Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 1 Solutions

A1.1. Suppose V is a vector space, and W, X are two subspaces of V. Then the set $Y = W + X = \{w + x \mid w \in W, x \in X\}$ (every element of Y is the sum of an element of W and an element of X) is also a subspace of V. (This can be shown using the Subspace Theorem, but you do **not** have to do that.)

If W and X are subspaces, $Y = W + X$, and $W \cap X = \{0\}$, then we say that Y is the *direct sum* of W and X. Prove that if Y is the direct sum of W and X then every $y \in Y$ can be written as $y = w + x$ with $w \in W$, $x \in X$ in **exactly one** way.

Solution: [8] Suppose Y is the direct sum of W and X, so that $Y = W + X$, where W and X are subspaces of V, and $W \cap X = \{0\}$.

First, since $Y = W + X$, every $y \in Y$ can be written in at least one way as $y = w + x$ with $w \in W$, $x \in X$.

Second, suppose that $y \in Y$ can be written as both $y = w + x$ and $y = w' + x'$, where $w, w' \in W$ and $x, x' \in X'$. We will show that these ways of writing y must be the same, i.e., that $w' = w$ and $x' = x$. We have $w+x = y = w'+x'$, so $w-w' = x'-x$. Since W is a subspace, $w-w' \in W$, because subspaces are closed under subtraction. (This follows either from closure under addition and scalar multiplication, $w-w' = w+(-1)w'$, or from closure under addition and additive inverse (because a subspace is a vector space in its own right), $w - w' = w + (-w')$.) Similarly, $x' - x \in X$ because X is a subspace. So $w - w' = x' - x \in W \cap X = \{0\}$, hence $w - w' = x' - x = 0$, and thus $w = w'$ and $x = x'$. Therefore, any two ways of writing y as $y = w + x$ with $w \in W$ and $x \in X$ are the same: y cannot be written like this in two different ways.

Since every $y \in Y$ can be written as $w + x$ with $w \in W$ and $x \in X$ in at least one way and not in two different ways, y can be written like this in exactly one way.

A1.2. The set $C(\mathbf{R})$ of continuous real-valued functions on the real numbers is a vector space under pointwise addition and scalar multiplication of functions. Let T be the set

$$T = \{f \in C(\mathbf{R}) \mid \int_1^3 f(x)\,dx = f(4)\}.$$

For example, the function $f(x) = x$ belongs to T. Prove that T is a subspace of $C(\mathbf{R})$.

Solution: We check the conditions of the Subspace Theorem.

(SS1) We must show that the function that is zero everywhere, which is the zero vector of $C(\mathbf{R})$, belongs to T. Suppose $z(x) = 0$ for every x. We have

$$\int_1^3 z(x)\,dx = \int_1^3 0\,dx = 0, \quad \text{and} \quad z(4) = 0,$$

which are equal, and therefore $z \in T$ and (SS1) holds.

(SS2) Suppose that $f, g \in T$. Then we know that

$$\int_1^3 f(x)\,dx = f(4), \quad \text{and} \quad \int_1^3 g(x)\,dx = g(4).$$

We must show that $f + g \in T$. We have

$$\begin{aligned}\int_1^3 (f+g)(x)\,dx &= \int_1^3 (f(x) + g(x))\,dx \\ &= \int_1^3 f(x)\,dx + \int_1^3 g(x)\,dx \\ &= f(4) + g(4) = (f+g)(4)\end{aligned}$$

and therefore $f + g \in T$, and (SS2) holds.

(SS3) Suppose that $f \in T$ and $\alpha \in \mathbf{R}$. Then we know that

$$\int_1^3 f(x)\,dx = f(4).$$

We must show that $\alpha f \in T$. We have

$$\begin{aligned}\int_1^3 (\alpha f)(x)\,dx &= \int_1^3 \alpha f(x)\,dx \\ &= \alpha \int_1^3 f(x)\,dx \\ &= \alpha f(4) = (\alpha f)(4)\end{aligned}$$

and therefore $\alpha f \in T$, and (SS3) holds.

Since (SS1)–(SS3) hold, T is a subspace of $C(\mathbf{R})$ by the Subspace Theorem.

A1.3. (a) Compute $17 + 15$ and 17×15 in $\mathbf{Z}_{19}$.

(b) Suppose F is a field, and $E \subseteq F$ satisfies the following four conditions:

(i) E is closed under subtraction: $\alpha - \beta \in E$ for all $\alpha, \beta \in E$.

(ii) E is closed under multiplication: $\alpha\beta \in E$ for all $\alpha, \beta \in E$.

(iii) If $\alpha \in E - \{0\}$ then $\alpha^{-1} \in E$.

(iv) $E - \{0\} \neq \emptyset$.

Prove that E is a subfield of F. (You may use the Subfield Theorem given on the handout. Make sure you use the correct version, not the incorrect version given out initially.)

Solution: (a) $17 + 15 = 32 = 1 \times 19 + 13$ in $\mathbf{R}$, so $17 + 15 = 13$ in $\mathbf{Z}_{19}$.
$17 \times 15 = 255 = 190 + 65 = 247 + 8 = 13 \times 19 + 8$ in $\mathbf{R}$, so $17 \times 15 = 8$ in $\mathbf{Z}_{19}$.

(b) Suppose E satisfies (i)–(iv). We check the conditions of the Subfield Theorem.

(SF1) By (iv), $E - \{0\} \neq \emptyset$, so there is $\alpha \in E - \{0\}$. By (i), E is closed under subtraction, so $\alpha - \alpha = 0 \in E$, and (SF1) holds.

(SF2) As in (SF1), we have some $\alpha \in E - \{0\}$. By (iii), $\alpha^{-1} \in E$. By (ii), E is closed under multiplication, so $\alpha\alpha^{-1} = 1 \in E$. Since we know that $0, 1 \in E$ and by (i) E is closed under subtraction, $0 - 1 = -1 \in E$ also. So (SF2) holds.

(SF3) Suppose $\alpha, \beta \in E$. We have already shown that $-1 \in E$, and by (ii) E is closed under multiplication, so $(-1)\beta = -\beta \in E$. By (i) E is closed under subtraction, so $\alpha - (-\beta) = \alpha + \beta \in E$. Hence, (SF3) holds.

(SF4) is just (ii), so (SF4) holds.

(SF5) is just (iii), so (SF5) holds.

Since (SF1)–(SF5) hold, E is a subfield of F by the Subfield Theorem.

A1.4. For both parts of this question we are working in the real vector space $C(\mathbf{R})$ as defined in A1.2 above.

(a) Suppose $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = 5\cos(x - \frac{\pi}{7})$. Prove that $h \in \operatorname{span}\{f, g\}$.

(b) Consider the functions $f_1(x) = \cos x$, $f_2(x) = \cos(2x)$, and $f_3(x) = \cos(3x)$. Prove that f_1, f_2, f_3 is a linearly independent collection of functions in $C(\mathbf{R})$ by setting a linear combination of these three functions equal to 0, and then substituting in the particular values $x = 0$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$. (You will need to solve a system of linear equations, but it is quite simple.)

Solution: (a) We must prove that h is a linear combination of f and g, i.e., that there are scalars a and b so that $h = \alpha f + \beta g$, which means (since operations are defined pointwise) that $h(x) = \alpha f(x) + \beta g(x)$ for all $x \in \mathbf{R}$. We use the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$:

$$\begin{aligned} h(x) = 5\cos(x - \frac{\pi}{7}) &= 5\left(\cos x \cos\frac{\pi}{7} + \sin x \sin\frac{\pi}{7}\right) \\ &= 5\sin\frac{\pi}{7}\sin x + 5\cos\frac{\pi}{7}\cos x = 5\sin\frac{\pi}{7} f(x) + 5\cos\frac{\pi}{7} g(x) \qquad \forall\, x \in \mathbf{R}. \end{aligned}$$

Hence, $h = (5\sin\frac{\pi}{7})f + (5\cos\frac{\pi}{7})g \in \text{span}\,\{f, g\}$, as required. (Note that we do not care what the values of $\sin\frac{\pi}{7}$, $\cos\frac{\pi}{7}$ are: all that matters is that they are scalars, i.e., real numbers.)

(b) To prove linear independence, we need to suppose that there are scalars α, β, γ such that $\alpha f_1 + \beta f_2 + \gamma f_3 = 0$,

$$\begin{array}{lrcrcrcll} \text{i.e.,} & \alpha f_1(x) & + & \beta f_2(x) & + & \gamma f_3(x) & = & 0 & \forall\, x \in \mathbf{R}, \\ \text{i.e.,} & \alpha \cos x & + & \beta \cos(2x) & + & \gamma \cos(3x) & = & 0 & \forall\, x \in \mathbf{R}. \end{array}$$

We must show that $\alpha = \beta = \gamma = 0$. One way to get information about α, β, γ is by substituting some particular values for x:

$$\begin{array}{rl} x = 0: & \alpha\cos 0 + \beta\cos 0 + \gamma\cos 0 = 0 \qquad \text{i.e.} \qquad \alpha + \beta + \gamma = 0 \quad (1) \\ x = \frac{\pi}{4}: & \alpha\cos\frac{\pi}{4} + \beta\cos\frac{\pi}{2} + \gamma\cos\frac{3\pi}{4} = 0 \qquad \frac{\sqrt{2}}{2}\alpha + 0\beta - \frac{\sqrt{2}}{2}\gamma = 0 \quad (2) \\ x = \frac{\pi}{2}: & \alpha\cos\frac{\pi}{2} + \beta\cos\pi + \gamma\cos\frac{3\pi}{2} = 0 \qquad 0\alpha - \beta + 0\gamma = 0 \quad (3) \end{array}$$

Dividing (2) by $\frac{\sqrt{2}}{2}$, we get the system

$$\begin{array}{rcrcrcll} \alpha & + & \beta & + & \gamma & = & 0 & (1) \\ \alpha & & & - & \gamma & = & 0 & (4) \\ & - & \beta & & & = & 0 & (3) \end{array}$$

This is such a simple system that we can solve it without using our heavy machinery: from (3), $\beta = 0$, so that (1) becomes $\alpha + \gamma = 0$. However, (4) says that $\alpha - \gamma = 0$ and adding we get $2\alpha = 0$ so that $\alpha = 0$, and then $\gamma = 0$.

Since the only possible scalars are $\alpha = \beta = \gamma = 0$, we conclude that the functions f_1, f_2 and f_3 are linearly independent, as required.