Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 2, due in class, Friday, 9th October

Remember:

- Solutions to problems should be fully explained, using clear English sentences where necessary.
- Solutions should be written (or typed) neatly on one side only of clean paper with straight (not ragged) edges.
- Multiple pages should be stapled (not clipped or folded) together.

A2.1. (a) Let P denote the vector space of continuous real 2π-periodic functions, so that $f(x+2\pi) = f(x)$ for every $f \in P$ and $x \in \mathbf{R}$. P is a vector space under pointwise addition and scalar multiplication. Let Q denote the set of real-valued infinite sequences $(s_1, s_2, s_3, \ldots)$. Q is a vector space under entrywise addition and scalar multiplication.

For $f \in P$, the *Fourier coefficients of* f are the numbers

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \quad \text{for } n = 0, 1, 2, \ldots, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \quad \text{for } n = 1, 2, 3, \ldots.$$

(We don't need b_0 because it would just be 0.) Let $R : P \to Q$ map the function $f \in P$ to the sequence $(a_0, a_1, b_1, a_2, b_2, a_3, b_3, \ldots) \in Q$ of Fourier coefficients of f. Prove that R is a linear transformation.

[Hint: to avoid dealing with cases you might just think of the trigonometric functions involved in computing the Fourier coefficients as $h_1(x)$, $h_2(x)$, $h_3(x)$, ...]

(b) If $z = a + bi \in \mathbf{C}$ with $a, b \in \mathbf{R}$, the *complex conjugate* of z is $\overline{z} = a - bi$. Is $J : \mathbf{C} \to \mathbf{C}$ with $J(z) = \overline{z}$ a linear transformation of complex vector spaces? Prove your answer.

A2.2. Find a basis for the nullspace and a basis for the range for the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^{2\times 2}$ by $T(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} x_1 + x_4 & x_2 - 2x_1 - x_5 \\ 0 & x_2 + 2x_4 - x_5 \end{bmatrix}$. In each case justify the fact that what you have found is a basis.

A2.3. All vector spaces in this problem are finite-dimensional vector spaces over the same field F.

(a) Suppose that $T \in L(V, W)$. Use the Rank-Nullity Theorem to prove that any two of the following three conditions imply the third.

(1) T is one-to-one.
(2) T is onto.
(3) $\dim V = \dim W$.

[It follows that T is an invertible, or an isomorphism, if any two of these three conditions hold.]

(b) Suppose that $T \in L(V, W)$ and $U \in L(W, X)$. Let $UT = U \circ T \in L(V, X)$ be the composition of T and U.

(i) Prove that $N(T) \subseteq N(UT)$, and use this to deduce that if UT is one-to-one, so is T.

(ii) Prove that $R(UT) \subseteq R(U)$, and use this to deduce that if UT is onto, so is U.

(iii) Use (a) and (i) and (ii) to show that if $\dim V = \dim W = \dim X = n$ (say) and UT is invertible, then T and U are both invertible.

A2.4. Suppose $A \in F^{m\times n}$ and $B \in F^{n\times p}$.

(a) Explain why $(AB)^{\mathrm{T}}$ and $B^{\mathrm{T}}A^{\mathrm{T}}$ both exist.

(b) Prove that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. (Use the definition of matrix multiplication given in class: $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$. Show that the ij entry of the first matrix is equal to the ij entry of the second matrix, for all suitable i and j. You should verify that the suitable i's and j's are the same for both matrices.)