Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 4, due in class, Friday, 6th November

Remember:

- Solutions to problems should be fully explained, using clear English sentences where necessary.
- Matrix calculations may be done using `LA` or `LAM`, or some other computational tool. However, your solutions should show details of your calculations, not just the final answers. Show intermediate steps in your calculations, including any matrix inverses you compute. If answers can be given exactly as fractions, do so; do not give decimal approximations.
- There is an automatic penalty of 50% of the value of a problem for arithmetic mistakes. Please use appropriate software for your computations.
- Solutions should be written (or typed) neatly on one side only of clean paper with straight (not ragged) edges.
- Multiple pages should be stapled (not clipped or folded) together.

A4.1. Use Gauss-Jordan elimination to find the inverse of the following matrix A, if it exists. Show all elementary row operations. If the inverse does not exist, explain how you know that.

$$A = \begin{bmatrix} 5 & -2 & 3 & -4 \\ 0 & 1 & -2 & 2 \\ 2 & -5 & 3 & 1 \\ -1 & 4 & 0 & -4 \end{bmatrix}.$$

A4.2. Use Gaussian elimination to solve the following system of equations in the field $\mathbf{Z}_3$. Show all elementary row operations. Explicitly list all elements of your final solution set individually. Do not use any minus signs in your final answer.

[Use `LAM`. Don't forget to use the `mo` command to set the modulus for computations to 3.]

$$\begin{array}{rcrcrcrcrcl} & & x_2 & + & 2x_3 & + & 2x_4 & + & x_5 & = & 1 \\ x_1 & + & 2x_2 & + & 2x_3 & & & + & 2x_5 & = & 1 \\ x_1 & & & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & + & x_2 & & & + & 2x_4 & & & = & 0 \end{array}$$

A4.3. The solution set of a system of linear equations in n variables is not always a vector subspace of F^n, but here we show that it has a related property. If U is a subset of a vector space V over a field F then we say that U is an *affine subspace* of V if for every $x, y \in U$ and every $\alpha \in F$ we also have $\alpha x + (1 - \alpha)y \in U$.

Prove that the solution set S of a system of linear equations $Ax = b$, where $A \in F^{m \times n}$ and $b \in F^m$, is an affine subspace of F^n.

For all of the remaining questions, you do NOT have to show details of using elementary row operations to reach row echelon or reduced row echelon form. Note that `LA` and `LAM` have built-in commands `ga` and `gj` which perform Gaussian elimination (forward pass only) and Gauss-Jordan elimination to get row echelon or reduced row echelon form, respectively. You may use these.

Remember that you still have to explain and provide details for other parts of your working.

A4.4. One way to determine whether a collection of vectors $v_1, v_2, \ldots, v_k$ is linearly independent is to consider the equation $a_1v_1 + a_2v_2 + \ldots + av_k = 0$. Usually this can be expressed as a system of linear equations in $a_1, a_2, \ldots, a_k$ which we can solve to determine whether we have linear independence.

Use the approach in the previous paragraph to determine whether the collection of vectors $1 + 2x - 3x^2 + x^3 - x^4$, $-1 + 7x^2 - 5x^3 + 3x^4$, $1 + x - x^2 + 3x^3$, $3 + 2x + 3x^2 + 11x^3 + 3x^4$ in $P_4(\mathbf{R})$ is linearly independent. If they are linearly dependent, provide a specific nontrivial linear combination that is equal to the zero vector.

A4.5. Another way to determine whether a collection of vectors $v_1, v_2, \ldots, v_k$ is linearly independent is to put the vectors themselves (if they belong to F^n) or their coordinate vectors with respect to a fixed basis B (more generally) as the rows of a matrix, and see if the matrix has *full row rank*, i.e., if its rank is equal to its number of rows. This can be done by reducing the matrix to row echelon (or reduced row echelon) form.

Working over the field $\mathbf{Z}_7$, use the approach in the previous paragraph to determine whether the collection of vectors (2, 1, 4, 5, 3, 6), (2, 0, 4, 0, 1, 0), (3, 1, 4, 5, 2, 0), (0, 6, 6, 3, 2, 5) in $\mathbf{Z}_7^6$ is linearly independent. What is the dimension of the subspace of $\mathbf{Z}_7^6$ spanned by these vectors?

A4.6. To determine whether a vector w belongs to the span of a set of vectors $\{v_1, v_2, \ldots, v_k\}$ we can try to solve the equation $a_1v_1 + a_2v_2 + \ldots a_kv_k = w$. Usually this can be expressed as a system of linear equations in $a_1, a_2, \ldots, a_k$.

Use the approach in the previous paragraph to determine whether the vector $w = (2, 2, 1, -3)$ belongs to $\operatorname{span}\{(5, 2, -1, 5), (1, 1, 2, -3), (3, 0, -1, 3), (2, 2, 3, -4), (1, 1, -1, 3), (3, 0, 1, -1)\}$. If it does belong to the span, give a specific linear combination of vectors in the set that equals w.

A4.7. We know that when we multiply an $m \times n$ matrix A on the left by an invertible $m \times m$ matrix P, L_P gives an isomorphism from colsp PA to colsp A. Thus, if a set of columns of PA is a basis for the column space of PA, then the corresponding columns in A (same column numbers) are a basis for the column space of A. In particular, if R is a row echelon matrix obtained from A by doing elementary row operations, then we know that $R = PA$ for some invertible P. The columns of R corresponding to its leading entries form an obvious basis for colsp R, so the corresponding columns of A form a basis for colsp A.

Working with the field $\mathbf{Z}_2$, use the approach in the previous paragraph to find a subset of $S = \{(1,1,1,1,0),\ (0,1,1,1,1),\ (1,0,0,0,1),\ (1,1,0,1,1),\ (0,0,1,0,1),\ (1,0,1,0,0),\ (0,1,0,1,0),\ (1,1,0,0,0)\}$ that is a basis for span S in $\mathbf{Z}_2^5$. What is the dimension of span S?