Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 2 Solutions

A2.1. (a) Let P denote the vector space of continuous real 2π-periodic functions, so that $f(x+2\pi)=f(x)$ for every $f\in P$ and $x\in\mathbf{R}$. P is a vector space under pointwise addition and scalar multiplication. Let Q denote the set of real-valued infinite sequences $(s_1,s_2,s_3,\ldots)$. Q is a vector space under entrywise addition and scalar multiplication.

For $f\in P$, the *Fourier coefficients of* f are the numbers

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \quad \text{for } n=0,1,2,\ldots, \qquad b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \quad \text{for } n=1,2,3,\ldots.$$

(We don't need b_0 because it would just be 0.) Let $R:P\to Q$ map the function $f\in P$ to the sequence $(a_0,a_1,b_1,a_2,b_2,a_3,b_3,\ldots)\in Q$ of Fourier coefficients of f. Prove that R is a linear transformation.

[Hint: to avoid dealing with cases you might just think of the trigonometric functions involved in computing the Fourier coefficients as $h_1(x)$, $h_2(x)$, $h_3(x)$, $\ldots$.]

(b) If $z=a+bi\in\mathbf{C}$ with $a,b\in\mathbf{R}$, the *complex conjugate* of z is $\overline{z}=a-bi$. Is $J:\mathbf{C}\to\mathbf{C}$ with $J(z)=\overline{z}$ a linear transformation of complex vector spaces? Prove your answer.

Solution: (a) Suppose $Rf=(s_1,s_2,s_3,\ldots)$. Then each s_i can be written as $s_i=\int_0^{2\pi} f(x)h_i(x)\,dx$ where $h_1(x)=\frac{1}{\pi}\cos(0x)$ so that $s_1=a_0$, $h_2(x)=\frac{1}{\pi}\cos(1x)$ so that $s_2=a_1$, $h_3(x)=\frac{1}{\pi}\sin(1x)$ so that $s_3=b_1$, $h_4(x)=\frac{1}{\pi}\cos(2x)$ so that $s_4=a_2$, $h_5(x)=\frac{1}{\pi}\sin(2x)$ so that $s_5=b_2$, and so on. Thus,

$$(Rf)_i=\int_0^{2\pi} f(x)h_i(x)\,dx.$$

Now we want to show that if $f,g\in P$ and $\alpha,\beta\in\mathbf{R}$, $R(\alpha f+\beta g)=\alpha Rf+\beta Rg$. To do this, we must show that the ith element of these two sequences are the same, for all $i=1,2,3,\ldots$. We have

$$\begin{aligned}(R(\alpha f+\beta g))_i &= \int_0^{2\pi}(\alpha f+\beta g)(x)h_i(x)\,dx=\int_0^{2\pi}[\alpha f(x)+\beta g(x)]h_i(x)\,dx\\ &\qquad \text{by definition of addition, scalar multiplication of functions}\\ &=\alpha\int_0^{2\pi} f(x)h_i(x)\,dx+\beta\int_0^{2\pi} g(x)h_i(x)\,dx\\ &=\alpha(Rf)_i+\beta(Rg)_i=(\alpha Rf+\beta Rg)_i\\ &\qquad \text{by definition of addition, scalar multiplication of sequences.}\end{aligned}$$

Since this holds for all $i=1,2,3,\ldots$, $R(\alpha f+\beta g)=\alpha Rf+\beta Rg$, so R is a linear transformation.

(b) No, this is not a linear transformation because it does not preserve scalar multiplication by complex scalars. For example, $J(1)=1$, but $J(i\times 1)=J(i)=-i\neq i=i\times J(1)$.

[It *is* a linear transformation if we think of $\mathbf{C}$ as a real vector space.]

A2.2. Find a basis for the nullspace and a basis for the range for the linear transformation $T:\mathbf{R}^5\to\mathbf{R}^{2\times 2}$ by $T(x_1,x_2,x_3,x_4,x_5)=\begin{bmatrix} x_1+x_4 & x_2-2x_1-x_5\\ 0 & x_2+2x_4-x_5\end{bmatrix}$. In each case justify the fact that what you have found is a basis.

Solution: First, the nullspace:

$$\begin{aligned}
N(T) &= \{x \in \mathbf{R}^5 \mid Tx = 0\} \\
&= \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{R}^5 \mid \begin{bmatrix} x_1+x_4 & x_2-2x_1-x_5 \\ 0 & x_2+2x_4-x_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\} \\
&= \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{R}^5 \mid x_1+x_4=0,\ x_2-2x_1-x_5=0,\ x_2+2x_4-x_5=0\} \\
&= \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{R}^5 \mid x_1=-x_4,\ x_2=2x_1+x_5,\ x_2=-2x_4+x_5\} \\
&= \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{R}^5 \mid x_1=-x_4,\ x_2=-2x_4+x_5 \text{ (since } x_1=-x_4),\ x_2=-2x_4+x_5\} \\
&= \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{R}^5 \mid x_1=-x_4,\ x_2=-2x_4+x_5\} \\
&= \{(-x_4,-2x_4+x_5,x_3,x_4,x_5) \mid x_3,x_4,x_5 \in \mathbf{R}\} \\
&= \{x_3(0,0,1,0,0)+x_4(-1,-2,0,1,0)+x_5(0,1,0,0,1) \mid x_3,x_4,x_5 \in \mathbf{R}\} \\
&= \text{span}\,\{(0,0,1,0,0),(-1,-2,0,1,0),(0,1,0,0,1)\}
\end{aligned}$$

and so we claim that $B = \{(0,0,1,0,0),(-1,-2,0,1,0),(0,1,0,0,1)\}$ is a basis for $N(T)$. In addition to spanning $N(T)$, it is linearly independent because from above a typical linear combination with coefficients x_3,x_4,x_5 is $(-x_4,-2x_4+x_5,x_3,x_4,x_5)$ which from the last three coordinates is only 0 if $x_3=x_4=x_5=0$.

Now the range. Using the standard basis $S = \{e_1,e_2,e_3,e_4,e_5\}$ of $\mathbf{R}^5$, we have

$$\begin{aligned}
R(T) &= \text{span}\left\{Te_1 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, Te_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, Te_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, Te_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, Te_5 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}\right\} \\
&= \text{span}\left\{Te_1 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, Te_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, Te_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right\} \\
&\quad \text{discarding } Te_3 = 0 \text{ and } Te_5 = -Te_2 \\
&= \text{span}\left\{Te_1 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, Te_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\} \\
&\quad \text{discarding } Te_4 = Te_1 + 2Te_2
\end{aligned}$$

We know $\{Te_1, Te_2\}$ is a basis because $\dim R(T) = \text{rank}\, T = \dim \mathbf{R}^5 - \text{null}\, T = 5-3 = 2$, and this is a spanning set with 2 elements. (Or it is not hard to check directly that it is linearly independent.)

A2.3. All vector spaces in this problem are finite-dimensional vector spaces over the same field F.

(a) Suppose that $T \in L(V,W)$. Use the Rank-Nullity Theorem to prove that any two of the following three conditions imply the third.

(1) T is one-to-one.
(2) T is onto.
(3) $\dim V = \dim W$.

[It follows that T is an invertible, or an isomorphism, if any two of these three conditions hold.]

(b) Suppose that $T \in L(V,W)$ and $U \in L(W,X)$. Let $UT = U \circ T \in L(V,X)$ be the composition of T and U.

(i) Prove that $N(T) \subseteq N(UT)$, and use this to deduce that if UT is one-to-one, so is T.

(ii) Prove that $R(UT) \subseteq R(U)$, and use this to deduce that if UT is onto, so is U.

(iii) Use (a) and (i) and (ii) to show that if $\dim V = \dim W = \dim X = n$ (say) and UT is invertible, then T and U are both invertible.

Solution: (a) Note that (1) is equivalent to $e_1 = 0$ where $e_1 = \text{null}\, T$, (2) is equivalent to $e_2 = 0$ where $e_2 = \text{rank}\, T - \dim W$, and (3) is equivalent to $e_3 = 0$, where $e_3 = \dim V - \dim W$. By the Rank-Nullity Theorem, $e_3 = \dim V - \dim W = \text{rank}\, T + \text{null}\, T - \dim W = \text{null}\, T + (\text{rank}\, T - \dim W) = e_1 + e_2$. Since $e_3 = e_1 + e_2$, if two of these quantities are zero the third must also be zero. The result follows.

(b)(i) Suppose $v \in N(T)$. Then $Tv = 0$, so $UTv = U0 = 0$. So $v \in N(UT)$. Thus, $N(T) \subseteq N(UT)$.

If UT is $1-1$ then $N(UT) = \{0\}$. Then $\{0\} \subseteq N(T) \subseteq N(UT) = \{0\}$, so $N(T) = \{0\}$ and T is also $1-1$.

(ii) Suppose $x \in R(UT)$. Then $x = UTv$ for some $v \in V$. Thus, $x = U(Tv) = Uw$ where $w = Tv \in W$. Hence $x \in R(U)$. Thus, $R(UT) \subseteq R(U)$. [Alternative proof: $R(UT) = (UT)(V) = U(T(V)) \subseteq U(W) = R(U)$ because $T(V) = R(T) \subseteq W$.]

If UT is onto, then $R(UT) = X$. Hence $X = R(UT) \subseteq R(U) \subseteq X$, so $R(U) = X$, and U is also onto.

(iii) Suppose UT is invertible.

Because UT is $1-1$, T is $1-1$ by (i), so (1) holds for T. Also, $\dim V = \dim W$, so (3) holds for T. Thus, T is invertible by (a).

Because UT is onto, U is onto by (ii), so (2) holds for U. Also, $\dim W = \dim X$, so (3) holds for U. Thus, U is invertible by (a).

[General comment: the second part of (b)(i), and all of (b)(ii), are true for compositions of functions in general, not just linear transformations.]

A2.4. Suppose $A \in F^{m\times n}$ and $B \in F^{n\times p}$.

(a) Explain why $(AB)^{\mathrm{T}}$ and $B^{\mathrm{T}}A^{\mathrm{T}}$ both exist.

(b) Prove that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. (Use the definition of matrix multiplication given in class: $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$. Show that the ij entry of the first matrix is equal to the ij entry of the second matrix, for all suitable i and j. You should verify that the suitable i's and j's are the same for both matrices.)

Solution: (a) Since A is $m \times n$ and B is $n \times p$ we can form AB, which is $m \times p$. We can transpose any matrix, so we can form $(AB)^{\mathrm{T}}$, which is $p \times m$.

Since B^{T} is $p \times n$ and A^{T} is $n \times m$, we can form $B^{\mathrm{T}}A^{\mathrm{T}}$, which is $p \times m$.

(b) From (a), both matrices are $p \times m$, so we can consider the ij entry where $1 \le i \le p$ and $1 \le j \le m$. Then we have

$$((AB)^{\mathrm{T}})_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} (A^{\mathrm{T}})_{kj}(B^{\mathrm{T}})_{ik} = \sum_{k=1}^{n} (B^{\mathrm{T}})_{ik}(A^{\mathrm{T}})_{kj} = (B^{\mathrm{T}}A^{\mathrm{T}})_{ij}$$

and so $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.