyclic diagonals)
$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31}$$

Again, each term corresponds to a permutation mapping rows → columns.

Perm.	Sign	Term
[123]	$+$	$+A_{11}A_{22}A_{33}$
[231]	$+$	$+A_{12}A_{23}A_{31}$
[312]	$+$	$+A_{13}A_{21}A_{32}$
[132]	$-$	$-A_{11}A_{23}A_{32}$
[213]	$-$	$-A_{12}A_{21}A_{33}$
[321]	$-$	$-A_{13}A_{22}A_{31}$

– so $\det A = \sum_{\pi \in S_3} \text{sgn}\,(\pi) A_{1,\pi(1)}A_{2,\pi(2)}A_{3,\pi(3)}$.

Definition: The *determinant* of an $n \times n$ matrix $A = [A_{ij}]$ is

$$\det A = |A| = \sum_{\pi \in S_n} \text{sgn}\,(\pi)\ A_{1,\pi(1)}A_{2,\pi(2)}A_{3,\pi(3)} \dots A_{n,\pi(n)}. \qquad (*)$$

Not very practical for computation: $n!$ terms. Will see we can use Gaussian elimination which needs around n^3 operations.

Properties: (D1) Permutation π is bijection, has inverse π^{-1}. And transposition in π corresponds to transposition in π^{-1}, so $\text{sgn}\,(\pi) = \text{sgn}\,(\pi^{-1})$. E.g.

$$[634521]^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 3 & 4 & 1 \end{bmatrix} = [652341]$$

swap 3 and 2   swap 2 and 5

$$\begin{bmatrix} 1 & \underline{2} & 3 & 4 & \underline{5} & 6 \\ 6 & \underline{2} & 4 & 5 & \underline{3} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \underline{2} & \underline{3} & 4 & 5 & 6 \\ 6 & \underline{5} & \underline{2} & 3 & 4 & 1 \end{bmatrix}$$

Looking at determinant using map π^{-1} from columns to rows, get

$$\begin{aligned} \det A &= \sum_{\pi \in S_n} \text{sgn}\,(\pi)\ A_{\pi^{-1}(1),1}A_{\pi^{-1}(2),2} \dots A_{\pi^{-1}(n),n} \\ &= \sum_{\sigma \in S_n} \text{sgn}\,(\sigma)\ A_{\sigma(1),1}A_{\sigma(2),2} \dots A_{\sigma(n),n} \quad (\sigma = \pi^{-1}) \text{ ranges over } S_n \text{ as } \pi \text{ ranges over } S_n \\ &= \sum_{\sigma \in S_n} \text{sgn}\,(\sigma)\ A^{\mathrm{T}}_{1,\sigma(1)}A^{\mathrm{T}}_{2,\sigma(2)} \dots A^{\mathrm{T}}_{n,\sigma(n)} = \det A^{\mathrm{T}}. \end{aligned}$$

Definition: A matrix A is *upper triangular* if $A_{ij} = 0$ whenever $j < i$ (all entries below main diagonal are zero) and *lower triangular* if $A_{ij} = 0$ whenever $j > i$ (all entries above main diagonal are zero). E.g.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ – upper tri.,} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 5 \end{bmatrix} \text{ – lower tri.}$$

OK to have zeroes in other places.

(D2) If A is upper triangular then only permutation that avoids entries below diagonal is e (forced to have $\pi(n) = n$, then $\pi(n-1) = n-1$, etc.) so only nonzero term in determinant is one for e, giving $\det A = \text{sgn}\,(e)A_{1,e(1)}A_{2,e(2)} \dots A_{n,e(n)} = A_{11}A_{22} \dots A_{nn}$ – product of main diagonal entries. By (D1), same is true when A is lower triangular. E.g.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24.$$

In particular, $\det I_n = 1$ and $\det 0_{n\times n} = 0$.

(D3) det is a *multilinear* function of the rows of a matrix: fix all rows but one (like partial derivatives), then det is linear function of varying row. E.g., for 3×3,

$$\begin{aligned}\det A = (A_{13}A_{32} - A_{12}A_{33})A_{21} + (A_{11}A_{33} - A_{13}A_{31})A_{22} \\ + (A_{12}A_{31} - A_{11}A_{32})A_{23} \quad = \quad c_{21}A_{21} + c_{22}A_{22} + c_{23}A_{23}\end{aligned}$$

– linear in $A_{2*} = (A_{21}, A_{22}, A_{23})$, second row. By (D1), also multilinear in columns.

Think of $D : (F^n)^n \to F$ with

$$D(x_1, x_2, \ldots, x_n) = \begin{vmatrix} x_1^{\mathrm{T}} \\ x_2^{\mathrm{T}} \\ \vdots \\ x_n^{\mathrm{T}} \end{vmatrix}, \quad \text{also} = |\, x_1 \ x_2 \ \ldots \ x_n \,| \text{ by (D1)} \qquad (x_1, x_2, \ldots \text{ are vectors}).$$

Then $D(x_1, \ldots, x_{i-1}, \alpha x_i + \beta y_i, x_{i+1}, \ldots, x_n)$
$= \alpha D(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + \beta D(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

E.g. $(7, 3) = 3(1, 1) + 4(1, 0)$ so $\begin{vmatrix} 15 & 8 \\ 7 & 3 \end{vmatrix} = 3\begin{vmatrix} 15 & 8 \\ 1 & 1 \end{vmatrix} + 4\begin{vmatrix} 15 & 8 \\ 1 & 0 \end{vmatrix}$.

(D4) If rows i and j in A are equal, then terms of (*) containing equal products $A_{ik}A_{j\ell}$ and $A_{i\ell}A_{jk}$ have opposite signs (permutations differ by one transposition $k \leftrightarrow \ell$) so cancel. Hence $\det A = 0$. Can also argue using (DE1) below that $\det A = -\det A$ but does not give $\det A = 0$ if char $F = 2$.

By (D1), if two equal columns also get $\det A = 0$.

(D5) If A has a row of zeroes then every product in (*) has a 0, so $\det A = 0$. By (D1), also $\det A = 0$ if column of zeroes.

Now examine effect of elementary row operations.

(DE1) If we swap two rows of A to get A', amounts to changing all permutations by one transposition, so $\det A' = -\det A$.

(DE2) If we multiply a row of A by constant α to get A', multilinearity gives $\det A' = \alpha \det A$.

(DE3) If get A' from A by adding α times row i to row j, then (taking $i < j$ without loss of generality)

$$\begin{aligned}\det A' &= D(\ldots, A_{i*}, \ldots, \alpha A_{i*} + A_{j*}, \ldots) \\ &= \alpha D(\ldots, A_{i*}, \ldots, A_{i*}, \ldots) + D(\ldots, A_{i*}, \ldots, A_{j*}, \ldots) \quad \text{by multilinearity} \\ &= 0 + D(\ldots, A_{i*}, \ldots, A_{j*}, \ldots) = \det A \quad \text{by (D4)}\end{aligned}$$

– determinant does not change.

By (D1), similar conclusions hold for elementary column operations.

Using elementary row operations to reduce matrix to upper triangular form and (DE1)–(DE3), get efficient way to compute determinants for matrices in general.

Example: Compute determinant by reducing matrix to row-echelon form, upper triangular.

$$\begin{vmatrix} 0 & 1 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 7 \\ -1 & 3 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 5 & 6 & 7 & 7 \\ -1 & 3 & 2 & 1 \end{vmatrix} \begin{matrix} R_1' = R_2 \\ R_2' = R_1 \\ \\ \\ \end{matrix} = -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & -4 & -8 & -13 \\ 0 & 5 & 5 & 5 \end{vmatrix} \begin{matrix} \\ \\ R_3' = R_3 - 5R_1 \\ R_4' = R_4 + R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 8 & 7 \\ 0 & 0 & -15 & -20 \end{vmatrix} \begin{matrix} \\ \\ R_3' = R_3 + 4R_2 \\ R_4' = R_4 - 5R_2 \end{matrix} = -8\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7/8 \\ 0 & 0 & -15 & -20 \end{vmatrix} \; R_3' = R_3/8$$

$$= -8\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7/8 \\ 0 & 0 & 0 & -55/8 \end{vmatrix} \begin{matrix} \\ \\ \\ R_4' = R_4 + 15R_3 \end{matrix} \qquad = -8 \cdot 1 \cdot 1 \cdot 1 \cdot (-55/8) = 55.$$

Another example:

$$\begin{vmatrix} 3 & 1 & 5 & 2 \\ -4 & 2 & 0 & -6 \\ 7 & 1 & 9 & 6 \\ 8 & 5 & 18 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1/3 & 5/3 & 2/3 \\ -4 & 2 & 0 & -6 \\ 7 & 1 & 9 & 6 \\ 8 & 5 & 18 & 3 \end{vmatrix} \begin{matrix} R_1' = R_1/3 \\ \\ \\ \\ \end{matrix}$$

$$= 3\begin{vmatrix} 1 & 1/3 & 5/3 & 2/3 \\ 0 & 10/3 & 20/3 & -10/3 \\ 0 & -4/3 & -8/3 & 4/3 \\ 0 & 7/3 & 14/3 & -7/3 \end{vmatrix} \begin{matrix} \\ R_2' = R_2 + 4R_1 \\ R_3' = R_3 - 7R_1 \\ R_4' = R_4 - 8R_1 \end{matrix}$$

$$= 3(10/3)\begin{vmatrix} 1 & 1/3 & 5/3 & 2/3 \\ 0 & 1 & 2 & -1 \\ 0 & -4/3 & -8/3 & 4/3 \\ 0 & 7/3 & 14/3 & -7/3 \end{vmatrix} \; R_2' = (3/10)R_2$$

$$= 10\begin{vmatrix} 1 & 1/3 & 5/3 & 2/3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 7/3 & 14/3 & -7/3 \end{vmatrix} = 10 \cdot 0 = 0 \quad \text{(zero row)}.$$

Elementary row operations also useful for theoretical results.

Representing each elementary row operation ε by an elementary matrix $E = \varepsilon(I)$, we can prove the following from (DE1)–(DE3):

Lemma DE: (i) An elementary matrix E of type 1 has $\det E = -1$, of type 2 has $\det E = \alpha$, and of type 3 has $\det E = 1$.

(ii) If $E, B \in F^{n\times n}$ where E is an elementary matrix, then $\det EB = \det E \det B$.

(D6) Suppose we reduce $A \in F^{n\times n}$ to its reduced row echelon form R using elementary row operations. Each operation multiplies the determinant by a nonzero constant, so $\det A = c \det R$ where $c \neq 0$.

If A is invertible, then $R = I$ and hence $\det A = c \det I = c \neq 0$. If A is not invertible then R has a row of zeroes and hence $\det A = c \det R = c \cdot 0 = 0$.

Thus: $\det A \neq 0 \Leftrightarrow A$ is invertible
- $\Leftrightarrow L_A$ (or any other $T \in L(V, W)$ with $[T]_B^C = A$) is an isomorphism
- $\Leftrightarrow$ rank $A = n$
- $\Leftrightarrow$ rows of A are linearly independent
- $\Leftrightarrow$ rows of A are a basis of F^n
- $\Leftrightarrow$ rows of A span F^n
- $\Leftrightarrow$ columns of A are linearly independent, etc.

Observation: For A, R as above, $R = E_k E_{k-1} \dots E_1 A$, or $A = E_1^{-1} E_2^{-1} \dots E_k^{-1} I$ where $E_1, \dots, E_k$ and their inverses are elementary matrices.

Theorem: If $A, B \in F^{n\times n}$ then $\det AB = \det A \det B$.

Proof: Suppose $\det A = 0$. Then the columns of A do not span F^n, so the columns of AB, which are linear combinations of the columns of A, do not span F^n, so $\det AB = 0$.

So suppose $\det A \neq 0$. Then A is invertible so $A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} I$ as above, and by Lemma DE (ii) $\det A = \det(E_1^{-1}) \det(E_2^{-1}) \ldots \det(E_k^{-1}) \det I = \det(E_1^{-1}) \ldots \det(E_k^{-1})$. But $AB = E_1^{-1} \ldots E_k^{-1} IB = E_1^{-1} \ldots E_k^{-1} B$ and hence by Lemma DE (ii) again, $\det AB = \det E_1^{-1} \ldots \det E_k^{-1}$ $\det B = \det A \det B$. ∎