**Math 2600/5600 - Linear Algebra - Fall 2015**

**Assignment 5, due in class, Friday, 20th November**

**Remember:**

- Solutions to problems should be fully explained, using clear English sentences where necessary.
- Matrix calculations may be done using `LA` or `LAM`, or some other computational tool. However, your solutions should show details of your calculations, not just the final answers. Show intermediate steps in your calculations, including any matrix inverses you compute. If answers can be given exactly as fractions, do so; do not give decimal approximations.
- There is an automatic penalty of 50% of the value of a problem for arithmetic mistakes. Please use appropriate software for your computations.
- Solutions should be written (or typed) neatly on one side only of clean paper with straight (not ragged) edges.
- Multiple pages should be stapled (not clipped or folded) together.

**A5.1.** The determinant is a multilinear alternating function. In this question we look at some other functions with similar properties.

If $V$, $W$ are vector spaces over a field $F$, then we can consider maps $f : V \times V \to W$, which take an ordered pair $(u, v)$ with $u, v \in V$ and give a vector $f(u, v) \in W$. Such a map $f$ is called *bilinear* (a special case of multilinear) if it is linear in each of $u$ and $v$ separately, with the other one held fixed. This means that $f(\alpha u_1 + \beta u_2, v) = \alpha f(u_1, v) + \beta f(u_2, v)$ and $f(u, \alpha v_1 + \beta v_2) = \alpha f(u, v_1) + \beta f(u, v_2)$ for all $\alpha, \beta \in F$ and $u, u_1, u_2, v, v_1, v_2 \in V$.

(a) If $A \in F^{n\times n}$ then we can define $f_A : F^n \times F^n \to F$ by $f_A(x, y) = x^{\mathrm{T}} Ay$. (As usual, vectors in $F^n$ are considered to be column vectors.)

(i) Find $f_A(x, y)$ if $F = \mathbf{R}$, $n = 3$, $x = (1, 2, 3)$, $y = (-1, 0, 1))$ and $A = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -3 \\ 5 & 1 & -4 \end{bmatrix}$.

(ii) Show that any $f_A$ as described above (*not* the specific $f_A$ from (i)) is bilinear.

(b) A bilinear map $f : V \times V \to W$ is called *alternating* if $f(v, v) = 0$ for all $v \in V$, and *antisymmetric* if $f(u, v) = -f(v, u)$ for all $u, v \in V$.

(i) Prove that if a bilinear map is alternating then it is also antisymmetric. [Hint: look at $f(u + v, u + v)$.]

(ii) Prove that if a bilinear map is antisymmetric then it is also alternating, provided the characteristic of the field we are working in is not equal to 2. Make it clear where you use the hypothesis about the characteristic of the field.

(c) (i) Use (a)(ii) to show that the ordinary dot product $d : F^n \times F^n \to F$ by $d(x, y) = x^{\mathrm{T}} y$ is bilinear.

(ii) Prove that if $F = \mathbf{Z}_2$ and $n \geq 1$ then the dot product $d : \mathbf{Z}_2^n \times \mathbf{Z}_2^n \to \mathbf{Z}_2$ is antisymmetric but not alternating. [This shows that (b)(ii) definitely does not work if $F$ has characteristic 2.]

**A5.2.** (a) Compute the determinant $|A|$ below using elementary row operations. Describe each individual elementary row operation you use.

$$|A| = \begin{vmatrix} 5 & 2 & -1 & 3 \\ 4 & 1 & 0 & -4 \\ -2 & 1 & 3 & 5 \\ 6 & 7 & 2 & 2 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1 & x^2 & 1 & x \\ x^2 & 0 & x & 0 \\ x & 1 & x & x \\ 1 & x & x^3 & 1 \end{vmatrix}$$

(b) Compute the determinant $|B|$ above first using a cofactor expansion and then using the quick method for $3 \times 3$ determinants (forwards/backwards diagonals). Simplify your answer as much as possible.

(c) Compute the following determinant $|C|$ as efficiently as you can, showing details. Show all elementary row or column operations used, or explicitly describe all cofactor expansions used, and describe other rules you use.

$$|C| = \begin{bmatrix} 2 & 2 & 1 & 5 & 3 & 1 & 4 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 & 3 & 5 & 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 & 4 & 5 & 2 & 2 & 1 & 0 \\ 4 & 0 & 3 & 2 & 5 & 3 & 4 & 0 & 1 & 0 \\ 5 & 0 & 3 & 4 & 2 & 3 & 0 & 0 & 1 & 0 \\ 7 & 0 & 1 & 4 & 7 & 0 & 0 & 0 & 3 & 0 \\ 2 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 3 & 0 \\ 5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 92 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**A5.3. Determinant of a block upper triangular matrix.** Recall the determinant formulae for $A \in F^{n \times n}$,

$$\det A = \sum_{\pi \in S_n} \text{sgn}\,(\pi)\, A_{1,\pi(1)} A_{2,\pi(2)} \cdots A_{n,\pi(n)} \qquad (*),$$

or the more general formula

$$\det A = \sum_{\pi \in S(R,C)} \text{sgn}\,(\pi) \prod_{r \in R} A_{r,\pi(r)} \qquad (**)$$

if the rows of $A$ are labelled by $R$ and $C$, respectively, in order, and $S(R,C)$ is the set of bijections from $R$ to $C$.

Suppose that $A \in F^{n \times n}$ has block form $A = \begin{bmatrix} B & D \\ 0 & G \end{bmatrix}$ where $B \in F^{k \times k}$ and $G \in F^{(n-k)\times(n-k)}$ for some $k$ with $1 \le k \le n-1$. Let $N = \{1, 2, \ldots, n\}$, let $K = \{1, 2, \ldots, k\}$ (which labels the rows and columns of $B$), and let $L = N - K = \{k+1, k+2, \ldots, n\}$ (which labels the rows and columns of $G$).

(a) Considering the expansion in $(*)$, show that $A_{1,\pi(1)} A_{2,\pi(2)} \cdots A_{n,\pi(n)}$ can only be nonzero if the permutation $\pi \in S_n$ has $\pi(K) = K$ and $\pi(L) = L$.

(b) If $\pi \in S_n$ has $\pi(K) = K$ and $\pi(L) = L$, show that $\text{sgn}\,(\pi) = \text{sgn}\,(\pi_K)\,\text{sgn}\,(\pi_L)$, where $\pi_K \in S(K,K)$ is the restriction of $\pi$ to domain and codomain $K$, and $\pi_L \in S(L,L)$ is defined similarly. [Something that may help: if a bijection $\pi$ can be put into order-preserving form using $t$ transpositions then $\text{sgn}\,(\pi) = (-1)^t$.]

(c) Show that $\det A = \det B \det G$ by starting with $(*)$, using (a) to discard some zero terms, splitting the sum over $\pi$ into a double sum over $\pi_K$ and $\pi_L$, and then using (b) and finally $(**)$. Remember that $A_{ij} = B_{ij}$ when $i, j \in K$ and $A_{ij} = G_{ij}$ when $i, j \in L$. Explain each step you take.