Math 2600/5600 - Linear Algebra - Fall 2015

Assignment 1, due in class, Friday, 18th September

Remember:

- Solutions to problems should be fully explained, using clear English sentences where necessary.
- Solutions should be written (or typed) neatly on one side only of clean paper with straight (not ragged) edges.
- Multiple pages should be stapled (not clipped or folded) together.

A1.1. Suppose V is a vector space, and W, X are two subspaces of V. Then the set $Y = W + X = \{w + x \mid w \in W, x \in X\}$ (every element of Y is the sum of an element of W and an element of X) is also a subspace of V. (This can be shown using the Subspace Theorem, but you do **not** have to do that.)

If W and X are subspaces, $Y = W + X$, and $W \cap X = \{0\}$, then we say that Y is the *direct sum* of W and X. Prove that if Y is the direct sum of W and X then every $y \in Y$ can be written as $y = w + x$ with $w \in W$, $x \in X$ in **exactly one** way.

A1.2. The set $C(\mathbf{R})$ of continuous real-valued functions on the real numbers is a vector space under pointwise addition and scalar multiplication of functions. Let T be the set

$$T = \{f \in C(\mathbf{R}) \mid \int_1^3 f(x)\,dx = f(4)\}.$$

For example, the function $f(x) = x$ belongs to T. Prove that T is a subspace of $C(\mathbf{R})$.

A1.3. (a) Compute $17 + 15$ and 17×15 in $\mathbf{Z}_{19}$.

(b) Suppose F is a field, and $E \subseteq F$ satisfies the following four conditions:

(i) E is closed under subtraction: $\alpha - \beta \in E$ for all $\alpha, \beta \in E$.

(ii) E is closed under multiplication: $\alpha\beta \in E$ for all $\alpha, \beta \in E$.

(iii) If $\alpha \in E - \{0\}$ then $\alpha^{-1} \in E$.

(iv) $E - \{0\} \neq \emptyset$.

Prove that E is a subfield of F. (You may use the Subfield Theorem given on the handout. Make sure you use the correct version, not the incorrect version given out initially.)

A1.4. For both parts of this question we are working in the real vector space $C(\mathbf{R})$ as defined in A1.2 above.

(a) Suppose $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = 5\cos(x - \frac{\pi}{7})$. Prove that $h \in \operatorname{span}\{f, g\}$.

(b) Consider the functions $f_1(x) = \cos x$, $f_2(x) = \cos(2x)$, and $f_3(x) = \cos(3x)$. Prove that f_1, f_2, f_3 is a linearly independent collection of functions in $C(\mathbf{R})$ by setting a linear combination of these three functions equal to 0, and then substituting in the particular values $x = 0$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$. (You will need to solve a system of linear equations, but it is quite simple.)