Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 5 Solutions

A5.1. The determinant is a multilinear alternating function. In this question we look at some other functions with similar properties.

If V, W are vector spaces over a field F, then we can consider maps $f : V \times V \to W$, which take an ordered pair (u, v) with $u, v \in V$ and give a vector $f(u, v) \in W$. Such a map f is called *bilinear* (a special case of multilinear) if it is linear in each of u and v separately, with the other one held fixed. This means that $f(\alpha u_1 + \beta u_2, v) = \alpha f(u_1, v) + \beta f(u_2, v)$ and $f(u, \alpha v_1 + \beta v_2) = \alpha f(u, v_1) + \beta f(u, v_2)$ for all $\alpha, \beta \in F$ and $u, u_1, u_2, v, v_1, v_2 \in V$.

(a) If $A \in F^{n\times n}$ then we can define $f_A : F^n \times F^n \to F$ by $f_A(x, y) = x^{\mathrm{T}} Ay$. (As usual, vectors in F^n are considered to be column vectors.)

(i) Find $f_A(x, y)$ if $F = \mathbf{R}$, $n = 3$, $x = (1, 2, 3)$, $y = (-1, 0, 1)$ and $A = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -3 \\ 5 & 1 & -4 \end{bmatrix}$.

(ii) Show that any f_A as described above (*not* the specific f_A from (i)) is bilinear.

(b) A bilinear map $f : V \times V \to W$ is called *alternating* if $f(v, v) = 0$ for all $v \in V$, and *antisymmetric* if $f(u, v) = -f(v, u)$ for all $u, v \in V$.

(i) Prove that if a bilinear map is alternating then it is also antisymmetric. [Hint: look at $f(u + v, u + v)$.]

(ii) Prove that if a bilinear map is antisymmetric then it is also alternating, provided the characteristic of the field we are working in is not equal to 2. Make it clear where you use the hypothesis about the characteristic of the field.

(c) (i) Use (a)(ii) to show that the ordinary dot product $d : F^n \times F^n \to F$ by $d(x, y) = x^{\mathrm{T}} y$ is bilinear.

(ii) Prove that if $F = \mathbf{Z}_2$ and $n \geq 1$ then the dot product $d : \mathbf{Z}_2^n \times \mathbf{Z}_2^n \to \mathbf{Z}_2$ is antisymmetric but not alternating. [This shows that (b)(ii) definitely does not work if F has characteristic 2.]

Solution: (a)(i) We have

$$f_A(x, y) = x^{\mathrm{T}} Ay = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -3 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 & 10 & -11 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -28.$$

(ii) Suppose $\alpha, \beta \in F$ and $x, x_1, x_2, y, y_1, y_2 \in F$. We have

$f_A(\alpha x_1 + \beta x_2, y) = (\alpha x_1 + \beta x_2)^{\mathrm{T}} Ay = (\alpha x_1^{\mathrm{T}} + \beta x_2^{\mathrm{T}})Ay$ (by linearity of transpose) $= (\alpha x_1^{\mathrm{T}})Ay + (\beta x_2)^{\mathrm{T}} Ay = \alpha(x_1^{\mathrm{T}} Ay) + \beta(x_2^{\mathrm{T}} Ay) = \alpha f_A(x_1, y) + \beta f_A(x_2, y)$.

Similarly,

$f_A(x, \alpha y_1 + \beta y_2) = x^{\mathrm{T}} A(\alpha y_1 + \beta y_2) = x^{\mathrm{T}} A(\alpha y_1) + x^{\mathrm{T}} A(\beta y_2) = \alpha(x^{\mathrm{T}} Ay_1) + \beta(x^{\mathrm{T}} Ay_2) = \alpha f(x, y_1) + \beta f(x, y_2)$.

Hence f_A is bilinear.

(b)(i) Suppose f is alternating. Then $0 = f(u+v, u+v)$ (since f is alternating) $= f(u, u+v) + f(v, u+v)$ (by linearity in first position) $= f(u, u) + f(u, v) + f(v, u) + f(v, v)$ (by linearity in second position) $= 0 + f(u, v) + f(v, u) + 0$ (since f is alternating). Hence, $f(u, v) = -f(v, u)$ and so f is antisymmetric.

(ii) Suppose f is antisymmetric. Then $f(v, v) = -f(v, v)$ and hence, adding $f(v, v)$ to both sides, $2f(v, v) = 0$. Since we are in a field whose characteristic is not 2, $2 \neq 0$, so we can divide by 2 to get $f(v, v) = 0$. Hence, f is alternating.

(c)(i) If we let $A = I \in F^{n\times n}$, the $n \times n$ identity matrix, then $d(x, y) = x^{\mathrm{T}} y = x^{\mathrm{T}} Iy = f_I(x, y)$, which we know is bilinear by (a)(ii) with $A = I$.

(ii) Working with field $\mathbf{Z}_2$ we know $1 = -1$. So we have $d(x, y) = x^{\mathrm{T}} y = x_1 y_1 + \ldots + x_n y_n = y^{\mathrm{T}} x = d(y, x) = -d(y, x)$ and hence d is antisymmetric (and symmetric at the same time!). But if we take $x = (1, 0, 0, \ldots, 0) \in \mathbf{Z}_2^n$ then $d(x, x) = 1 \neq 0$, so d is not alternating.

A5.2. (a) Compute the determinant $|A|$ below using elementary row operations. Describe each individual elementary row operation you use.

$$|A| = \begin{vmatrix} 5 & 2 & -1 & 3 \\ 4 & 1 & 0 & -4 \\ -2 & 1 & 3 & 5 \\ 6 & 7 & 2 & 2 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1 & x^2 & 1 & x \\ x^2 & 0 & x & 0 \\ x & 1 & x & x \\ 1 & x & x^3 & 1 \end{vmatrix}$$

(b) Compute the determinant $|B|$ above first using a cofactor expansion and then using the quick method for 3×3 determinants (forwards/backwards diagonals). Simplify your answer as much as possible.

(c) Compute the following determinant $|C|$ as efficiently as you can, showing details. Show all elementary row or column operations used, or explicitly describe all cofactor expansions used, and describe other rules you use.

$$|C| = \begin{vmatrix} 2 & 2 & 1 & 5 & 3 & 1 & 4 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 & 3 & 5 & 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 & 4 & 5 & 2 & 2 & 1 & 0 \\ 4 & 0 & 3 & 2 & 5 & 3 & 4 & 0 & 1 & 0 \\ 5 & 0 & 3 & 4 & 2 & 3 & 0 & 0 & 1 & 0 \\ 7 & 0 & 1 & 4 & 7 & 0 & 0 & 0 & 3 & 0 \\ 2 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 3 & 0 \\ 5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 92 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Solution: (a) Using elementary row operations to reduce to an upper triangular matrix,

$$|A| = \begin{vmatrix} 5 & 2 & -1 & 3 \\ 4 & 1 & 0 & -4 \\ -2 & 1 & 3 & 5 \\ 6 & 7 & 2 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 2/5 & -1/5 & 3/5 \\ 4 & 1 & 0 & -4 \\ -2 & 1 & 3 & 5 \\ 6 & 7 & 2 & 2 \end{vmatrix} \; R_1' = R_1/5$$

$$= 5 \begin{vmatrix} 1 & 2/5 & -1/5 & 3/5 \\ 0 & -3/5 & 4/5 & -32/5 \\ 0 & 9/5 & 13/5 & 31/5 \\ 0 & 23/5 & 16/5 & -8/5 \end{vmatrix} \begin{matrix} \\ R_2' = R_2 - 4R_1 \\ R_3' = R_3 + 2R_1 \\ R_4' = R_4 - 6R_1 \end{matrix} = (-3/5)5 \begin{vmatrix} 1 & 2/5 & -1/5 & 3/5 \\ 0 & 1 & -4/3 & 32/3 \\ 0 & 9/5 & 13/5 & 31/5 \\ 0 & 23/5 & 16/5 & -8/5 \end{vmatrix} \; R_2' = (-5/3)R_2$$

$$= -3 \begin{vmatrix} 1 & 2/5 & -1/5 & 3/5 \\ 0 & 1 & -4/3 & 32/3 \\ 0 & 0 & 5 & -13 \\ 0 & 0 & 28/3 & -152/3 \end{vmatrix} \begin{matrix} \\ \\ R_3' = R_3 - (9/5)R_2 \\ R_4' = R_4 - (23/5)R_2 \end{matrix} = -3(5) \begin{vmatrix} 1 & 2/5 & -1/5 & 3/5 \\ 0 & 1 & -4/3 & 32/3 \\ 0 & 0 & 1 & -13/5 \\ 0 & 0 & 28/3 & -152/3 \end{vmatrix} \; R_3' = R_3/5$$

$$= -15 \begin{vmatrix} 1 & 2/5 & -1/5 & 3/5 \\ 0 & 1 & -4/3 & 32/3 \\ 0 & 0 & 1 & -13/5 \\ 0 & 0 & 0 & -132/5 \end{vmatrix} \; R_4' = R_4 - (28/3)R_3 \qquad = -15(-132/5) = 396$$

since the determinant of an upper triangular matrix is the product of the main diagonal entries.

(b) Expanding first across row 2 we get

$$|B| = \begin{vmatrix} 1 & x^2 & 1 & x \\ x^2 & 0 & x & 0 \\ x & 1 & x & x \\ 1 & x & x^3 & 1 \end{vmatrix} = (-1)^{2+1}x^2 \begin{vmatrix} x^2 & 1 & x \\ 1 & x & x \\ x & x^3 & 1 \end{vmatrix} + 0 + (-1)^{2+3}x \begin{vmatrix} 1 & x^2 & x \\ x & 1 & x \\ 1 & x & 1 \end{vmatrix} + 0$$

$$= -x^2(x^3 + x^2 + x^4 - x^6 - 1 - x^3) - x(1 + x^3 + x^3 - x^2 - x^3 - x)$$

using quick 3×3 determinants

$$= -x^2(x^2 + x^4 - x^6 - 1) - x(1 + x^3 - x^2 - x) = -x^4 - x^6 + x^8 + x^2 - x - x^4 + x^3 + x^2$$

$$= x^8 - x^6 - 2x^4 + x^3 + 2x^2 - x.$$

(c) The most efficient way is to observe that we can obtain an upper triangular matrix by 4 independent column swaps: $C_1 \leftrightarrow C_{10}$, $C_3 \leftrightarrow C_8$, $C_4 \leftrightarrow C_7$ and $C_5 \leftrightarrow C_6$. Each column swap changes the sign. Therefore,

$$|C| = \begin{vmatrix} 2 & 2 & 1 & 5 & 3 & 1 & 4 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 & 3 & 5 & 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 & 4 & 5 & 2 & 2 & 1 & 0 \\ 4 & 0 & 3 & 2 & 5 & 3 & 4 & 0 & 1 & 0 \\ 5 & 0 & 3 & 4 & 2 & 3 & 0 & 0 & 1 & 0 \\ 7 & 0 & 1 & 4 & 7 & 0 & 0 & 0 & 3 & 0 \\ 2 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 3 & 0 \\ 5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 92 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = (-1)^4 \begin{vmatrix} 5 & 2 & 2 & 4 & 1 & 3 & 5 & 1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 5 & 3 & 5 & 4 & 1 & 2 \\ 0 & 0 & 2 & 2 & 5 & 4 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 3 & 5 & 2 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 & 3 & 2 & 4 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 7 & 4 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 92 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= 5 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 7 \cdot 6 \cdot 5 \cdot 2 \cdot 1 = 2^5 3^3 5^2 7 = 100 \cdot 8 \cdot 27 \cdot 7 = 151,200$$

again using the fact that the determinant of an upper triangular matrix is the product of the main diagonal entries. Cofactor expansion is also a reasonable approach, but requires ten steps and careful attention to signs.

A5.3. Determinant of a block upper triangular matrix. Recall the determinant formulae for $A \in F^{n\times n}$,

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, A_{1,\pi(1)} A_{2,\pi(2)} \dots A_{n,\pi(n)} \qquad (*),$$

or the more general formula

$$\det A = \sum_{\pi \in S(R,C)} \operatorname{sgn}(\pi) \prod_{r\in R} A_{r,\pi(r)} \qquad (**)$$

if the rows of A are labelled by R and C, respectively, in order, and $S(R,C)$ is the set of bijections from R to C.

Suppose that $A \in F^{n\times n}$ has block form $A = \begin{bmatrix} B & D \\ 0 & G \end{bmatrix}$ where $B \in F^{k\times k}$ and $G \in F^{(n-k)\times(n-k)}$ for some k with $1 \le k \le n-1$. Let $N = \{1, 2, \dots, n\}$, let $K = \{1, 2, \dots, k\}$ (which labels the rows and columns of B), and let $L = N - K = \{k+1, k+2, \dots, n\}$ (which labels the rows and columns of G).

(a) Considering the expansion in $(*)$, show that $A_{1,\pi(1)} A_{2,\pi(2)} \dots A_{n,\pi(n)}$ can only be nonzero if the permutation $\pi \in S_n$ has $\pi(K) = K$ and $\pi(L) = L$.

(b) If $\pi \in S_n$ has $\pi(K) = K$ and $\pi(L) = L$, show that $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi_K)\operatorname{sgn}(\pi_L)$, where $\pi_K \in S(K,K)$ is the restriction of π to domain and codomain K, and $\pi_L \in S(L,L)$ is defined similarly. [Something that may help: if a bijection π can be put into order-preserving form using t transpositions then $\operatorname{sgn}(\pi) = (-1)^t$.]

(c) Show that $\det A = \det B \det G$ by starting with $(*)$, using (a) to discard some zero terms, splitting the sum over π into a double sum over π_K and π_L, and then using (b) and finally $(**)$. Remember that $A_{ij} = B_{ij}$ when $i, j \in K$ and $A_{ij} = G_{ij}$ when $i, j \in L$. Explain each step you take.

Solution: (a) Take $\pi \in S_n$. Suppose that for some $i \in L$ we have $\pi(i) \notin L$. Then the product $A_{1,\pi(1)} \dots A_{n,\pi(n)}$ contains $A_{i,\pi(i)}$ which is 0 because $A_{ij} = 0$ if $i \in L$ and $j \notin L$. Therefore if π gives a nonzero product, for every $i \in L$ we have $\pi(i) \in L$; in other words, $\pi(L) \subseteq L$. Since π is a bijection L and $\pi(L)$ are the same finite size, so $\pi(L) = L$. Since π is a bijection we also have $\pi(K) = \pi(N - L) = \pi(N) - \pi(L) = N - L = K$.

(b) If we want to reorder $\pi = [\pi(1)\ \pi(2)\ \dots\ \pi(n)]$ to be the identity permutation using transpositions, we can do that independently on the first k entries (since $\pi(K) = K$) and the last $n-k$ entries (since $\pi(L) = L$). Therefore, if we can reduce π_K to $e_{K,K}$ with t transpositions and we can reduce π_L to $e_{L,L}$ with u transpositions, we can reduce π to $e \in S_n$ using $t+u$ transpositions. Hence $\operatorname{sgn}(\pi) = (-1)^{t+u} = (-1)^t(-1)^u = \operatorname{sgn}(\pi_K)\operatorname{sgn}(\pi_L)$.

(c) We have

$$\det A = \sum_{\pi \in S_n} \text{sgn}\,(\pi)\; A_{1,\pi(1)} A_{2,\pi(2)} \ldots A_{n,\pi(n)}$$

$$= \sum_{\pi \in S_n, \pi(K)=K \text{ and } \pi(L)=L} \text{sgn}\,(\pi)\; A_{1,\pi(1)} A_{2,\pi(2)} \ldots A_{n,\pi(n)} \quad \text{by (a), since other terms zero}$$

$$= \sum_{\pi \in S_n, \pi(K)=K \text{ and } \pi(L)=L} \text{sgn}\,(\pi)\; A_{1,\pi_K(1)} \ldots A_{k,\pi_K(k)} A_{k+1,\pi_L(k+1)} \ldots A_{n,\pi_L(n)}$$

since $\pi(i) = \pi_K(i)$ for $i \leq k$, and $\pi(i) = \pi_L(i)$ for $i \geq k+1$

$$= \sum_{\pi_K \in S(K,K)} \sum_{\pi_L \in S(L,L)} \text{sgn}\,(\pi_K)\text{sgn}\,(\pi_L)\; A_{1,\pi_K(1)} \ldots A_{k,\pi_K(k)} A_{k+1,\pi_L(k+1)} \ldots A_{n,\pi_L(n)}$$

because each π with $\pi(K) = K$ and $\pi(L) = L$ corresponds to a unique pair π_K and π_L, and using (b)

$$= \sum_{\pi_K \in S(K,K)} \left(\text{sgn}\,(\pi_K)\; A_{1,\pi_K(1)} \ldots A_{k,\pi_K(k)} \sum_{\pi_L \in S(L,L)} \text{sgn}\,(\pi_L)\; A_{k+1,\pi_L(k+1)} \ldots A_{n,\pi_L(n)} \right)$$

pulling things that do not depend on π_L outside the second sum

$$= \left(\sum_{\pi_K \in S(K,K)} \text{sgn}\,(\pi_K)\; A_{1,\pi_K(1)} \ldots A_{k,\pi_K(k)} \right) \left(\sum_{\pi_L \in S(L,L)} \text{sgn}\,(\pi_L)\; A_{k+1,\pi_L(k+1)} \ldots A_{n,\pi_L(n)} \right)$$

pulling second sum out on right, since it does not depend on π_K

$$= \left(\sum_{\pi_K \in S(K,K)} \text{sgn}\,(\pi_K)\; B_{1,\pi_K(1)} \ldots B_{k,\pi_K(k)} \right) \left(\sum_{\pi_L \in S(L,L)} \text{sgn}\,(\pi_L)\; G_{k+1,\pi_L(k+1)} \ldots G_{n,\pi_L(n)} \right)$$

since $A_{ij} = B_{ij}$ for $i, j \in K$ and $A_{ij} = G_{ij}$ for $i, j \in L$

$$= \left(\sum_{\pi_K \in S(K,K)} \text{sgn}\,(\pi_K) \prod_{r \in K} B_{r,\pi_K(r)} \right) \left(\sum_{\pi_L \in S(L,L)} \text{sgn}\,(\pi_L) \prod_{r \in L} G_{r,\pi_L(r)} \right) = \det B\; \det G \qquad \text{by } (**).$$