Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 4 Solutions

A4.1. Use Gauss-Jordan elimination to find the inverse of the following matrix A, if it exists. Show all elementary row operations. If the inverse does not exist, explain how you know that.

$$A = \begin{bmatrix} 5 & -2 & 3 & -4 \\ 0 & 1 & -2 & 2 \\ 2 & -5 & 3 & 1 \\ -1 & 4 & 0 & -4 \end{bmatrix}.$$

Solution: We augment A with a 4×4 identity matrix and apply Gauss-Jordan elimination:

$$\left[\begin{array}{cccc|cccc} 5 & -2 & 3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 1 & 0 & 0 \\ 2 & -5 & 3 & 1 & 0 & 0 & 1 & 0 \\ -1 & 4 & 0 & -4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1' = R_1/5} \left[\begin{array}{cccc|cccc} 1 & -2/5 & 3/5 & -4/5 & 1/5 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 1 & 0 & 0 \\ 2 & -5 & 3 & 1 & 0 & 0 & 1 & 0 \\ -1 & 4 & 0 & -4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_3' = R_3 - 2R_1 \\ R_4' = R_4 + R_1}} \left[\begin{array}{cccc|cccc} 1 & -2/5 & 3/5 & -4/5 & 1/5 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -21/5 & 9/5 & 13/5 & -2/5 & 0 & 1 & 0 \\ 0 & 18/5 & 3/5 & -24/5 & 1/5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1' = R_1 + (2/5)R_2 \\ R_3' = R_3 + (21/5)R_2 \\ R_4' = R_4 - (18/5)R_2}} \left[\begin{array}{cccc|cccc} 1 & 0 & -1/5 & 0 & 1/5 & 2/5 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -33/5 & 11 & -2/5 & 21/5 & 1 & 0 \\ 0 & 0 & 39/5 & -12 & 1/5 & -18/5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3' = (-5/33)R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & -1/5 & 0 & 1/5 & 2/5 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5/3 & 2/33 & -7/11 & -5/33 & 0 \\ 0 & 0 & 39/5 & -12 & 1/5 & -18/5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1' = R_1 + (1/5)R_3 \\ R_2' = R_2 + 2R_3 \\ R_4' = R_4 - (39/5)R_3}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1/3 & 7/33 & 3/11 & -1/33 & 0 \\ 0 & 1 & 0 & -4/3 & 4/33 & -3/11 & -10/33 & 0 \\ 0 & 0 & 1 & -5/3 & 2/33 & -7/11 & -5/33 & 0 \\ 0 & 0 & 0 & 1 & -3/11 & 15/11 & 13/11 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1' = R_1 + (1/3)R_4 \\ R_2' = R_2 + (4/3)R_4 \\ R_3' = R_3 + (5/3)R_4}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4/33 & 8/11 & 4/11 & 1/3 \\ 0 & 1 & 0 & 0 & -8/33 & 17/11 & 14/11 & 4/3 \\ 0 & 0 & 1 & 0 & -13/33 & 18/11 & 20/11 & 5/3 \\ 0 & 0 & 0 & 1 & -3/11 & 15/11 & 13/11 & 1 \end{array}\right].$$

Hence, we conclude that

$$A^{-1} = \begin{bmatrix} 4/33 & 8/11 & 4/11 & 1/3 \\ -8/33 & 17/11 & 14/11 & 4/3 \\ -13/33 & 18/11 & 20/11 & 5/3 \\ -3/11 & 15/11 & 13/11 & 1 \end{bmatrix}.$$

A4.2. Use Gaussian elimination to solve the following system of equations in the field $\mathbf{Z}_3$. Show all elementary row operations. Explicitly list all elements of your final solution set individually. Do not use any minus signs in your final answer.

[Use `LAM`. Don't forget to use the `mo` command to set the modulus for computations to 3.]

$$\begin{array}{rcrcrcrcrcl} & & x_2 & + & 2x_3 & + & 2x_4 & + & x_5 & = & 1 \\ x_1 & + & 2x_2 & + & 2x_3 & & & + & 2x_5 & = & 1 \\ x_1 & & & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & + & x_2 & & & + & 2x_4 & & & = & 0 \end{array}$$

Solution: We form the augmented matrix and proceed with Gaussian elimination:

Forward pass:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 0 & 0 \end{array}\right] \quad \rightarrow \quad \begin{array}{l} R_1' = R_2 \\ R_2' = R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad \begin{array}{l} \\ \\ R_3' = R_3 - R_1 \\ R_4' = R_4 - R_1 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{array}\right] \quad \rightarrow \quad \begin{array}{l} \\ \\ R_3' = R_3 - R_2 \\ R_4' = R_4 - 2R_2 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\rightarrow \quad \begin{array}{l} \\ \\ R_3' = R_3/2 \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \quad \rightarrow \quad \begin{array}{l} \\ \\ \\ R_4' = R_4 - R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and now backward pass:

$$\rightarrow \quad \begin{array}{l} R_2' = R_2 - 2R_3 \\ \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \rightarrow \quad \begin{array}{l} R_1' = R_1 - 2R_2 \\ \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which now has the LHS in reduced row echelon form. Converting back into a system and solving for the leading variables, we get

$$\begin{array}{rcrcrcrcl} x_1 & & & + & x_3 & & + & 2x_5 & = 2 \\ & & x_2 & + & 2x_3 & & & & = 1 \\ & & & & & x_4 & + & 2x_5 & = 0 \\ & & & & & & & 0 & = 0 \end{array} \quad \rightarrow \quad \begin{array}{rcrcr} x_1 & = & 2 - x_3 - 2x_5 & = & 2 + 2x_3 + x_5 \\ x_2 & = & 1 - 2x_3 & = & 1 + x_3 \\ x_4 & = & -2x_5 & = & x_5 \end{array}$$

Hence the solution set is

$$\begin{aligned} &\{(2 + 2x_3 + x_5, 1 + x_3, x_3, x_5, x_5) \mid x_3, x_5 \in \mathbf{Z}_3\} \\ &\qquad = \{(2,1,0,0,0), (0,1,0,1,1), (1,1,0,2,2), (1,2,1,0,0), (2,2,1,1,1), \\ &\qquad\quad (0,2,1,2,2), (0,0,2,0,0), (1,0,2,1,1), (2,0,2,2,2)\}. \end{aligned}$$

A4.3. The solution set of a system of linear equations in n variables is not always a vector subspace of F^n, but here we show that it has a related property. If U is a subset of a vector space V over a field F then we say that U is an *affine subspace* of V if for every $x, y \in U$ and every $\alpha \in F$ we also have $\alpha x + (1 - \alpha)y \in U$.

Prove that the solution set S of a system of linear equations $Ax = b$, where $A \in F^{m \times n}$ and $b \in F^m$, is an affine subspace of F^n.

Solution: We have that $S = \{x \in F^m \mid Ax = b\}$. Suppose that $x, y \in S$ and $\alpha \in F$. Then $Ax = b$ and $Ay = b$. Let $z = \alpha x + (1-\alpha)y$. Then

$$Az = A(\alpha x + (1-\alpha)y) = \alpha Ax + (1-\alpha)Ay = \alpha b + (1-\alpha)b = (\alpha + (1-\alpha))\, b = b$$

and hence $z \in S$. Thus, we have shown that for every $x, y \in S$ and $\alpha \in F$ we have $z = \alpha x + (1-\alpha)y \in S$. Hence, S is an affine subspace of F^n.

A4.4. One way to determine whether a collection of vectors $v_1, v_2, \dots, v_k$ is linearly independent is to consider the equation $a_1 v_1 + a_2 v_2 + \dots + a v_k = 0$. Usually this can be expressed as a system of linear equations in $a_1, a_2, \dots, a_k$ which we can solve to determine whether we have linear independence.

Use the approach in the previous paragraph to determine whether the collection of vectors $1 + 2x - 3x^2 + x^3 - x^4$, $-1 + 7x^2 - 5x^3 + 3x^4$, $1 + x - x^2 + 3x^3$, $3 + 2x + 3x^2 + 11x^3 + 3x^4$ in $P_4(\mathbf{R})$ is linearly independent. If they are linearly dependent, provide a specific nontrivial linear combination that is equal to the zero vector.

Solution: We set up the equation

$$a_1(1+2x-3x^2+x^3-x^4)+a_2(-1+7x^2-5x^3+3x^4)+a_3(1+x-x^2+3x^3)+a_4(3+2x+3x^2+11x^3+3x^4) = 0.$$

We convert this to a system of linear equations by taking coefficients of powers of x, and set up the augmented matrix.

$$\begin{array}{rrcrcrcrcl} [1]: & a_1 & - & a_2 & + & a_3 & + & 3a_4 & = & 0 \\ [x]: & 2a_1 & & & + & a_3 & + & 2a_4 & = & 0 \\ [x^2]: & -3a_1 & + & 7a_2 & - & a_3 & + & 3a_4 & = & 0 \\ [x^3]: & a_1 & - & 5a_2 & + & 3a_3 & + & 11a_4 & = & 0 \\ [x^4]: & -a_1 & + & 3a_2 & & & + & 3a_4 & = & 0 \end{array} \quad \to \quad \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 3 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ -3 & 7 & -1 & 3 & 0 \\ 1 & -5 & 3 & 11 & 0 \\ -1 & 3 & 0 & 3 & 0 \end{array}\right].$$

Using `gj` in `LA` to get the reduced row echelon form, converting back into a system (ignoring zero rows), and solving for leading variables we have

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -3/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \to \quad \begin{array}{l} a_1 - \frac{3}{2}a_4 = 0 \\ a_2 + \frac{1}{2}a_4 = 0 \\ a_3 + 5a_4 = 0 \end{array} \quad \to \quad \begin{array}{rcl} a_1 & = & \frac{3}{2}a_4 \\ a_2 & = & -\frac{1}{2}a_4 \\ a_3 & = & -5a_4 \end{array}$$

So the solution set is $\{(\frac{3}{2}a_4, -\frac{1}{2}a_4, -5a_4, a_4) \mid a_4 \in \mathbf{R}\}$. In particular, taking $a_4 = 2$ we get $(3, -1, -10, 2)$ and we have the nontrivial linear combination

$$3(1+2x-3x^2+x^3-x^4) - (-1+7x^2-5x^3+3x^4) - 10(1+x-x^2+3x^3) + 2(3+2x+3x^2+11x^3+3x^4) = 0$$

so these vectors are not linearly independent.

A4.5. Another way to determine whether a collection of vectors $v_1, v_2, \dots, v_k$ is linearly independent is to put the vectors themselves (if they belong to F^n) or their coordinate vectors with respect to a fixed basis B (more generally) as the rows of a matrix, and see if the matrix has *full row rank*, i.e., if its rank is equal to its number of rows. This can be done by reducing the matrix to row echelon (or reduced row echelon) form.

Working over the field $\mathbf{Z}_7$, use the approach in the previous paragraph to determine whether the collection of vectors (2, 1, 4, 5, 3, 6), (2, 0, 4, 0, 1, 0), (3, 1, 4, 5, 2, 0), (0, 6, 6, 3, 2, 5) in $\mathbf{Z}_7^6$ is linearly independent. What is the dimension of the subspace of $\mathbf{Z}_7^6$ spanned by these vectors?

Solution: Label the vectors as v_1, v_2, v_3, v_4 and let $V = \operatorname{span}\{v_1, v_2, v_3, v_4\}$. Putting the vectors as rows of a matrix A and reducing to reduced row echelon form R using `gj` in `LAM` gives

$$A = \begin{bmatrix} 2 & 1 & 4 & 5 & 3 & 6 \\ 2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 4 & 5 & 2 & 0 \\ 0 & 6 & 6 & 3 & 2 & 5 \end{bmatrix} \quad \to \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 6 & 1 \\ 0 & 1 & 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & 6 & 3 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

(You could also just reduce to row echelon form using `ga` in `LAM`.) Since elementary row operations preserve the rowspace, rowsp R = rowsp $A = V$. Since R is in row echelon form, its rank is the number of nonzero rows, which is 4: R has full row rank. Hence $4 = \dim \operatorname{rowsp} R = \dim V$. Our original set of vectors spans V and there are $4 = \dim V$ of them, so they are a basis for V and hence linearly independent.

A4.6. To determine whether a vector w belongs to the span of a set of vectors $\{v_1, v_2, \ldots, v_k\}$ we can try to solve the equation $a_1v_1 + a_2v_2 + \ldots a_kv_k = w$. Usually this can be expressed as a system of linear equations in $a_1, a_2, \ldots, a_k$.

Use the approach in the previous paragraph to determine whether the vector $w = (2, 2, 1, -3)$ belongs to $\operatorname{span}\{(5, 2, -1, 5), (1, 1, 2, -3), (3, 0, -1, 3), (2, 2, 3, -4), (1, 1, -1, 3), (3, 0, 1, -1)\}$. If it does belong to the span, give a specific linear combination of vectors in the set that equals w.

Solution: If w is in the span of the vectors then for some $a_1, a_2, \ldots, a_6$ we have

$$\begin{aligned} &a_1(5, 2, -1, 5) + a_2(1, 1, 2, -3) + a_3(3, 0, -1, 3) + a_4(2, 2, 3, -4) + a_5(1, 1, -1, 3) + a_6(3, 0, 1, -1) \\ &\qquad = (2, 2, 1, -3). \end{aligned}$$

We convert this into a system of linear equations by examining each coordinate, and set up the augmented matrix (notice that this has the vectors in our set as the columns on the left, and w as the extra column):

$$\begin{array}{rcrcrcrcrcrcr} 5a_1 &+& a_2 &+& 3a_3 &+& 2a_4 &+& a_5 &+& 3a_6 &=& 2 \\ 2a_1 &+& a_2 && &+& 2a_4 &+& a_5 && &=& 2 \\ -a_1 &+& 2a_2 &-& a_3 &+& 3a_4 &-& a_5 &+& a_6 &=& 1 \\ 5a_1 &-& 3a_2 &+& 3a_3 &-& 4a_4 &+& 3a_5 &-& a_6 &=& -3 \end{array} \quad \to \quad \left[\begin{array}{rrrrrr|r} 5 & 1 & 3 & 2 & 1 & 3 & 2 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 \\ -1 & 2 & -1 & 3 & -1 & 1 & 1 \\ 5 & -3 & 3 & -4 & 3 & -1 & -3 \end{array}\right]$$

We then reduce the augmented matrix to reduced row echelon form using `gj` in `LA`.

$$\left[\begin{array}{rrrrrr|r} 1 & 0 & 0 & 1/4 & 3/4 & -1/2 & 0 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1 & -1/4 & -3/4 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We see that the last row will give the impossible equation $0 = 1$, so the system is inconsistent, and w does not belong to the span of the given vectors.

A4.7. We know that when we multiply an $m \times n$ matrix A on the left by an invertible $m \times m$ matrix P, L_P gives an isomorphism from colsp PA to colsp A. Thus, if a set of columns of PA is a basis for the column space of PA, then the corresponding columns in A (same column numbers) are a basis for the column space of A. In particular, if R is a row echelon matrix obtained from A by doing elementary row operations, then we know that $R = PA$ for some invertible P. The columns of R corresponding to its leading entries form an obvious basis for colsp R, so the corresponding columns of A form a basis for colsp A.

Working with the field $\mathbf{Z}_2$, use the approach in the previous paragraph to find a subset of $S = \{(1, 1, 1, 1, 0),\ (0, 1, 1, 1, 1),\ (1, 0, 0, 0, 1),\ (1, 1, 0, 1, 1),\ (0, 0, 1, 0, 1),\ (1, 0, 1, 0, 0),\ (0, 1, 0, 1, 0),\ (1, 1, 0, 0, 0)\}$ that is a basis for span S in $\mathbf{Z}_2^5$. What is the dimension of span S?

Solution: Putting the vectors of S as the columns of a matrix A, we have span S = colsp A. Reducing A to its reduced row echelon form R using `gj` in `LAM`, we get

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \to \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that if P is the matrix representing the elementary row operations that transformed A into R, then L_P is an isomorphism from span S = colsp A to colsp R. We see that R has four nonzero rows and hence rank $R = 4$. The four columns of R corresponding to leading entries (columns $1, 2, 4, 8$) are linearly independent (they are standard basis vectors) and hence form a basis for colsp R. Hence, from the isomorphism, a basis for span S is given by columns $1, 2, 4, 8$ of A, so a basis is $\{(1, 1, 1, 1, 0),\ (0, 1, 1, 1, 1),\ (1, 1, 0, 1, 1),\ (1, 1, 0, 0, 0)\}$, so that dim span $S = 4$.