The zig-zag property and exponential cancellation of ordered sets

Ralph McKenzie
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240

December, 2003

1 Introduction

Around 1940, in an effort to unify the arithmetic of cardinal and ordinal numbers, Garrett Birkhoff [3, 4] defined several operations on ordered sets (i.e. posets), of which at least one, (cardinal) exponentiation, is of considerable independent interest: for ordered sets $\mathbf{P}$ and $\mathbf{Q}$, let $\mathbf{P}^{\mathbf{Q}}$ denote the ordered set $\langle \mathrm{M}(\mathbf{Q},\mathbf{P}), \leq \rangle$ where $\mathrm{M}(\mathbf{Q},\mathbf{P})$ is the set of all monotone functions from $\mathbf{Q}$ into $\mathbf{P}$ and for two monotone functions f, g we put $f \leq g$ iff $f(x) \leq g(x)$ for each $x \in Q$.

G. Birkhoff discussed the cancellation problems for this operation in [4] (see page 300). He observed that $\mathbf{C}^{\mathbf{A}} \cong \mathbf{C}^{\mathbf{B}}$ does not imply $\mathbf{A} \cong \mathbf{B}$. (For example, if $\mathbf{C}$ is totally unordered and $\mathbf{A}$ is connected, then $\mathbf{C}^{\mathbf{A}} \cong \mathbf{C}$.) He conjectured that $\mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{C}}$ implies $\mathbf{A} \cong \mathbf{B}$ whenever $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are finite. (We always assume that ordered sets are non-empty, unless otherwise stated.) This "exponential cancellation conjecture", in the case that $\mathbf{C}$ is totally unordered, was finally proved by L. Lovasz [19] in 1967. {L. Lovasz's celebrated theorem is, of course, much more general than this. His result applies to finite direct powers of arbitrary structures with a finite number of elements.} It interests me that G. Birkhoff ([4], p. 300) attributes the special conjecture,

$\mathbf{A}^2 \cong \mathbf{B}^2$ implies $\mathbf{A} \cong \mathbf{B}$ for finite ordered sets $\mathbf{A}$ and $\mathbf{B}$, to Stanislaw Ulam.

This paper is a survey of work on G. Birkhoff's exponential cancellation conjecture, leading up to its proof in 2000. It will be largely the tale of two important ideas, the logarithmic property and the zig-zag property. The proof revealed an interesting algebraic structure, the algebra of isomorphism types of finite ordered sets under cardinal sum, cardinal product, and an operation $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$ which is a refinement of Birkhoff exponentiation. (This algebra is described here in §9.) The machinery of the proof supplies tools which can be used to explore the structure of the groups $\mathrm{Aut}(\mathbf{A}^{\mathbf{B}})$ for finite $\mathbf{A}$ and $\mathbf{B}$. These groups have interesting properties, and we pursue this topic in §10.

Many results from the literature will be proved here, and several new results are proved. Theorems 5.2 and 5.3 are strengthened versions of results from B. Jónsson, R. McKenzie [17]. The proof given for Theorem 9.2 is new. Theorems 10.3, 10.4, 10.5, 10.6 and 10.7 are new.

The paper was conceived while I conducted a graduate seminar on this subject at Vanderbilt University in Spring 2003, and I am indebted to all the members of this seminar—William Funk, Nikolaos Galatos, Marcin Koczik, Ashot Minasyan, Alexey Muranov, David Stanovsky and Annika Wille—especially for their fresh perspective, but also for their enthusiasm and good ideas.

Except for L. Lovasz [19] and two other papers, M. M. Day [8] and M. Novotný [22], nothing more was written on exponentiation of ordered sets between 1942 and 1978. The subject was revived in 1978 by I. Rival who, in collaboration with others, wrote three papers on various aspects of this operation: [7], [9], [10]. In the first of these papers, I. Rival and D. Duffus [9] defined the "logarithm" of an ordered set, $\log(\mathbf{P})$, to be the ordered subset consisting of the strictly join-irreducible elements of $\mathbf{P}$, i.e., the elements p for which the set of elements strictly less than p has a largest member. They proved the "logarithmic property": $\log(\mathbf{P}^{\mathbf{Q}}) \cong \log(\mathbf{P}) \times \mathbf{Q}^{\partial}$ in the case where $\mathbf{P}$ has a least element. Using this, they proved that if $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are finite ordered sets and $\mathbf{C}$ is not an anti-chain, then $\mathbf{C}^{\mathbf{A}} \cong \mathbf{C}^{\mathbf{B}}$ implies $\mathbf{A} \cong \mathbf{B}$. They also proved that if $\mathbf{A}$ and $\mathbf{B}$ are finite lattice-ordered sets, and $\mathbf{C}$ is finite and bounded—i.e., has a least and a greatest element—then $\mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{C}}$ implies $\mathbf{A} \cong \mathbf{B}$. The authors stated that their work was inspired by G. Birkhoff's fundamental representation theorem for finite distributive lattices: If $\mathbf{D}$ is a finite distributive lattice, then $\mathbf{D}$ is order-isomorphic with $\mathbf{2}^{\mathbf{Q}}$ where $\mathbf{2}$ denotes the two-element chain and, in this representation, $\mathbf{Q}$ is, up to isomorphism, uniquely determined as the dual of the ordered set $\log(\mathbf{D})$.

D. Duffus and I. Rival [9] inspired other people to study this topic, and a half-dozen papers in the arithmetic of ordered sets appeared over the next four years: [1], [2], [11], [12], [17], [18]. The idea that ordered sets have logarithms obeying the logarithmic property was the principal contribution of Ivan Rival to this subject—and it is fundamental, although the specific logarithm employed has varied through the years. (The logarithm introduced in B. Jónsson, R. McKenzie [17] and employed in the paper you are reading is different; and two other logarithms are employed in D. Duffus, R. Wille [11] and J. D. Farley [13].)

B. Jónsson and R. McKenzie [17] introduced a host of new techniques, including an improved concept of the logarithm of an ordered set, and proved an assortment of mixed refinement and cancellation results relating the operations of direct product, disjoint union, and exponentiation on ordered sets. Notably, they obtained cancellation of the exponent, $\mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{C}}$ implies $\mathbf{A} \cong \mathbf{B}$, whenever these ordered sets are all connected, one of $\mathbf{A}$ or $\mathbf{C}$ has a least or greatest element, and $\mathbf{A}^{\mathbf{C}}$ has the descending chain condition.

B. Jónsson [18] proved that the automorphism group of $\mathbf{A}^{\mathbf{B}}$ is naturally isomorphic to $\mathrm{Aut}(\mathbf{A}) \times \mathrm{Aut}(\mathbf{B})$, provided that $\mathbf{A}^{\mathbf{B}}$ is directly indecomposable, bounded, and has the descending chain condition, and $\mathbf{A}$ is exponentially indecomposable.

Following the publication in 1982 of D. Duffus [12], B. Jónsson and R. McKenzie [17] and B. Jónsson [18], and the Darmstadt dissertation of H. Bauer [1], our subject became dormant until 1996. In that year, J. D. Farley [13] appeared, with significant extensions of most of the known results about groups of the form $\mathrm{Aut}(\mathbf{A}^{\mathbf{B}})$. J. D. Farley dealt primarily with function lattices $\mathbf{A}^{\mathbf{B}}$—that is, he assumed that $\mathbf{A}$ is a lattice-ordered set—but obtained B. Jónsson's result cited above through the use of Dedekind-MacNeille completions. Under the assumptions in B. Jónsson's result, the Dedekind-MacNeille completion of $\mathbf{A}^{\mathbf{B}}$, or $\mathbf{DM}(\mathbf{A}^{\mathbf{B}})$, is isomorphic to $\mathbf{DM}(\mathbf{A})^{\mathbf{B}}$ and he showed that if the natural map $\mathrm{Aut}(\mathbf{DM}(\mathbf{A})) \times \mathrm{Aut}(\mathbf{B}) \to \mathrm{Aut}(\mathbf{DM}(\mathbf{A})^{\mathbf{B}})$ is an isomorphism, then the same is true for the natural map $\mathrm{Aut}(\mathbf{A}) \times$

$\mathrm{Aut}(\mathbf{B}) \to \mathrm{Aut}(\mathbf{A}^{\mathbf{B}})$.

J. Farley pointed out that G. Birkhoff's 1942 conjecture—$\mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{C}}$ implies $\mathbf{A} \cong \mathbf{B}$ for finite ordered sets $\mathbf{A}, \mathbf{B}, \mathbf{C}$—was still unsolved, and stated that, in his opinion, this was the most important open problem in the theory of finite ordered sets. He observed that the best result on the problem up to that time was to be found in B. Jónsson, R. McKenzie [17] and it amounted to this: the conjectured implication is true whenever $\mathbf{A}$ or $\mathbf{C}$ has a least element or a greatest element and both are connected. Spurred by Farley's results, and his interest in the problem, I began to work on the Birkhoff conjecture once more. I succeeded in proving the full conjecture early in 2000.

R. McKenzie [21] contains my proof of the Birkhoff conjecture. That proof is not as transparent as it should be, due to the abrupt way in which I introduced a result from B. Jónsson, R. McKenzie [17], a result with an intricate proof which was stated in [17] in a different formalism and not in the form needed in [21]. For that reason and others, I think there is value in re-visiting this proof. The key idea of the proof is the "zig-zag property" of an isomorphism $\varphi : \mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{C}}$. This property had been introduced by B. Jónsson in 1977 in his unpublished proof that in this situation, if $\mathbf{C}$ is the two-element chain, then $\mathbf{A} \cong \mathbf{B}$. It was Jónsson's proof that inspired C. Bergman, R. McKenzie and Sz. Nagy [2] to show that φ has the zig-zag property whenever $\mathbf{C}$ is a chain (this is rather non-trivial), thus proving that $\mathbf{A} \cong \mathbf{B}$ in this case. The zig-zag property played a central role for obtaining results on exponential cancellation also in [17], [20] and [21].

To formulate the zig-zag property, we introduce some notation. Let $\mathbf{A}$ and $\mathbf{P}$ be ordered sets. A *diagonal element* in $\mathbf{A}^{\mathbf{P}}$ is a function which for some $x \in A$ is the unique function in $\{x\}^P$. This constant function is usually denoted $\langle x \rangle$, with $\mathbf{P}$ understood. Suppose that we have an isomorphism $\varphi : \mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$ where $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are ordered sets.

Definition 1.1 We say that φ has the *zig-zag property* provided that for all $a \in A$, there are $a_t \in A, b_t \in B$ for all $t \in P$ and there is $b \in B$ such that

$$\begin{aligned} \varphi(\langle a \rangle) &= \langle b_t : t \in P \rangle \\ \varphi(\langle a_t \rangle) &= \langle b_t \rangle \ \text{ for all } t \in P \text{ and} \\ \varphi(\langle a_t : t \in P \rangle) &= \langle b \rangle \,. \end{aligned} \tag{1}$$

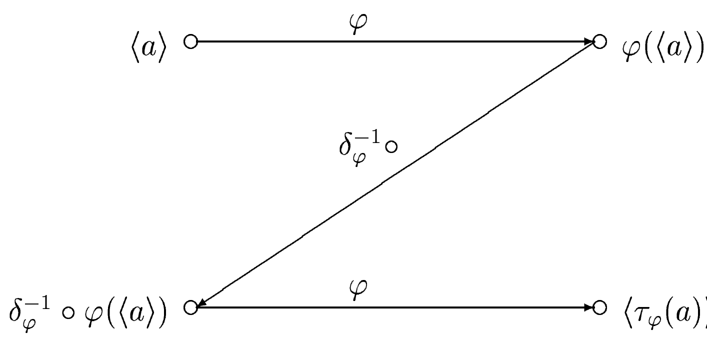

Figure 1: The zig-zag property

If φ has the zig-zag property, we write τ_φ for the function $A \to B$ such that $\tau_\varphi(a) = b$ when a_t, b_t $(t \in P)$ exist satisfying (1) above.

$R(\varphi)$ denotes the set of all $a \in A$ such that there is $b \in B$ with $\varphi(\langle a\rangle) = \langle b\rangle$. For such a and b, we write $\delta_\varphi(a) = b$. Thus δ_φ is defined as a function mapping $R(\varphi)$ onto $R(\varphi^{-1})$.

It is clear that δ_φ is an order-isomorphism between the ordered sets induced by $\mathbf{A}$ on $R(\varphi)$ and induced by $\mathbf{B}$ on $R(\varphi^{-1})$. If φ has the zig-zag property, then we have the equation

$$\langle\tau_\varphi(a)\rangle \quad = \quad \varphi(\delta_\varphi^{-1}\circ\varphi(\langle a\rangle))\,, \tag{2}$$

for $a \in A$. (Here, $f \circ g$ denotes the composition of functions f and g.) The reader will verify that if both φ and φ^{-1} possess the zig-zag property, then $\tau_\varphi : \mathbf{A} \cong \mathbf{B}$.

I call it the zig-zag property because b is produced from a by a sequence of three steps that move back and forth between $\mathbf{A}^\mathbf{P}$ and $\mathbf{B}^\mathbf{P}$ ("zigging and zagging"): compute $\varphi(\langle a\rangle) \in \mathrm{M}(\mathbf{P},\mathbf{B})$, compose this monotone function from $\mathbf{P}$ to $R(\varphi^{-1})$ with δ_φ^{-1} to produce the monotone function $\delta_\varphi^{-1}\circ\varphi(\langle a\rangle) \in \mathrm{M}(\mathbf{P}, R(\varphi))$, finally apply φ a second time to produce $\langle b\rangle = \varphi(\delta_\varphi^{-1}\circ\varphi(\langle a\rangle))$.

Here is the original result that introduced the zig-zag property.

Theorem 1.1 (B. Jónsson, 1977) $\mathbf{A^2} \cong \mathbf{B^2}$ *implies* $\mathbf{A} \cong \mathbf{B}$.

PROOF. Suppose that $\varphi : \mathbf{A^2} \cong \mathbf{B^2}$. We show that φ has the zig-zag property. The same proof of course will show that φ^{-1} has the zig-zag property. Thus τ_φ will be an isomorphism of $\mathbf{A}$ with $\mathbf{B}$.

We write elements of $\mathbf{A^2}$ (or $\mathbf{B^2}$) as (x, y) where $\{x, y\} \subseteq A$ (or $\{x, y\} \subseteq B$) and $x \leq y$. We have to show that for each $a \in A$, there are $a_i \in A$, $b, b_i \in B$ ($i \in \{0, 1\}$) such that $\varphi(a, a) = (b_0, b_1)$, $\varphi(a_i, a_i) = (b_i, b_i)$, $\varphi(a_0, a_1) = (b, b)$.

So let $\varphi(a, a) = (b_0, b_1)$, $b_0 \leq b_1$. Suppose that $\varphi(c, d) = (b_0, b_0)$. Let $\varphi(c, c) = (p, q)$, and then put $\varphi(x, y) = (p, b_1)$. Now $(p, b_1) \vee (b_0, b_0) = (b_0, b_1)$ in $\mathbf{B^2}$, hence $(x, y) \vee (c, d) = (a, a)$ in $\mathbf{A^2}$. However, clearly, $a \geq x \geq c$ and so (x, a) is an upper bound of both (x, y) and (c, d). It follows that $x = a$, implying $y = a$ and $p = b_0 = q$. This means that $\varphi(c, c) = (b_0, b_0)$. Now write a_0 for c.

The dual argument shows that $\varphi(a_1, a_1) = (b_1, b_1)$ for a unique $a_1 \in A$.

Now write $\varphi(a_0, a_1) = (u, v)$. We need to show that $u = v$, so put $\varphi(a_0', a_1') = (v, v)$. Then $a_0 \leq a_0' \leq a_1' = a_1$ and $b_0 \leq u \leq v \leq b_1$. We have

$$\mathbf{B^2} \models (b_0, b_1) \wedge (u, v) = (b_0, v) = (b_0, b_1) \wedge (v, v)\,;$$

and so

$$\mathbf{A^2} \models (a, a) \wedge (a_0, a_1) = (a_0, a) = (a, a) \wedge (a_0', a_1')\,.$$

This means that $a \wedge a_0' = a_0$ in $\mathbf{A}$. Now by what was proved above, we have that $\varphi(a_0', a_0') = (w, w)$ for some w. Then $(a, a) \wedge (a_0', a_0') = (a_0, a_0)$ yields $(b_0, b_1) \wedge (w, w) = (b_0, b_0)$. Since $w \leq b_1$, this means that $w = b_0$, or $a_0' = a_0$, or finally, $u = v$ as desired. •

This paper will tell the story of the zig-zag property, a surprisingly ubiquitous fact that underlies exponential cancellation theory. We know a variety of situations in which the property is guaranteed to hold, but we have only a slender set of examples of its failure (see Example 5.1). To delineate more clearly the true domain of validity of this property is a most interesting open problem.

We shall also be examining the arithmetic of ordered sets under combined operations $\mathbf{A} + \mathbf{B}$ (disjoint union), $\mathbf{A} \times \mathbf{B}$ (direct product) and $\mathbf{A}^{\mathbf{B}}$. All of the "high school identities"—simple equational laws relating addition, multiplication and exponentiation over positive integers—are valid in this arithmetic. They are also valid in the arithmetic obtained when Birkhoff's exponentiation $\mathbf{A}^{\mathbf{B}}$ is replaced by a related construction, $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$. Moreover, the algebra of isomorphism types of finite ordered sets under the operations $\mathbf{A}+\mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ and $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$ is an interesting recursive algebra. The construction $\mathcal{C}(\mathcal{A}^{\mathbf{B}})$, defined in the next section, is the second key idea in my proof of Birkhoff's conjecture.

2 Basic Notions

We use boldface capital letters to denote ordered sets, i.e. posets, throughout this paper; thus $\mathbf{P} = \langle P, \leq \rangle$ where P is the set of elements (or points) of $\mathbf{P}$. Unless otherwise noted, P is assumed to be non-empty.

A *subposet* $\mathbf{Q}$ of a poset $\mathbf{P}$ is a subset Q of P together with the restriction to Q of the order relation of $\mathbf{P}$. Frequently, we write Q, rather than $\mathbf{Q}$, to denote this subposet.

Let $\mathbf{A}$ be an ordered set and $x, y \in A$. A *path* connecting x to y is a finite sequence $x_0, \ldots, x_n$ of elements of $\mathbf{A}$ such that $x_0 = x$, $x_n = y$ and for all $i < n$, $x_i \leq x_{i+1}$ or $x_{i+1} \leq x_i$ (we say that x_i and x_{i+1} are *comparable*). Define the distance between x, and y, $d_{\mathbf{A}}(x, y)$ to be the least n such that there exists a path $x = x_0, \ldots, x_n = y$, or if there is no such path, $d_{\mathbf{A}}(x, y) = \infty$. Define the *diameter* of $\mathbf{A}$, or $\operatorname{diam}(\mathbf{A})$, to be the supremum of all numbers $d_{\mathbf{A}}(x, y)$, $(x, y \in A)$.

We write $\mathbf{A} + \mathbf{B}$ for the (cardinal) *sum* of two ordered sets $\mathbf{A}$ and $\mathbf{B}$. It is determined only up to isomorphism. If $A \cap B = \emptyset$ then the universe of $\mathbf{A} + \mathbf{B}$ is $A \cup B$, and $\mathbf{A}$ and $\mathbf{B}$ are subposets of $\mathbf{A} + \mathbf{B}$ such that no element of $\mathbf{A}$ is comparable to any element of $\mathbf{B}$. If $A \cap B \neq \emptyset$, then $\mathbf{A} + \mathbf{B} \cong \mathbf{A}' + \mathbf{B}'$ where $\mathbf{A} \cong \mathbf{A}'$, $\mathbf{B} \cong \mathbf{B}'$, and $A' \cap B' = \emptyset$. An ordered set $\mathbf{A}$ is *connected*, or $+$ *-indecomposable*, iff there do not exist non-empty ordered sets $\mathbf{C}$, $\mathbf{D}$ with $\mathbf{A} \cong \mathbf{C} + \mathbf{D}$. The *connected components* of $\mathbf{A}$ are the maximal connected subposets of $\mathbf{A}$. Note that the connected component of $\mathbf{A}$ containing an element a is nothing other than $\{x \in A : d_{\mathbf{A}}(x, a) < \infty\}$, and $\mathbf{A}$ is connected iff for all $x, y \in A$, $d_{\mathbf{A}}(x, y) < \infty$.

We write $\mathbf{A} \times \mathbf{B}$ (or sometimes, $\mathbf{AB}$) for the (cardinal) *product* of two ordered sets $\mathbf{A}$ and $\mathbf{B}$. It is the direct product, $A \times B$, ordered so that $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$. An ordered set is *trivial* iff its universe is a one-element set. An ordered set $\mathbf{A}$ is *directly indecomposable*, or $\times$-*indecomposable*, iff $\mathbf{A}$ is not trivial, and $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ always implies that $\mathbf{B}$ or $\mathbf{C}$ is trivial.

Two other operations introduced by G. Birkhoff play almost no role in this paper. Assuming that the ordered sets $\mathbf{A}$ and $\mathbf{B}$ are disjoint, their *ordinal sum*, $\mathbf{A} \oplus \mathbf{B}$, is the set-union of A and B, ordered so that $\mathbf{A}$ and $\mathbf{B}$ are sub-posets and for all $x \in A$ and $y \in B$ holds $x < y$. The *ordinal product*, $\mathbf{A} \otimes \mathbf{B}$, is the direct product ordered lexicographically, so that $(x, y) < (x', y')$ iff $x < x'$ or $x = x'$ and $y < y'$.

Definition 2.1 For ordered sets $\mathbf{A}$ and $\mathbf{B}$, we use $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$ to denote the sub-poset of $\mathbf{A}^{\mathbf{B}}$ consisting of all monotone functions f such that

$$d_{\mathbf{A}^{\mathbf{B}}}(f, g) < \infty$$

for some function g which is constant on each connected component of $\mathbf{B}$.

3 Hashimoto Theorem

The Hashimoto theorem is the basic result about isomorphic direct product decompositions of ordered sets.

Theorem 3.1 (J. Hashimoto [16]) *Suppose that* $\mathbf{C}$ *is a connected ordered set and* $\mathbf{C} \cong \prod_{s \in S} \mathbf{A}_s \cong \prod_{t \in T} \mathbf{B}_t$. *Then there are ordered sets* $\mathbf{C}_{s,t}$ *for* $(s, t) \in S \times T$ *so that for all* $s_0 \in S$ *and* $t_0 \in T$, $\mathbf{A}_{s_0} \cong \prod_{t \in T} \mathbf{C}_{s_0,t}$ *and* $\mathbf{B}_{t_0} \cong \prod_{s \in S} \mathbf{C}_{s,t_0}$.

J. Hashimoto's theorem implies that every finite, connected ordered set has the unique factorization property, i.e., its decomposition as a direct product of directly indecomposable ordered sets is unique up to isomorphism of factors and order of their occurrence.

The theorem was earlier proved by G. Birkhoff for the case that $\mathbf{C}$ is bounded. Most proofs of it actually prove that $\mathbf{C}$ has the strict refinement property of C. C. Chang, B. Jónsson and A. Tarski [6]. The strict refinement property is equivalent to the conclusion in the next lemma. The proposition that follows the lemma is the version of the Hashimoto theorem that will be most convenient for us.

Lemma 3.1 *Let* $\varphi : \mathbf{A} \times \mathbf{B} \cong \mathbf{C} \times \mathbf{D}$ *where* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ *are connected ordered sets. Suppose that* $a_i \in A, b_i \in B, c \in C, d_i \in D$ *and* $\varphi(a_0, b_0) = (c, d_0)$ *and* $\varphi(a_1, b_1) = (c, d_1)$*. Then for all* $a \in A$*, there are* $c' \in C$*,* $\{d'_0, d'_1\} \subseteq D$ *so that* $\varphi(a, b_0) = (c', d'_0)$ *and* $\varphi(a, b_1) = (c', d'_1)$*.*

PROOF. We first prove this under the assumption that d_0 and d_1 are comparable. We write p_0 for the function $C \times D \to C$ defined by $p_0(x, y) = x$. We can suppose that $d_0 \le d_1$. Then $a_0 \le a_1$ and $b_0 \le b_1$ and so $(a_0, b_0) \le (a_0, b_1) \le (a_1, b_1)$, implying that $\varphi(a_0, b_1) = (x, y)$ where $c \le x \le c$—i.e., $\varphi(a_0, b_1) = (c, y)$ for some y. Now we prove that $p_0\varphi(a, b_0) = p_0\varphi(a, b_1)$ for all $a \in A$ by induction on the distance $d_{\mathbf{A}}(a, a_0)$. If $a = a_0$, we have the result: $p_0\varphi(a_0, b_0) = c = p_0\varphi(a_0, b_1)$. Suppose that $n > 0$ and it is true for all a with $d_{\mathbf{A}}(a, a_0) < n$, and let $a \in A$ with $d_{\mathbf{A}}(a, a_0) = n$. Then there is $a' \in A$ with $d_{\mathbf{A}}(a', a_0) = n - 1$ and $d_{\mathbf{A}}(a, a') = 1$, i.e., a and a' are comparable. Without loss of generality, suppose that $a' \le a$. By induction, we have that $\varphi(a', b_0) = (u, e)$ and $\varphi(a', b_1) = (u, f)$ for some $u \in C$ and $e, f \in D$. Recall that $b_0 \le b_1$. Now $\varphi(a, b_0) = (u', e') \ge (u, e)$; moreover $(a, b_1) = (a', b_1) \vee (a, b_0)$ in $\mathbf{A} \times \mathbf{B}$. Thus $\varphi(a, b_1) = (u, f) \vee (u', e')$ in $\mathbf{C} \times \mathbf{D}$, which implies that $\varphi(a, b_1) = (u', f')$ where $f' = e' \vee f$ in $\mathbf{D}$. The inductive proof for the case that d_0, d_1 are comparable is now complete.

To complete the proof, we proceed by induction on $d_{\mathbf{D}}(d_0, d_1)$. The case where $d_{\mathbf{D}}(d_0, d_1) \le 1$ has already been treated above. Suppose that $n > 1$ and the desired conclusion is true whenever $d_{\mathbf{D}}(d_0, d_1) < n$. Let $\varphi(a_i, b_i) = (c, d_i)$ for $i \in \{0, 1\}$ where $d_{\mathbf{D}}(d_0, d_1) = n$. Choose $d \in D$ such that $d_{\mathbf{D}}(d_0, d) = n-1$ and d, d_1 are comparable. We can write $(c, d) = \varphi(a, b)$, for some $(a, b) \in A \times B$. Choosing any $q \in A$, we have by two applications of the induction assumption, that $p_0\varphi(q, b_0) = p_0\varphi(q, b) = p_0\varphi(q, b_1)$. This completes the proof. •

Proposition 3.1 *Let* $\mathbf{C}$ *be a connected ordered set, and assume that* $\varphi : \mathbf{C} \cong \mathbf{A}_0 \times \mathbf{A}_1$ *and* $\psi : \mathbf{C} \cong \mathbf{B}_0 \times \mathbf{B}_1$*. There are ordered sets and isomorphisms*

$$\beta_0 : \mathbf{X} \times \mathbf{Y} \;\cong\; \mathbf{A}_0 \,,$$

$$\begin{aligned} \beta_1 : \mathbf{Z} \times \mathbf{W} &\cong \mathbf{A}_1\,, \\ \gamma_0 : \mathbf{X} \times \mathbf{Z} &\cong \mathbf{B}_0\,, \\ \gamma_1 : \mathbf{Y} \times \mathbf{W} &\cong \mathbf{B}_1\,, \end{aligned}$$

so that for all $(x, y, z, w) \in X \times Y \times Z \times W$,

$$\psi \circ \varphi^{-1}(\beta_0(x, y), \beta_1(z, w)) = (\gamma_0(x, z), \gamma_1(y, w))\,.$$

PROOF. Write λ for $\psi \circ \varphi^{-1}$. Choose $c \in C$ and put $\varphi(c) = (a_0, a_1)$ and $\psi(c) = (b_0, b_1)$. Applying Lemma 3.1 to the isomorphism λ, we see that the sets $\lambda(A_0 \times \{a_1\})$ and $\lambda(\{a_0\} \times A_1)$ are rectangles: there are sets $X, Z \subseteq B_0$ and $Y, W \subseteq B_1$ such that $\lambda(A_0 \times \{a_1\}) = X \times Y$ and $\lambda(\{a_0\} \times A_1) = Z \times W$. There are sets $X', Y' \subseteq A_0$ and $Z', W' \subseteq A_1$ so that $\lambda(X' \times Z') = B_0 \times \{b_1\}$ and $\lambda(Y' \times W') = \{b_0\} \times B_1$. The lemma also implies that

$$\begin{aligned} \lambda(X' \times \{a_1\}) &= X \times \{b_1\}\,, \\ \lambda(Y' \times \{a_1\}) &= \{b_0\} \times Y\,, \\ \lambda(\{a_0\} \times Z') &= Z \times \{b_1\}\,, \\ \lambda(\{a_0\} \times W') &= \{b_0\} \times W\,. \end{aligned}$$

Letting $\mathbf{X}, \mathbf{Z}$ be the ordered sets induced by $\mathbf{B}_0$ on X, Z and similarly defining $\mathbf{Y}, \mathbf{W}, \mathbf{X}', \ldots, \mathbf{W}'$, it is clear from the displayed formulas that $\mathbf{X} \cong \mathbf{X}', \ldots, \mathbf{W} \cong \mathbf{W}'$, and we have isomorphisms

$$\begin{aligned} \beta_0 : \mathbf{X} \times \mathbf{Y} &\cong \mathbf{A}_0\,, \\ \beta_1 : \mathbf{Z} \times \mathbf{W} &\cong \mathbf{A}_1\,, \\ \gamma_0 : \mathbf{X} \times \mathbf{Z} &\cong \mathbf{B}_0\,, \\ \gamma_1 : \mathbf{Y} \times \mathbf{W} &\cong \mathbf{B}_1\,. \end{aligned}$$

To demonstrate the equation

$$\lambda(\beta_0(x, y), \beta_1(z, w)) = (\gamma_0(x, z), \gamma_1(y, w))\,,$$

we use of course Lemma 3.1. Suppose that $\lambda(u, a_1) = (x, y), \lambda(a_0, v) = (z, w)$ and $\lambda(u, v) = (r, s)$. Since $p_1\lambda^{-1}(b_0, b_1) = p_1\lambda^{-1}(x, y) = a_1$, then $\lambda(x', a_1) = (x, b_1)$ for some $x' \in X'$. Since $p_0\lambda^{-1}(z, w) = p_0\lambda^{-1}(b_0, b_1) = a_0$, then $\lambda(a_0, z') = (z, b_1)$ for some $z' \in Z'$. Write $\lambda(x'', z'') = (r, b_1)$. Now

$p_0\lambda^{-1}(r,s) = p_0\lambda^{-1}(x,y) = u$ implies that $p_0\lambda^{-1}(r,b_1) = p_0\lambda^{-1}(x,b_1)$, i.e., $x' = x''$. And $p_1\lambda^{-1}(r,s) = p_1\lambda^{-1}(z,w) = v$ implies that $p_1\lambda^{-1}(r,b_1) = p_1\lambda^{-1}(z,b_1)$, i.e., $z' = z''$. Thus $\lambda(x',z') = (r,b_1)$, which means that $\gamma_0(x,z) = r$. The analogous calculation shows that $\gamma_1(y,w) = s$. •

4 The algebra of finite ordered sets

In the arithmetic of ordered sets, we have the following laws. The proofs are straightforward, using various natural bijections.

Proposition 4.1 *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ *be ordered sets.*

(1) $\mathbf{A} \cong \mathbf{1} \times \mathbf{A} \cong \mathbf{A}^{\mathbf{1}}$ *and* $\mathbf{1}^{\mathbf{A}} \cong \mathbf{1}$.

(2) $\mathbf{A} + \mathbf{B} \cong \mathbf{B} + \mathbf{A}$.

(3) $\mathbf{A} \times \mathbf{B} \cong \mathbf{B} \times \mathbf{A}$.

(4) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) \cong (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

(5) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \cong (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

(6) $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) \cong \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

(7) $\mathbf{A}^{\mathbf{C} \times \mathbf{D}} \cong (\mathbf{A}^{\mathbf{C}})^{\mathbf{D}}$.

(8) $\mathbf{A}^{\mathbf{C} + \mathbf{D}} \cong \mathbf{A}^{\mathbf{C}} \times \mathbf{A}^{\mathbf{D}}$.

(9) $(\mathbf{A} \times \mathbf{B})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{C}} \times \mathbf{B}^{\mathbf{C}}$.

(10) *If* $\mathbf{C}$ *is connected, then* $(\mathbf{A} + \mathbf{B})^{\mathbf{C}} = \mathbf{A}^{\mathbf{C}} + \mathbf{B}^{\mathbf{C}}$.

We leave it to the reader to establish that all parts of this proposition remain true when G. Birkhoff's operation of exponentiation is replaced by $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$ defined in Definition 2.1. The latter operation has the nice property that when $\mathbf{A}$ is connected and finite then $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$ is also connected, which is not the case for $\mathbf{A}^{\mathbf{B}}$ in general.

Example 4.1 Let $\mathbf{A} = \mathbf{B} = \mathbf{C}_2$ be the four-element crown $2 \oplus 2$. Then we have $\mathbf{C}_2^{\mathbf{C}_2} \cong 1+1+1+1+\mathcal{C}(\mathbf{C}_2^{\mathbf{C}_2})$. It is a good exercise to draw $\mathcal{C}(\mathbf{C}_2^{\mathbf{C}_2})$. This connected poset has 32 elements and its automorphism group has order 2^6. The automorphism group of $\mathbf{C}_2^{\mathbf{C}_2}$ is quite a bit larger than $\mathrm{Aut}(\mathbf{C}_2)\times\mathrm{Aut}(\mathbf{C}_2)$, even though $\mathbf{C}_2$ is additively-, directly- and exponentially-indecomposable. Every $\varphi \in \mathrm{Aut}(\mathbf{C}_2^{\mathbf{C}_2})$ has the zig-zag property; in fact, $R(\varphi) = C_2$.

Let $\mathbf{Z}[\bar{x}]$ denote the ring of polynomials in commuting indeterminates $\bar{x} = (x_n : n \in \omega)$ with integer coefficients. Let $\mathbf{Z}^+[\bar{x}]$ denote the algebra consisting of the non-zero polynomials with non-negative coefficients, under operations of addition and multiplication and with the constant 1. This is a commutative semi-ring—i.e., it satisfies the equations (where f, g, h are arbitrary members of $Z^+[\bar{x}]$):

(1) $1 \cdot f = f$.

(2) $f + g = g + f$.

(3) $f \cdot g = g \cdot f$.

(4) $f + (g + h) = (f + g) + h$.

(5) $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

Let $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n, \ldots,$ be a denumerable list of finite $\times$-indecomposable and connected ordered sets such that for every finite $\times$-indecomposable, connected, ordered set $\mathbf{P}$ there is a unique $n \in \omega$ with $\mathbf{P} \cong \mathbf{P}_n$. Now, just as $\mathbf{Z}[\bar{x}]$ is the free commutative-associative ring with 1 generated by the indeterminates $\{x_n\}$, the algebra $\mathbf{Z}^+[\bar{x}]$ is the free commutative semi-ring generated by $\{x_n\}$. This has the consequence, in view of Proposition 4.1, (1)–(7), that there is a mapping τ defined on $Z^+[\bar{x}]$ satisfying: (i) for every f, $\tau(f)$ is a finite ordered set; (ii) for every n, $\tau(x_n) = \mathbf{P}_n$; (iii) for every finite ordered set $\mathbf{A}$ there is a unique $f \in Z^+[\bar{x}]$ with $\tau(f) \cong \mathbf{A}$; (iv) for every $f, g \in Z^+[\bar{x}]$, $\tau(f + g) \cong \tau(f) + \tau(g)$, and $\tau(fg) \cong \tau(f) \times \tau(g)$; (v) $\tau(1) = \mathbf{1}$.

This establishes the following theorem.

Theorem 4.1 *The algebra of isomorphism types of finite ordered sets under the operations induced by cardinal sum and cardinal product, is isomorphic with* $\mathbf{Z}^+[\bar{x}]$.

PROOF. It is trivial to see that the mapping τ exists satisfying (i), (ii), (iv) and (v). We need to show that τ satisfies (iii). To see that it is onto, note that every ordered set is a sum of connected ordered sets, and every finite connected ordered set other than those isomorphic to **1** is isomorphic to the product of a finite nonvoid sequence of $\times$-indecomposable connected ordered sets.

It remains to show that $\tau(f) \cong \tau(g)$ implies $f = g$. So suppose that f and g belong to $Z^+[\bar{x}]$ and $\tau(f) \cong \tau(g)$. By a "monomial" we mean an element of $Z^+[\bar{x}]$ either equal to 1 or to a finite product of elements x_n. For example, $1, x_2, x_0^3 x_4 x_5^6$ are monomials, $2x_1x_2$ is not. Clearly, the monomials are precisely the non-zero elements t of $Z^+[\bar{x}]$ such that $\tau(t)$ is connected. Now each of f and g can naturally be expressed as a sum of monomials, say,

$$\begin{aligned} f &= \sum_{i=0}^{i=m-1} u_i \,, \\ g &= \sum_{j=0}^{j=n-1} v_j \,. \end{aligned}$$

Then

$$\begin{aligned} \tau(f) &\cong \sum_{i=0}^{i=n-1} \tau(u_i) \,, \\ \tau(g) &\cong \sum_{j=0}^{j=m-1} \tau(v_j) \,, \end{aligned}$$

The isomorphism between $\tau(f)$ and $\tau(g)$ must match up the connected components, so we have $m = n$ and there is a permutation π of $\{0, \ldots, m-1\}$ such that $\tau(u_i) \cong \tau(v_{\pi(i)})$ for all $i < m$.

Fixing $i = i_0$, we proceed to show that $u_{i_0} = v_{\pi(i_0)}$. Let $x_{r_1}, \ldots, x_{r_k}$ be the indeterminates that actually occur in at least one of $u_{i_0}, v_{\pi(i_0)}$. We can write

$$\begin{aligned} u_{i_0} &= x_{r_1}^{a_1} x_{r_2}^{a_2} \cdot \cdots \cdot x_{r_k}^{a_k} \,, \\ v_{\pi(i_0)} &= x_{r_1}^{b_1} x_{r_2}^{b_2} \cdot \cdots \cdot x_{r_k}^{b_k} \,, \end{aligned}$$

where for all $1 \leq i \leq k$, $a_i + b_i > 0$. Then we have

$$\begin{aligned} \tau(u_{i_0}) &= \mathbf{P}_{r_1}^{a_1} \times \mathbf{P}_{r_2}^{a_2} \times \cdots \times \mathbf{P}_{r_k}^{a_k}, \\ \tau(v_{\pi(i_0)}) &= \mathbf{P}_{r_1}^{b_1} \times \mathbf{P}_{r_2}^{b_2} \times \cdots \times \mathbf{P}_{r_k}^{b_k}. \end{aligned}$$

Now the isomorphism $\tau(u_{i_0}) \cong \tau(v_{\pi(i_0})$ and the unique factorization property of $\tau(u_{i_0})$ (following from Theorem 3.1) implies that $a_i = b_i$ for all i. This means that $u_{i_0} = v_{\pi(i_0)}$.

Since i_0 was arbitrary, we have shown that $u_i = v_{\pi(i)}$ for all $i < m$. Clearly, this entails $f = g$. •

Corollary 4.1 *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be finite ordered sets. If* $\mathbf{A} + \mathbf{B} \cong \mathbf{A} + \mathbf{C}$ *then* $\mathbf{B} \cong \mathbf{C}$. *If* $\mathbf{A} \times \mathbf{B} \cong \mathbf{A} \times \mathbf{C}$, *then* $\mathbf{B} \cong \mathbf{C}$.

Proof. This follows from Theorem 4.1, since the semi-ring $\mathbf{Z}^+[\bar{x}]$ has cancellation for addition and for multiplication by non-zero elements. •

As is well-known, L. Lovasz [19] has a completely different proof that $\mathbf{E} \cong \mathbf{A} \times \mathbf{C} \cong \mathbf{A} \times \mathbf{C}$ implies $\mathbf{B} \cong \mathbf{C}$ for finite posets $\mathbf{E}$, and more generally, that this implication is true when $\mathbf{E}$ is a finite structure and $\mathbf{A}$ has a one-element full substructure—i.e., an element a such that for every basic operation F, $\mathbf{A} \models F(a, \ldots, a) = a$ and for every basic relation R, $\mathbf{A} \models R(a, \ldots, a)$.

In Section §9, we shall extend Theorem 4.1 to the algebra of isomorphism types of finite ordered sets under the operations induced by $\mathbf{A} + \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ and $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$.

5 The logarithm and the zig-zag property

In this section, we show that the zig-zag property of Definition 1.1 is valid under some very modest assumptions. This section reproduces some of the definitions and results of B. Jónsson, R. McKenzie [17], §7. The results rely on a certain notion of logarithm of an ordered set. The ordered sets mentioned in the definitions and results of this section are not assumed to be finite, except where a finiteness assumption is explicitly stated.

We begin with an example of an isomorphism $\varphi : \mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$ that does not possess the zig-zag property. Except for some fairly obvious variants of it, this appears to be the only known such example at the present time, in which $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{P}$ are finite and connected and $\mathbf{A}$ and $\mathbf{B}$ are directly indecomposable. The example should be compared with Theorem 5.3.

Example 5.1 For a positive integer n, we write $\mathbf{n}$ for the n-element chain $\langle\{0, 1, \dots, n-1\}, \leq\rangle$. Observe that $\mathbf{2^n} \cong \mathbf{n} \oplus \mathbf{1}$. The ordered set $\mathbf{3^{2\times 2}}$ has an automorphism that fails to possess the zig-zag property. Let

$$\alpha : \mathbf{3^{2\times 2}} \cong \mathbf{2^{2\times 2\times 2}}$$

be the natural isomorphism arising from writing $\mathbf{3}$ as $\mathbf{2^2}$. It is easy to check that where $f \in \mathrm{M}(\mathbf{2}\times\mathbf{2}, \mathbf{3})$ and $\alpha(f) = g$, we have $g(x, y, z) = x$ unless $x = 0$ and $f(y, z) = 2$ (in which case $g(x, y, z) = 1$) or $x = 1$ and $f(y, z) = 0$ (in which case $g(x, y, z) = 0$). The automorphism of our example is determined by a cyclic automorphism of $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$. Namely, for $f \in \mathrm{M}(\mathbf{2} \times \mathbf{2} \times \mathbf{2}, \mathbf{2})$ let $\psi(f)$ be the function $g \in \mathrm{M}(\mathbf{2} \times \mathbf{2} \times \mathbf{2}, \mathbf{2})$ with $g(x, y, z) = f(z, x, y)$. Then for $f \in \mathrm{M}(\mathbf{2} \times \mathbf{2}, \mathbf{3})$, put $\varphi(f) = \alpha^{-1}\psi\alpha(f)$.

To see that the zig-zag property fails for φ, consider $f = \langle 1\rangle \in \mathrm{M}(\mathbf{2}\times\mathbf{2}, \mathbf{3})$. Then $\alpha(f)(x, y, z) = x$, and it is easy to calculate that where $\bar{f} = \varphi(f)$, we have $\bar{f}(y, 0) = 0$ and $\bar{f}(y, 1) = 2$. From this, one can demonstrate that $\varphi(\langle\bar{f}(p)\rangle) = \langle\bar{f}(p)\rangle$ for all $p \in \{0, 1\} \times \{0, 1\}$. Thus if φ had the zig-zag property, we would have that $\varphi^2(f)$ is constant. However, $\varphi^2(f)$ maps $(0, 0)$ to 0 and $(1, 1)$ to 2.

As I remarked earlier, C. Bergman, Sz. Nagy, R. McKenzie [2] proved that $\varphi : \mathbf{A^P} \cong \mathbf{B^P}$ has the zig-zag property if $\mathbf{P}$ is linearly ordered. B. Jónsson, R. McKenzie [17] proved among other things that if $\varphi : \mathbf{A^P} \cong \mathbf{B^P}$ and $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are finite and connected, and if $\mathbf{P}$ is directly indecomposable and one of $\mathbf{A}$ or $\mathbf{P}$ has a least or greatest element, then φ has the zig-zag property. Part of this result is essential in my proof of the Birkhoff conjecture and will be proved in this section, as Theorem 5.1, with a re-organized argument. Two new results, Theorems 5.2 and 5.3, will be deduced from the argument.

In [20], I generalized the cited results of [17] as follows: if $\varphi : \mathbf{A^P} \cong \mathbf{B^P}$ and $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are finite and connected, and if $\mathbf{P}$ is directly indecomposable, and if either $\mathbf{P}$ is dismantlable to each of its points or else each of $\mathbf{A}$ and $\mathbf{B}$ is dismantlable to each of its covering pairs, then φ has the zig-zag property.

Definition 5.1 Let $\mathbf{A}$ and $\mathbf{B}$ be ordered sets and choose $c \leq d$ in $\mathbf{A}$. For $x \in B$, we write $\langle c, d \rangle_x$ for the element $f \in \mathbf{A}^{\mathbf{B}}$ such that $f(y) = d$ for all $y \geq x$ and $f(y) = c$ for all $y \not\geq x$.

For any ordered set $\mathbf{B}$, $\mathbf{2}^{\mathbf{B}}$ is a lattice-ordered set. The lattice is complete and distributive, and its non-zero strictly join-irreducible elements are precisely the functions $\langle 0, 1 \rangle_x$ for $x \in B$. The ordered set that $\mathbf{2}^{\mathbf{B}}$ induces on this set of join-irreducible elements is naturally isomorphic to $\mathbf{B}^{\partial}$, and for finite $\mathbf{B}$, is the dual object associated to the lattice $\mathbf{2}^{\mathbf{B}}$ under Birkhoff duality. Every finite distributive lattice $\mathbf{L}$ is order-isomorphic to $\mathbf{2}^{\mathbf{J}}$ where $\mathbf{J}$ is the Birkhoff-dual of $\mathbf{L}$, i.e., the ordered set of non-zero join-irreducible elements of $\mathbf{L}$. It is natural to think of $\mathbf{J}$ as the logarithm of $\mathbf{2}^{\mathbf{J}}$.

The logarithm of $\mathbf{A}$ defined in D. Duffus and I. Rival [9] is the sub-poset of $\mathbf{A}$ consisting of the elements that possess a unique lower cover. This is a natural generalization of the Birkhoff-dual of a finite distributive lattice. However, this notion is not adequate to obtain the results of this section, because it does not satisfy Lemma 5.1 below. We shall now define our own concept of logarithm for any ordered set with zero. It is a bit complex to define, but if $\mathbf{A}$ is a lattice-ordered set with zero, our logarithm is identical with that of D. Duffus and I. Rival.

Definition 5.2 Let $\mathbf{A}$ be an ordered set. For $X \subseteq A$, write $X{\downarrow}$ for the set of all elements $y \in A$ such that $y \leq x$ for some $x \in X$. If $X = \{x\}$, we abbreviate this to $x{\downarrow}$. Analogously, we write $X{\uparrow}$ for the set of all elements $y \in A$ such that $y \geq x$ for some $x \in X$; and we put $x{\uparrow} = \{x\}{\uparrow}$. We say that two elements x, y in A are $\uparrow$-compatible provided that $x{\uparrow} \cap y{\uparrow} \neq \emptyset$.

Suppose that $X \cup \{x\} \subseteq A$. We write $X \gg x$, and say that X *dominates* x, provided that whenever $u \in A$ and $X \subseteq u{\downarrow}$ and u, x are $\uparrow$-compatible, then we have $u \geq x$.

Now suppose that $\mathbf{A}$ has least element 0. We define $L(\mathbf{A})$ to be the set of all $a \neq 0$ in A such that $(a{\downarrow}) \setminus \{a\} \not\gg a$. We define $\mathbf{L}(\mathbf{A})$ to be $L(\mathbf{A})$ with the induced order from $\mathbf{A}$.

The next two lemmas give the two properties of this logarithm that are critical for the application we wish to make of it.

Lemma 5.1 *Let* $\mathbf{A}$ *be an ordered set with zero, which satisfies the descending chain condition. Then for all* $a \in A$*,* $(a\downarrow) \cap L(\mathbf{A}) \gg a$*. Thus whenever* $a < b$ *in* $\mathbf{A}$ *we have that* $a\downarrow \cap L(\mathbf{A}) \neq b\downarrow \cap L(\mathbf{A})$*.*

PROOF. Suppose that the first assertion of the lemma fails. Then there is a minimal member u among the set of all $a \in A$ such that the condition $(a\downarrow) \cap L(\mathbf{A}) \gg a$ fails. Trivially, $u > 0$, and $u \notin L(\mathbf{A})$. We get a contradiction by showing that, in fact, $(u\downarrow) \cap L(\mathbf{A}) \gg u$. To that end, let b be any element $\uparrow$-compatible with u such that $(u\downarrow) \cap L(\mathbf{A}) \subseteq b\downarrow$. Then for each $x < u$, b is $\uparrow$-compatible with x and $(x\downarrow) \cap L(\mathbf{A}) \subseteq b\downarrow$. Since $x < u$ we have that $(x\downarrow) \cap L(\mathbf{A}) \gg x$ and thus it follows that $x \leq b$. This is true for all $x < u$. Now since $u \notin L(\mathbf{A})$, it follows that $b \geq u$, the desired contradiction.

For the second statement, suppose that a and b are elements of $\mathbf{A}$ with $a < b$. Then since $(b\downarrow) \cap L(\mathbf{A}) \gg b$ and $a \ngeq b$ while a is certainly $\uparrow$-compatible with b, we can conclude the existence of some $d \in (b\downarrow) \cap L(\mathbf{A})$ with $d \nleq a$. •

Definition 5.3 An ordered set $\mathbf{A}$ with zero which satisfies the conclusion of Lemma 5.1, that for all $a \in A$, $(a\downarrow) \cap L(\mathbf{A}) \gg a$, will be said to be *L-dense.*

Remark 5.1 If $\mathbf{A}$ is the ordered set underlying a finite distributive lattice, then $\mathbf{L}(\mathbf{A})$ is just the usual ordered set of non-zero join-irreducible elements of $\mathbf{A}$, and we have $\mathbf{A} \cong \mathbf{2}^{\mathbf{L}(\mathbf{A})^\partial}$ as we remarked above. Thus if $\mathbf{B}$ is the underlying ordered set of another finite distributive lattice, then $\mathbf{A} \cong \mathbf{B}$ iff $\mathbf{L}(\mathbf{A}) \cong \mathbf{L}(\mathbf{B})$. Unfortunately, this fails in general. For example consider $\mathbf{A} = 1 \oplus 2 \oplus 2$ (one element 0 below two unordered elements $\{a, b\}$, which in turn are below two unordered elements $\{c, d\}$) and $\mathbf{B} = 1 \oplus 2 \oplus 3$.

Recall that $\langle 0, u \rangle_x \in A^P$ denotes the function f with $f(y) = u$ when $y \geq x$ and $f(y) = 0$ when $y \ngeq x$.

Lemma 5.2 *Let* $\mathbf{A}, \mathbf{P}$ *be ordered sets and suppose that* $\mathbf{A}$ *has a zero element* 0*. Then* $L(\mathbf{A}^\mathbf{P})$ *is the set of all functions* $\langle 0, u \rangle_x$ *where* $u \in L(\mathbf{A})$ *and* $x \in P$*. For two such functions* $f = \langle 0, u \rangle_x$ *and* $g = \langle 0, v \rangle_y$*, we have that* $f \leq g$ *in* $\mathbf{A}^\mathbf{P}$ *iff* $u \leq v$ *and* $y \leq x$*. Thus* $\mathbf{L}(\mathbf{A}^\mathbf{P}) \cong \mathbf{L}(\mathbf{A}) \times \mathbf{P}^\partial$*; and if* $\mathbf{A}$ *is L-dense, so is* $\mathbf{A}^\mathbf{P}$*.*

PROOF. The proof is straightforward and left to the reader. •

The next definition is Definition 3.1 in [17], and partially repeats our Definition 1.1. Lemmas 5.3–5.9 and Corollary 5.1, which follow the definition, are taken from §7 in [17].

Definition 5.4 Suppose that $\varphi : \mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{D}}$ where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are posets.

(i) $\Delta_A \subseteq A^C$ is the set of all constant functions $\langle a \rangle$ $(a \in A)$; $\Delta_B \subseteq B^D$ is analogously defined.

(ii) $\Delta(\varphi) = \{f \in \mathrm{M}(\mathbf{C}, \mathbf{A}) : \varphi(f) \in \Delta_B\}$.

(iii) $R(\varphi) = \{x \in A : \langle x \rangle \in \Delta(\varphi)\}$.

(iv) $x \leq_\varphi y$ iff $x, y \in R(\varphi)$, $x \leq y$, and every $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$ with $f(C) \subseteq \{x, y\}$ belongs to $\Delta(\varphi)$.

(v) δ_φ is the map from $R(\varphi)$ to $R(\varphi^{-1})$ such that $\varphi(\langle x \rangle) = \langle \delta_\varphi(x) \rangle$ for all $x \in R(\varphi)$.

We investigate four potential properties of an isomorphism $\varphi : \mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{D}}$:

$(\varphi, 1)$ $\leq_\varphi$ is a partial ordering of $R(\varphi)$.

$(\varphi, 2)$ $\delta_\varphi : \langle R(\varphi), \leq_\varphi \rangle \cong \langle R(\varphi^{-1}), \leq_{\varphi^{-1}} \rangle$.

$(\varphi, 3)$ $\Delta(\varphi)$ is the set of all order-preserving maps from $\mathbf{C}$ into $\langle R(\varphi), \leq_\varphi \rangle$.

$(\varphi, 4)$ $\leq_\varphi$ agrees with $\leq$ on $R(\varphi)$.

The chief results of this section are the three theorems below. The reader should observe that every finite ordered set with zero is L-dense (a consequence of 5.1).

Theorem 5.1 *Let $\varphi : \mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{D}}$ where $\mathbf{A}$ and $\mathbf{B}$ are L-dense with 0 and $\mathbf{C}$ and $\mathbf{D}$ are connected and directly indecomposable. Then $(\varphi, 1)$, $(\varphi, 2)$, $(\varphi, 3)$ and $(\varphi^{-1}, 3)$ hold. Hence if $\mathbf{C} = \mathbf{D}$ then φ and φ^{-1} have the zig-zag property, and consequently, $\tau_\varphi : \mathbf{A} \cong \mathbf{B}$.*

Theorem 5.2 *Let $\varphi : \mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{D}}$ where $\mathbf{A}$ and $\mathbf{B}$ are L-dense with 0 and $\mathbf{C}$ and $\mathbf{D}$ are connected and have no common direct factor with more than one element. Then $(\varphi, 1)$, $(\varphi, 2)$, $(\varphi, 3)$, $(\varphi^{-1}, 3)$ and $(\varphi, 4)$ hold. Where $\mathbf{E} = \langle R(\varphi), \leq \rangle$, we have that there are isomorphisms $\psi_A : \mathbf{A} \cong \mathbf{E}^{\mathbf{D}}$ and $\psi_B : \mathbf{B} \cong \mathbf{E}^{\mathbf{C}}$ so that where π is the natural isomorphism of $(\mathbf{E}^{\mathbf{D}})^{\mathbf{C}}$ with $(\mathbf{E}^{\mathbf{C}})^{\mathbf{D}}$, then for all $f \in M(\mathbf{C}, \mathbf{A})$, $\varphi(f) = \psi_B^{-1} \circ \pi(\psi_A \circ f)$.*

Theorem 5.2 asserts that after rewriting $\mathbf{A}^{\mathbf{C}}$ as $(\mathbf{E}^{\mathbf{D}})^{\mathbf{C}}$ and rewriting $\mathbf{B}^{\mathbf{D}}$ as $(\mathbf{E}^{\mathbf{C}})^{\mathbf{D}}$ with the aid of ψ_A and ψ_B, φ becomes the natural isomorphism π.

Theorem 5.3 *Let $\varphi : \mathbf{A}^{\mathbf{P}} \cong \mathbf{A}^{\mathbf{P}}$ where $\mathbf{A}$ is L-dense with 0 and $\mathbf{P}$ is connected, finitely factorable, and has no nontrivial direct divisor of the form $\mathbf{Q}^2$. Then φ has the zig-zag property.*

Theorem 5.3 should be compared with Example 5.1. That $\mathbf{P}$ is finitely factorable means that $\mathbf{P} \cong \mathbf{Q}_1 \times \cdots \times \mathbf{Q}_n$ for some integer $n \geq 0$ where $\mathbf{Q}_i$ are directly indecomposable.

The proof of the above theorems is easily obtained following the sequence of seven lemmas and a corollary below, in which we assume that $\varphi : \mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{D}}$, that the ordered sets $\mathbf{A}$ and $\mathbf{B}$ are L-dense with 0, and that the ordered sets $\mathbf{C}$ and $\mathbf{D}$ are connected.

Clearly, $\varphi(L(\mathbf{A}^{\mathbf{C}})) = L(\mathbf{B}^{\mathbf{D}})$, hence by Lemma 5.2, there is an isomorphism

$$\psi : \mathbf{L}(\mathbf{A}) \times \mathbf{C}^{\partial} \cong \mathbf{L}(\mathbf{B}) \times \mathbf{D}^{\partial}$$

so that for all $a \in L(\mathbf{A})$ and $c \in C$,

$$\psi(a, c) = (b, d) \leftrightarrow \varphi(\langle 0, a \rangle_c) = \langle 0, b \rangle_d \,.$$

Let A_t $(t \in T)$ be the connected components of $\mathbf{L}(\mathbf{A})$. Since $\mathbf{C}^{\partial}$ is connected, then $A_t \times C$ are the connected components of $\mathbf{L}(\mathbf{A}) \times \mathbf{C}^{\partial}$ and $\psi(A_t \times C)$ are the connected components of $\mathbf{L}(\mathbf{B}) \times \mathbf{D}^{\partial}$. Thus the connected components of $\mathbf{L}(\mathbf{B})$ can be listed as B_t $(t \in T)$ with $\psi(A_t \times C) = B_t \times D$ for all $t \in T$.

Moreover, by Proposition 3.1, for each t we can choose posets $\mathbf{X}_t, \mathbf{Y}_t, \mathbf{Z}_t, \mathbf{W}_t$ and isomorphisms

$$\begin{aligned} \alpha_t(\mathbf{X}_t \times \mathbf{Y}_t) &\cong \mathbf{A}_t\,, \\ \gamma_t(\mathbf{Z}_t \times \mathbf{W}_t) &\cong \mathbf{C}^\partial\,, \\ \beta_t(\mathbf{X}_t \times \mathbf{Z}_t) &\cong \mathbf{B}_t\,, \\ \delta_t(\mathbf{Y}_t \times \mathbf{W}_t) &\cong \mathbf{D}^\partial \end{aligned}$$

such that for all $(x, y, z, w) \in X_t \times Y_t \times Z_t \times W_t$,

$$\psi(\alpha_t(x, y), \gamma_t(z, w)) = (\beta_t(x, z), \delta_t(y, w))\,, \quad \text{i.e.,}$$

$$\varphi(\langle 0, \alpha_t(x, y)\rangle_{\gamma_t(z,w)}) = \langle 0, \beta_t(x, z)\rangle_{\delta_t(y,w)}\,.$$

Necessarily, the posets $\mathbf{X}_t, \mathbf{Y}_t, \mathbf{Z}_t, \mathbf{W}_t$ are connected.

Until the end of this section, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \varphi$ and the auxiliary isomorphisms ψ and $\alpha_t, \beta_t, \gamma_t, \delta_t$ $(t \in T)$ will be held fixed. For $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$, we will usually write $\bar{f}$ for the element $\varphi(f)$ of $\mathrm{M}(\mathbf{D}, \mathbf{B})$. Note that it follows from the displayed formulas above that for all $t \in T$ and for $(x, y, z, w) \in X_t \times Y_t \times Z_t \times W_t$,

$$\alpha_t(x, y) \leq f(\gamma_t(z, w)) \ \text{ iff } \ \beta_t(x, z) \leq \bar{f}(\delta_t(y, w))\,. \tag{3}$$

Lemma 5.3 *An element $a \in A$ belongs to $R(\varphi)$ iff for all t and $x \in X_t$, the condition $\alpha_t(x, y) \leq a$ either holds for all $y \in Y_t$ or else for none.*

Proof. Let $f = \langle a\rangle$ and consider any $m = \gamma_t(z, w)$ in C. Then $\alpha_t(x, y) \leq a$ is equivalent to $\alpha_t(x, y) \leq f(m)$, i.e., to $\beta_t(x, z) \leq \bar{f}(\delta_t(y, w))$. If $\bar{f}$ is constant, then the latter condition is independent of y (and of w). On the other hand, if $\bar{f}$ is not constant, then since $\mathbf{D}$ is connected, it follows from Lemma 5.1 that x, z can be chosen so that these formulas hold for one pair (y, w) but not for all. •

Lemma 5.4 *A function $f \in M(\mathbf{C}, \mathbf{A})$ belongs to $\Delta(\varphi)$ iff, for all t, $x \in X_t$ and $z \in Z_t$, the condition $\alpha_t(x, y) \leq f(\gamma_t(z, w))$ either holds for all $y \in Y_t$ and $w \in W_t$, or else for none.*

PROOF. Let $\bar{f} = \varphi(f)$. Then $\alpha_t(x,y) \leq f(\gamma_t(z,w))$ iff $\beta_t(x,z) \leq \bar{f}(\delta_t(y,w))$. If $\bar{f}$ is constant, then the second condition is independent of y, w. The converse follows from Lemma 5.1. •

Lemma 5.5 *Given $a_0, a_1 \in R(\varphi)$, we have $a_0 \leq_\varphi a_1$ iff $a_0 \leq a_1$ and, for all t with $|W_t| > 1$, and for all $a \in A_t$, $a \leq a_1$ implies $a \leq a_0$. Thus $(\varphi, 1)$ holds.*

PROOF. We can assume that $a_0 < a_1$. Suppose first that we have $|W_t| > 1$ and $a = \alpha_t(x,y)$ satisfies $a \leq a_1$, $a \not\leq a_0$. Since $|W_t| > 1$ and $\mathbf{W}_t$ is connected, there are $m_j = \gamma_t(z, w_j) \in C$ ($j \in \{0,1\}$) with $m_0 < m_1$. Let $f = \langle a_0, a_1 \rangle_{m_1} \in \mathrm{M}(\mathbf{C}, \mathbf{A})$. Now $\alpha_t(x,y) \leq f(m_1)$ and $\alpha_t(x,y) \not\leq f(m_0)$, leading to $\beta_t(x,z) \leq \bar{f}(\delta_t(y,w_1)$ and $\beta_t(x,z) \not\leq \bar{f}(\delta_t(y,w_0)$. This implies that $\bar{f}$ is not constant, and so $a_0 \not\leq_\varphi a_1$.

Next, suppose that for some $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$ with $f(C) \subseteq \{a_0, a_1\}$, we have that $\bar{f}$ is not constant. Then there are $n_0 < n_1 \in D$ with $\bar{f}(n_0) < \bar{f}(n_1)$. By Lemma 5.1, we can choose $b = \beta_t(x,z) \in B_t$ for some t, so that $b \leq \bar{f}(n_1)$ and $b \not\leq \bar{f}(n_0)$. Say $n_j = \delta_t(y_j, w_j)$ ($j \in \{0,1\}$). Thus formula (3) yields $\alpha_t(x, y_1) \leq f(\gamma_t(z, w_1))$ and $\alpha_t(x, y_0) \not\leq f(\gamma_t(z, w_0))$. By Lemma 5.3, we have $\alpha_t(x, y_0) \leq f(\gamma_t(z, w_1))$, since $f(\gamma_t(z, w_1)) \in R(\varphi)$. These relations imply that $f(\gamma_t(z, w_1)) = a_1$ and $f(\gamma_t(z, w_0)) = a_0$. Thus $w_0 > w_1$, hence $|W_t| > 1$. Finally, where $a = \alpha_t(x, y_0)$, we have $a \leq a_1$ and $a \not\leq a_0$, as desired. •

Lemma 5.6 $\delta_\varphi : \langle R(\varphi), \leq_\varphi \rangle \cong \langle R(\varphi^{-1}), \leq_{\varphi^{-1}} \rangle$. *Thus $(\varphi, 2)$ holds.*

PROOF. Let $a_0 \leq a_1$, $\{a_0, a_1\} \subseteq R(\varphi)$, and put $b_j = \delta_\varphi(a_j)$ so that $b_0 \leq b_1$. For any $(x,y,z) \in X_t \times Y_t \times Z_t$ and $j \in \{0,1\}$ we have $\alpha_t(x,y) \leq a_j$ iff $\beta_t(x,z) \leq b_j$ by formula (3) (since $\varphi(\langle a_j \rangle) = \langle b_j \rangle$). Now it readily follows by Lemma 5.5 that $a_0 \leq_\varphi a_1$ iff $b_0 \leq_\varphi b_1$. •

Lemma 5.7 *For all $f \in \Delta(\varphi)$, $f(C) \subseteq R(\varphi)$.*

PROOF. Suppose that $\varphi(f) = \langle b \rangle$. Let $m \in C$ and for any given t let $(x,y) \in X_t \times Y_t$, and write $m = \gamma_t(z,w)$ for some $(z,w) \in Z_t \times W_t$. Then $\alpha_t(x,y) \leq f(m)$ iff $\beta_t(x,z) \leq b$ (by formula (3)). Since this condition is independent of y, it follows by Lemma 5.3 that $f(m) \in R(\varphi)$. •

Lemma 5.8 *Every order-preserving function from C into $\langle R(\varphi), \leq_\varphi \rangle$ belongs to $\Delta(\varphi)$.*

PROOF. Lemma 5.5 makes it apparent that $\leq_\varphi$ is a partial order on $R(\varphi)$. Let f be an order-preserving function from C into $\langle R(\varphi), \leq_\varphi \rangle$. Clearly, $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$. Choose t and consider the condition

$$\alpha_t(x, y) \leq f(\gamma_t(z, w))$$

for $(x, y, z, w) \in X_t \times Y_t \times Z_t \times W_t$. Since $f(C) \subseteq R(\varphi)$, then by Lemma 5.3, this condition is independent of y (for fixed x, z, w). If it is dependent on w, then we can find x, y, z and $w_0 > w_1$ such that

$$a = \alpha_t(x, y) \leq f(\gamma_t(z, w_1)) = a_1 \quad \text{and}$$

$$a \not\leq f(\gamma_t(z, w_0)) = a_0 \, .$$

But this contradicts Lemma 5.5 since $a_0 \leq_\varphi a_1$. •

Lemma 5.9 *The set $\Delta(\varphi)$ coincides with the set of all order-preserving functions from C into $\langle R(\varphi), \leq_\varphi \rangle$ iff, for all t, either $|Z_t| = 1$ or $|W_t| = 1$.*

PROOF. Left as an exercise. •

Corollary 5.1 *The conditions $(\varphi, 1)$ and $(\varphi, 2)$ hold. If $\mathbf{C}$ is directly indecomposable, then $(\varphi, 3)$ holds. If $\mathbf{C}$, $\mathbf{D}$ have no common direct factor of more than one element, then $(\varphi, 3)$, $(\varphi^{-1}, 3)$ and $(\varphi, 4)$ hold.*

PROOF. This follows from Lemmas 5.3–5.9 and the facts that if $\mathbf{C}$ is directly indecomposable, then for all t, either $|Z_t| = 1$ or $|W_t| = 1$, and if $\mathbf{C}$ and $\mathbf{D}$ have no non-trivial common factor, then for all t, $|W_t| = 1$. •

Theorem 5.1 follows immediately from Corollary 5.1.

PROOF OF THEOREM 5.2. By Corollary 5.1, the properties $(\varphi, 1)$, $(\varphi, 2)$, $(\varphi, 3)$, $(\varphi, 4)$ hold. It follows that $\psi_A(a) = \delta_\varphi^{-1} \circ \varphi(\langle a \rangle)$ defines an isomorphism $\mathbf{A} \cong \mathbf{E}^{\mathbf{D}}$ and $\psi_B(b) = \varphi^{-1}(\langle b \rangle)$ gives an isomorphism $\mathbf{B} \cong \mathbf{E}^{\mathbf{D}}$. What remains to be proved amounts to this: for all $\bar{f} = \varphi(f) \in \mathrm{M}(\mathbf{D}, \mathbf{B})$ and for all $c \in C$ and $d \in D$, we have that

$$\varphi(\langle f(c) \rangle)[d] = \delta_\varphi(\varphi^{-1}(\langle \bar{f}[d] \rangle)[c]) . \tag{4}$$

To see that this holds, let $(c, d) \in C \times D$ be fixed. Define, for every $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$, $v^f_{c,d} = \varphi(\langle f(c) \rangle)[d]$, and where $\bar{f} = \varphi(f)$, define $u^f_{c,d} = \varphi^{-1}(\langle \bar{f}(d) \rangle)[c]$. By Lemma 5.7, $u^f_{c,d} \in R(\varphi)$ and $v^f_{c,d} \in R(\varphi^{-1})$. Our goal is to prove that $\delta_\varphi(u^f_{c,d}) = v^f_{c,d}$ for all $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$.

In this proof, we write $\gamma_t(z)$ when $t \in T$ and $z \in Z_t$, to denote $\gamma_t(z, w)$ where w is the unique element of W_t. (Since $\mathbf{C}, \mathbf{D}$ possess no nontrivial common direct factors, then $|W_t| = 1$ for all $t \in T$.) Likewise, when $y \in Y_t$ we write $\delta_t(y)$ for $\delta_t(y, w)$ where w is the unique element in W_t. Moreover, for $t \in T$, we denote by $z^t_c \in Z_t$ and $y^t_d \in Y_t$ the elements such that $c = \gamma_t(z^t_c)$ and $d = \delta_t(y^t_d)$.

<u>Claim:</u> For $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$ and for $b \in L(\mathbf{B})$, we have $b \leq v^f_{c,d}$ iff $b \leq \delta_\varphi(u^f_{c,d})$.

To prove this claim, suppose that $b \in B_t$, say $b = \beta_t(x, z)$. Then we have

$$\begin{aligned} b \leq v^f_{c,d} &\leftrightarrow \beta_t(x, z) \leq \varphi(\langle f(c) \rangle)[\delta_t(y^t_d)] \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq \langle f(c) \rangle[\gamma_t(z)] \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq f(c) . \end{aligned}$$

But we also have

$$\begin{aligned} b \leq \delta_\varphi(u^f_{c,d}) &\leftrightarrow \beta_t(x, z) \leq \varphi(\langle u^f_{c,d} \rangle)[\delta_t(y^t_d)] \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq \langle u^f_{c,d} \rangle[\gamma_t(z)] \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq u^f_{c,d} \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq \varphi^{-1}(\langle \bar{f}(d) \rangle)[\gamma_t(z^t_c)] \\ &\leftrightarrow \beta_t(x, z^t_c) \leq \langle \bar{f}(d) \rangle[\delta_t(y^t_d)] \\ &\leftrightarrow \beta_t(x, z^t_c) \leq \bar{f}(\delta_t(y^t_d)) \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq f(\gamma_t(z^t_c)) \\ &\leftrightarrow \alpha_t(x, y^t_d) \leq f(c) . \end{aligned}$$

Thus the claim is proved.

Now given $f \in \mathrm{M}(\mathbf{C}, \mathbf{A})$, let $g = \langle 0, f(c)\rangle_c \in \mathrm{M}(\mathbf{C}, \mathbf{A})$, and put $\bar{g} = \varphi(g)$. Here we have $g \leq f$, and thus $\bar{g} \leq \bar{f}$. Clearly, $g(c) = f(c)$ yields $v^g_{c,d} = v^f_{c,d}$, and $g \leq f$ yields $u^g_{c,d} \leq u^f_{c,d}$.

We also have $g \leq \langle g(c)\rangle$ and so $\bar{g} \leq \varphi(\langle g(c)\rangle)$ and consequently $\bar{g}(d) \leq v^g_{c,d}$, $\varphi^{-1}(\langle \bar{g}(d)\rangle) \leq \langle \delta^{-1}_\varphi(v^g_{c,d})\rangle$ and $u^g_{c,d} \leq \delta^{-1}_\varphi(v^g_{c,d})$. Thus $\delta_\varphi(u^g_{c,d}) \leq v^g_{c,d}$. Therefore, the L-density of $\mathbf{B}$ and the above claim, applied to g, yield that $\delta_\varphi(u^g_{c,d}) = v^g_{c,d}$.

Since $u^g_{c,d} \leq u^f_{c,d}$, then

$$v^f_{c,d} = v^g_{c,d} = \delta_\varphi(u^g_{c,d}) \leq \delta_\varphi(u^f_{c,d})\,.$$

Now the L-density of $\mathbf{B}$ and the above claim, applied to f, yield that $\delta_\varphi(u^f_{c,d}) = v^f_{c,d}$ as required. •

Proof of Theorem 5.3. We have $\varphi : \mathbf{A}^\mathbf{P} \cong \mathbf{A}^\mathbf{P}$ where $\mathbf{A}$ is L-dense with 0, and we can assume that $\mathbf{P} = \mathbf{P}_0 \times \cdots \times \mathbf{P}_{n-1}$ where the $\mathbf{P}_i$ are connected and directly indecomposable and pairwise non-isomorphic.

Let $a \in A$, $f = \langle a\rangle \in \mathrm{M}(\mathbf{P}, \mathbf{A})$ and write $\varphi(f) = \bar{f}$. By Lemma 5.7, $\bar{f}(P) \subseteq R(\varphi^{-1})$. Thus we have elements $a_p \in A$ for $p \in P$ with $\varphi(\langle a_p\rangle) = \langle \bar{f}(p)\rangle$. We write $g = \langle a_p : p \in P\rangle$ and $\bar{g} = \varphi(g)$. Our task is to prove that $\bar{g}$ is constant.

We can regard $\bar{g}$ as a monotone function $g(x_0, \ldots, x_{n-1})$ of n variables, with x_i ranging over P_i. We say that g is P_i-constant if it does not depend on the variable x_i. It will suffice, of course, to show that $\bar{g}$ is P_i-constant for all $i \in \{0, \ldots, n-1\}$. We prove this for $i = 0$. The same proof works for any i.

Thus we put $\mathbf{D} = \mathbf{P}_0$ and $\mathbf{C} = \mathbf{P}_1 \times \cdots \times \mathbf{P}_{n-1}$, so that $\mathbf{P} \cong \mathbf{C} \times \mathbf{D}$ and $\mathbf{C}, \mathbf{D}$ have no nontrivial common direct divisor. To simplify notation, we henceforth assume that $\mathbf{P} = \mathbf{C} \times \mathbf{D}$.

Now for each $f \in \mathrm{M}(\mathbf{P}, \mathbf{A})$, denote by f^D the member of of $\mathrm{M}(\mathbf{D}, \mathbf{A}^\mathbf{C})$ resulting from applying the natural isomorphism $\mathbf{A}^{\mathbf{C}\times\mathbf{D}} \cong (\mathbf{A}^\mathbf{C})^\mathbf{D}$ to f, so that $f^D(y)[x] = f(x, y)$ for all $(x, y) \in P$. Similarly, write f^C for the member of $\mathrm{M}(\mathbf{C}, \mathbf{A}^\mathbf{D})$ resulting from rewriting $\mathbf{A}^{\mathbf{C}\times\mathbf{D}}$ as $(\mathbf{A}^\mathbf{D})^\mathbf{C}$. Through these identifications, we can rewrite φ as φ^{DD}, an automorphism of $(\mathbf{A}^\mathbf{C})^\mathbf{D}$, or as φ^{CC}, an automorphism of $(\mathbf{A}^\mathbf{D})^\mathbf{C}$, or as φ^{CD}, an isomorphism of $(\mathbf{A}^\mathbf{D})^\mathbf{C}$ with

$(\mathbf{A^C})^\mathbf{D}$, or as φ^{DC}, an isomorphism of $(\mathbf{A^C})^\mathbf{D}$ with $(\mathbf{A^D})^\mathbf{C}$. Recall that, by Lemma 5.2, each of $\mathbf{A^C}$ and $\mathbf{A^D}$ is L-dense, and of course has zero. Hence the conclusions in Lemmas 5.3 - 5.9 and Theorems 5.1 - 5.2 apply to all four of these isomorphisms, as well as to φ.

To begin the proof, we fix $a \in A$ and put $f = \langle a \rangle \in \mathrm{M}(\mathbf{C} \times \mathbf{D}, \mathbf{A})$, and we define $\bar{f} = \varphi(f)$, $g = \langle a_{xy} : (x,y) \in C \times D \rangle$, $\bar{g} = \varphi(g)$ just as before, with $\varphi(\langle a_{xy} \rangle) = \langle \bar{f}(x,y) \rangle$ for all $(x,y) \in C \times D$. Our task now is to prove that $\bar{g}$ does not depend on its second variable—i.e., that $\bar{g}$ is D-constant.

Now f^C is constant, and $\varphi^{CC}(f^C) = \bar{f}^C$, hence by Lemma 5.7 applied to $(\varphi^{CC})^{-1}$, for each $c \in C$ the function $(\varphi^{CC})^{-1}(\langle \bar{f}^C(c) \rangle)$ is constant—i.e., $\varphi^{-1}(\langle \bar{f}(c,y) : (x,y) \in C \times D \rangle)$ is C-constant. Replacing φ^{CC} in this argument by φ^{DC}, we conclude that also $\varphi^{-1}(\langle \bar{f}(c,y) : (x,y) \in C \times D \rangle)$ is D-constant. Thus there is an element $u_c \in A$ such that

$$\varphi(\langle u_c \rangle) = \langle \bar{f}(c,y) : (x,y) \in C \times D \rangle .$$

The isomorphism φ^{DD} has the zig-zag property by Theorem 5.1 (as $\mathbf{D}$ is connected and directly indecomposable). Thus we have the corresponding isomorphism $\tau_{\varphi^{DD}} : \mathbf{A^C} \cong \mathbf{A^C}$. An easy calculation, using the facts we have already compiled, establishes that for $c \in C$,

$$\varphi^{CD}(\langle g^C(c) \rangle) = \langle \tau_{\varphi^{DD}}(\langle u_c \rangle) \rangle .$$

Thus the element $\varphi^{CD}(\langle g^C(c) \rangle)$, which can be rewritten as $\varphi(\langle g(c,y) : (x,y) \in C \times D \rangle)$, is constant, meaning that it is D-constant.

We can apply Theorem 5.2 to the isomorphism φ^{CD}, since $\mathbf{C}$ and $\mathbf{D}$ have no nontrivial common direct factor. This gives that for the function $g^C \in \mathrm{M}(\mathbf{C}, \mathbf{A^D})$, and for all $(c,d) \in C \times D$, we have

$$\langle \varphi^{CD}(\langle g^C(c) \rangle)[d] \rangle = \varphi^{CD}(\langle ((\varphi^{CD})^{-1}(\langle \bar{g}^D(d) \rangle)[c] \rangle) .$$

Since $\varphi^{CD}(\langle g^C(c) \rangle)$ is D-constant, as noted in the last paragraph, then the displayed equation asserts that

$$\varphi^{CD}(\langle g^C(c) \rangle) = \varphi^{CD}(\langle ((\varphi^{CD})^{-1}(\langle \bar{g}^D(d) \rangle)[c] \rangle) \text{ for all } (c,d) \in C \times D ;$$

which can be rewritten as

$$\langle g^C(c) \rangle = \langle (\varphi^{CD})^{-1}(\langle \bar{g}^D(d) \rangle)[c] \rangle ,$$

and is equivalent to

$$g^C(c) = (\varphi^{CD})^{-1}(\langle \bar{g}^D(d)\rangle)[c] \text{ for all } (c,d) \in C \times D\,.$$

This means that $\varphi^{CD}(g^C) = \langle \bar{g}^D(d)\rangle$; equivalently, $\bar{g}$ is D-constant, just as we desired. •

6 Chains in $\mathbf{A}^\mathbf{P}$

It follows immediately from Theorem 5.1 (and its dual) that G. Birkhoff's cancellation conjecture for isomorphisms $\varphi : \mathbf{A}^\mathbf{P} \cong \mathbf{B}^\mathbf{P}$ with $\mathbf{A}, \mathbf{B}, \mathbf{P}$ finite is true in the case where $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are connected, $\mathbf{A}$ has a least or a largest element, and $\mathbf{P}$ is directly indecomposable. The decisive step on the way to the proof of the full conjecture was the removal of the hypothesis that $\mathbf{A}$ has a least or a largest element. From this, it follows by an inductive argument that requires less than a page that the conjecture is true when $\mathbf{P}$ is connected (no special assumptions on $\mathbf{A}, \mathbf{B}$). The remaining argument to remove the hypothesis that $\mathbf{P}$ is connected is lengthy, and involves establishing some unusual statements about matrix arithmetic over rings of polynomials in many variables with coefficients from an integral domain, but otherwise requires no new ideas.

We shall require some facts about chains in $\mathbf{A}^\mathbf{P}$, which were first observed by G. Birkhoff.

When $x \leq y$ in $\mathbf{A}$, $I_\mathbf{A}[x,y] = x{\uparrow} \cap y{\downarrow}$ is the interval determined by x and y. We say that *y covers x*, written $x \prec y$, provided that $|I_\mathbf{A}[x,y]| = 2$. We write $f \prec_b g$, where $f, g \in \mathbf{A}^\mathbf{B}$, $b \in B$, to denote that $f(b) \prec g(b)$ and for all $x \in B \setminus \{b\}$, $f(x) = g(x)$. It is easy to see that $f \prec g$ in $\mathbf{A}^\mathbf{B}$ is always equivalent to: $f \prec_b g$ for some $b \in B$. The *height* of $\mathbf{A}$, written $\mathrm{ht}(\mathbf{A})$, is the largest cardinality of a chain in $\mathbf{A}$, minus one. The *height of an element x in $\mathbf{A}$* is the height of the ordered set $x{\downarrow}$.

Lemma 6.1 *Let $\mathbf{A}, \mathbf{B}$ be finite and $f < g$ in $\mathbf{A}^\mathbf{B}$.*

(1) *If for all $b \in B$, C_b is a maximal chain in $I_\mathbf{A}[f(b), g(b)]$, then there exists a maximal chain C in $I_{\mathbf{A}^\mathbf{B}}[f,g]$ such that for all $b \in B$, $C_b = \{h(b) : h \in C\}$.*

(2) *The height of the interval* $I_{\mathbf{A}^\mathbf{B}}[f,g]$ *is the sum of the heights of the intervals* $I_\mathbf{A}[f(b), g(b)]$, $b \in B$.

(3) $\mathrm{ht}(I_{\mathbf{A}^\mathbf{B}}[f,g]) = \mathrm{ht}(\mathbf{A}^\mathbf{B})$ *iff* f *and* g *are constant on each connected component of* $\mathbf{B}$ *and* $\mathrm{ht}(I_\mathbf{A}[f(b), g(b)]) = \mathrm{ht}(\mathbf{A})$ *for all* $b \in B$.

(4) $\mathrm{ht}(\mathbf{A}^\mathbf{B}) = \mathrm{ht}(\mathbf{A}) \cdot |B|$.

PROOF. For (1), suppose that we have constructed $f = f_0 \prec f_1 \prec \cdots \prec f_k \leq g$ where $f_i(b) \in C_b$ for all $i \leq k$ and $b \in B$. If $f_k < g$, let $b \in B$ be maximal such that $f_k(b) < g(b)$. There is a unique element $c \in C_b$ such that $f_k(b) \prec c$. Define f_{k+1} so that $f_{k+1}(x) = f_k(x)$ for $x \in B \setminus \{b\}$ and $f_{k+1}(b) = c$. It is easy to see that $f_{k+1} \in \mathbf{A}^\mathbf{B}$ and $f_k \prec_b f_{k+1} \leq g$. This construction will produce the desired maximal chain.

For (2), if the C_b in (1) are chosen so that $|C_b| = \mathrm{ht}(I_\mathbf{A}[f(b), g(b)])$, then the maximal chain produced in (1) will have size one greater than the sum of the numbers $|C_b|$. On the other hand, if $f = f_0 \prec f_1 \prec \cdots \prec f_m = g$ is any maximal chain in the interval, then for each $b \in B$, the number of $i < m$ such that $f_i \prec_b f_{i+1}$ can be at most $\mathrm{ht}(I_\mathbf{A}[f(b), g(b)])$, so that m is at most the sum of the numbers $\mathrm{ht}(I_\mathbf{A}[f(b), g(b)])$, $b \in B$.

Statements (3) and (4) follow readily from (2). •

Recall that $R(\varphi)$ is defined in Definition 1.1 and $\mathcal{C}(\mathbf{A}^\mathbf{B})$ is defined in Definition 2.1.

Lemma 6.2 (G. Birkhoff) *Suppose that* $\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{Q}$ *are finite. If* $\varphi : \mathbf{A}^\mathbf{P} \cong \mathbf{B}^\mathbf{Q}$, *or if* $\varphi : \mathcal{C}(\mathbf{A}^\mathbf{P}) \cong \mathcal{C}(\mathbf{B}^\mathbf{Q})$, *then there is* $f \in M(\mathbf{P}, \mathbf{A})$ *such that* f *is constant on each connected component of* $\mathbf{P}$ *and* $\varphi(f)$ *is constant on each connected component of* $\mathbf{Q}$. *Thus if* $\mathbf{P}$ *and* $\mathbf{Q}$ *are connected, then* $R(\varphi) \neq \emptyset$.

PROOF. Suppose first that $\varphi : \mathbf{A}^\mathbf{P} \cong \mathbf{B}^\mathbf{Q}$. Let f be any element of $\mathrm{M}(\mathbf{P}, \mathbf{A})$ such that $\mathrm{ht}(f\downarrow) = \mathrm{ht}(\mathbf{A}^\mathbf{P})$. Then $\mathrm{ht}(\varphi(f)\downarrow) = \mathrm{ht}(\mathbf{B}^\mathbf{Q})$; and by Lemma 6.1, f and $\varphi(f)$ have the required property. The same proof works if $\varphi : \mathcal{C}(\mathbf{A}^\mathbf{P}) \cong \mathcal{C}(\mathbf{B}^\mathbf{Q})$. •

Corollary 6.1 *If* $\varphi : \mathbf{A}^\mathbf{P} \cong \mathbf{B}^\mathbf{Q}$, *where* $\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{Q}$ *are finite and* $\mathbf{A}$ *and* $\mathbf{B}$ *are connected, then* φ *induces by restriction an isomorphism of* $\mathcal{C}(\mathbf{A}^\mathbf{P})$ *with* $\mathcal{C}(\mathbf{B}^\mathbf{Q})$.

PROOF. Let $f \in \mathrm{M}(\mathbf{P}, \mathbf{A})$ be such that f is constant on each connected component of $\mathbf{P}$ and $\varphi(f)$ is constant on each connected component of $\mathbf{Q}$. Since $\mathbf{A}$ is connected, then $\mathcal{C}(\mathbf{A}^{\mathbf{P}})$ is the subposet of $\mathbf{A}^{\mathbf{P}}$ induced on the connected component containing f. Likewise, $\mathcal{C}(\mathbf{B}^{\mathbf{Q}})$ is the subposet of $\mathbf{B}^{\mathbf{Q}}$ induced on the connected component containing the element $\varphi(f)$. Thus $\varphi(\mathcal{C}(\mathbf{A}^{\mathbf{P}}) = \mathcal{C}(\mathbf{B}^{\mathbf{Q}})$. •

The hypothesis of connected bases in Corollary 6.1 is necessary.

Example 6.1 Let $\mathbf{D} = \mathbf{C}_2 + 1$ where $\mathbf{C}_2$ is the four-element crown $2 \oplus 2$ of example 4.1. Then $\mathbf{D}^{\mathbf{C}_2} = \mathcal{C}(\mathbf{C}_2^{\mathbf{C}_2}) + 1 + 1 + 1 + 1 + 1$ while $\mathcal{C}(\mathbf{D}^{\mathbf{C}_2}) = \mathcal{C}(\mathbf{C}_2^{\mathbf{C}_2}) + 1$. There is an automorphism $\varphi : \mathbf{D}^{\mathbf{C}_2} \cong \mathbf{D}^{\mathbf{C}_2}$ which exchanges one of the one-element connected components inside $\mathcal{C}(\mathbf{D}^{\mathbf{C}_2})$ with one of those outside $\mathcal{C}(\mathbf{D}^{\mathbf{C}_2})$. Thus $\varphi(\mathcal{C}(\mathbf{D}^{\mathbf{C}_2})) \neq \mathcal{C}(\mathbf{D}^{\mathbf{C}_2})$.

7 $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, n)$-isomorphisms

The key idea for extending the applicability of Theorem 5.1 to the case of an isomorphism $\varphi : \mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$ where $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are connected but $\mathbf{A}$ does not possess a least or a greatest element is to consider the induced isomorphisms $\varphi|_X : \mathbf{X} \cong \mathbf{Y}$ for a certain family of sub-posets $\mathbf{X} \subseteq \mathcal{C}(\mathbf{A}^{\mathbf{P}})$.

Definition 7.1 Suppose that $\mathbf{A}, \mathbf{B}$ are ordered sets, $S \subseteq A$ and $n \geq 0$. By $R_{n,S}(\mathbf{A})$ we mean the set of all $x \in A$ such that there exists $x_0, \ldots, x_n = x$ with $x_0 \in S$ and x_i, x_{i+1} comparable for all $i < n$. We write $R_{n,S}(\mathbf{A}^{\mathbf{B}})$ for the set $R_{n,\Delta_S}(\mathbf{A}^{\mathbf{B}})$, where $\Delta_S = \{\langle s \rangle : s \in S\} \subseteq \mathrm{M}(\mathbf{B}, \mathbf{A})$. In otherwords, $R_{n,S}(\mathbf{A}^{\mathbf{B}})$ is the set of all $f \in \mathrm{M}(\mathbf{B}, \mathbf{A})$ such that there exist $f_0, f_1, \ldots, f_n = f$ in $\mathrm{M}(\mathbf{B}, \mathbf{A})$ for which $f_0 = \langle s \rangle$ for some $s \in S$, and for $i < n$, the functions f_i, f_{i+1} are comparable.

We remark that with respect to the metric $d_{\mathbf{A}}(x, y)$ defined in §2, $R_{n,\{x\}}(\mathbf{A})$ is the ball of radius $n + 1$ about x. For connected $\mathbf{A}$ and $x \in A$, we have $A = \bigcup_n R_{n,\{x\}}(\mathbf{A})$. Note that in general, $R_{n,S}(\mathbf{A}^{\mathbf{B}})$ is a subset of the set of monotone functions from $\mathbf{B}$ into $R_{n,S}(\mathbf{A})$. It may fail to be identical with this set of monotone functions.

Definition 7.2 Assume that $\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{Q}$ are finite ordered sets. Let $n \geq 0$, $S \subseteq A$, $S' \subseteq B$, and $\delta : S \cong S'$. By an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{Q}}, n)$*-isomorphism* over δ we mean an isomorphism $\psi : R_{n,S}(\mathbf{A}^{\mathbf{P}}) \cong R_{n,S'}(\mathbf{B}^{\mathbf{Q}})$ with $\psi(\langle x \rangle) = \langle \delta(x) \rangle$ for all $x \in S$.

The reader should observe that when $\psi : \mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{Q}}$, $n \geq 0$, $S \subseteq R(\psi)$, $S' = \delta_\psi(S) \subseteq R(\psi^{-1})$ and $\delta = \delta_\psi|_S$, then ψ induces by restriction an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{Q}}, n)$-isomorphism over δ.

Definition 7.3 Suppose that $\lambda : \mathbf{X} \cong \mathbf{Y}$ where $\mathbf{X} \subseteq \mathbf{A}^{\mathbf{P}}$ and $\mathbf{Y} \subseteq \mathbf{B}^{\mathbf{Q}}$. We write $R(\lambda)$ for the set of $a \in A$ such that $\langle a \rangle$ belongs to X and $\lambda(\langle a \rangle) = \langle b \rangle$ for some b. For such an a, we put $\delta_\lambda(a) = b$. Thus $\delta_\lambda : R(\lambda) \cong R(\lambda^{-1})$.

We remark that if ψ is an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{Q}}, n)$-isomorphism over δ, then $S \subseteq R(\psi)$, $S' \subseteq R(\psi^{-1})$, and δ_ψ extends δ. Note also that ψ induces by restriction $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{Q}}, m)$-isomorphisms over δ, for all $0 \leq m < n$.

For the remainder of this section, we assume that $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are finite ordered sets and $\mathbf{P}$ is connected and directly indecomposable.

Lemma 7.1 *Suppose that $S \subseteq A$, $S' \subseteq B$, $R_{1,S}(\mathbf{A}) \subseteq X \subseteq \mathbf{A}^{\mathbf{P}}$, $R_{1,S'}(\mathbf{B}) \subseteq Y \subseteq \mathbf{B}^{\mathbf{P}}$, and $\lambda : X \cong Y$ with $S \subseteq R(\lambda)$, $S' \subseteq R(\lambda^{-1})$ and $\delta_\lambda(S) = S'$. Then for all $a \in R_{1,S}(\mathbf{A})$, $\lambda(\langle a \rangle) \in R(\lambda^{-1})^P$ and $\lambda(\delta_\lambda^{-1} \circ \lambda(\langle a \rangle)) = \langle b \rangle$ for some $b \in R_{1,\{b\}}(\mathbf{B})$. The function that maps $a \mapsto b$ is an isomorphism of $R_{1,S}(\mathbf{A})$ with $R_{1,S'}(\mathbf{B})$.*

Proof. For $a \in A$, let $\mathbf{A}(a\uparrow)$, $\mathbf{A}(a\downarrow)$ denote the posets that $\mathbf{A}$ induces on the sets $a\uparrow$, $a\downarrow$ respectively. Clearly, for $a \in S$ and $b = \delta_\lambda(a)$, λ induces by restriction isomorphisms $\mathbf{A}(a\uparrow) \cong \mathbf{B}(b\uparrow)$, $\mathbf{A}(a\downarrow) \cong \mathbf{B}(b\downarrow)$. Now this lemma follows directly from Theorem 5.1 and its dual applied to these isomorphisms. •

Lemma 7.2 *Suppose that ψ is an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, n)$-isomorphism over δ, $n \geq 1$. Then we have an $(S, S', \mathbf{A}, \mathbf{B}, 1)$-isomorphism $\tau_{1,\psi} : R_{1,S}(\mathbf{A}) \cong R_{1,S'}(\mathbf{B})$ extending δ, defined by $\tau_{1,\psi}(x) = y$, where $\langle y \rangle = \psi(\delta_\psi^{-1} \circ \psi(\langle x \rangle))$.*

PROOF. By Lemma 7.1. •

For the next three lemmas, we suppose that ψ is an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, n+1)$-isomorphism over δ, $n \geq 1$. Then we put $K = R_{1,S}(\mathbf{A})$ and $K' = R_{1,S'}(\mathbf{B})$ and set $T = R(\psi) \cap K$, $T' = R(\psi^{-1}) \cap K'$. Notice that

$$S \subseteq T \subseteq K \text{ and } S' \subseteq T' \subseteq K'.$$

Lemma 7.3 *Suppose that ψ is an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, n+1)$-isomorphism over δ, $n \geq 1$. Let T, T', K, K' be as defined above. Then*

$$R_{n,T}(\mathbf{A}^{\mathbf{P}}) \subseteq R_{n,K}(\mathbf{A}^{\mathbf{P}}) \subseteq R_{n+1,S}(\mathbf{A}^{\mathbf{P}}),$$

$$R_{n,T'}(\mathbf{B}^{\mathbf{P}}) \subseteq R_{n,K'}(\mathbf{B}^{\mathbf{P}}) \subseteq R_{n+1,S'}(\mathbf{B}^{\mathbf{P}})$$

and the map ψ' which is ψ restricted to $R_{n,T}(\mathbf{A}^{\mathbf{P}})$ is a $(T, T', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, n)$-isomorphism over δ_ψ restricted to T. Moreover, $R(\psi) \subseteq R_{n,T}(\mathbf{A})$ so that $R(\psi) = R(\psi')$. Likewise $R(\psi^{-1}) = R(\psi'^{-1})$ and we have that $\delta_\psi = \delta_{\psi'}$.

PROOF. The first statement should be perfectly obvious. Now suppose that $\psi(\langle x \rangle) = \langle y \rangle$, $x \in R(\psi)$, $y \in R(\psi^{-1})$. Take a chain $x_0, \ldots, x_{n+1} = x$ with $x_0 \in S$ and x_i, x_{i+1} comparable for $i \leq n$. Now $x_1 \in R_{1,S}(\mathbf{A})$ and so $\psi(\langle x_1 \rangle) \in T'^P$ by Lemma 7.2. Choose any $p \in P$ and consider the sequence $b_1, b_2, \ldots, b_{n+1} = y$ where $b_i = \psi(\langle x_i \rangle)[p]$. These elements are pairwise comparable and $b_1 \in T'$, hence $y \in R_{n,T'}(\mathbf{B})$. Now it immediately follows that $y \in R(\psi'^{-1})$. The same proof with ψ^{-1} replacing ψ shows that $x \in R_{n,T}(\mathbf{A})$, and $x \in R(\psi')$. •

Lemma 7.4 *Suppose that ψ is an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, n+1)$-isomorphism over δ, $n \geq 1$. Let T, T', K, K' be as above. If $f \in R_{n,K}(\mathbf{A}^{\mathbf{P}})$ (or $g \in R_{n,K'}(\mathbf{B}^{\mathbf{P}})$) then $\psi(f) \in (R_{n,T'}(\mathbf{B}))^P$ (or respectively, $\psi^{-1}(g) \in (R_{n,T}(\mathbf{A}))^P$).*

PROOF. Let $f \in R_{n,K}(\mathbf{A}^{\mathbf{P}})$, say we have $\langle a \rangle = f_1, f_2, \ldots, f_{n+1} = f$ with $a \in K$ and f_i, f_{i+1} comparable for $1 \leq i \leq n$. For any $p \in P$, we have the sequence of pairwise comparable elements $b_1, \ldots, b_{n+1}$ where $b_i = \psi(f_i)[p]$. By Lemma 7.2, $b_1 \in T'$, hence $b_{n+1} \in R_{n,T'}(\mathbf{B})$. •

Lemma 7.5 *Suppose that ψ is an $(S, S', \mathbf{A^P}, \mathbf{B^P}, n+1)$-isomorphism over δ, $n \geq 1$. Let T, T', K, K' be as above and suppose that $\tau' : R_{n,T}(\mathbf{A}) \cong R_{n,T'}(\mathbf{B})$ and τ' extends δ_ψ. For $f \in R_{n,K}(\mathbf{A^P})$, put $\psi''(f) = \psi(\tau'^{-1} \circ \psi(f))$. Then ψ'' is an $(K, K', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphism over the map $\tau_{1,\psi}$ of Lemma 7.2, and $\delta_{\psi''}$ extends δ_ψ.*

PROOF. Let $f \in R_{n,K}(\mathbf{A^P})$. Then we have, say, $\{f_0, f_1, \ldots, f_{n+1}\} \subseteq \mathbf{A^P}$ with $f_0 = \langle a_0 \rangle$, $f_1 = \langle a_1 \rangle$, $f_{n+1} = f$, $a_0 \in S$, $a_1 \in K$ and f_i, f_{i+1} comparable for all $i \leq n$. Put $g_i = \psi(f_i)$. Since all the functions f_i belong to $R_{n,K}(\mathbf{A^P})$, then by Lemma 7.4, $g_i : P \to R_{n,T'}(\mathbf{B})$. Thus, we can define $h_i = \tau'^{-1} \circ g_i$, and we have that $h_i \in \mathbf{A^P}$ and h_i, h_{i+1} are comparable for $i \leq n$. Moreover, $g_0 = \langle \delta_\psi(a_0) \rangle = \langle \tau'(a_0) \rangle$, so $h_0 = f_0 = \langle a_0 \rangle$. Thus it follows that $\{h_0, \ldots, h_{n+1}\} \subseteq R_{n+1,S}(\mathbf{A^P})$. Thus we have $\ell_i = \psi(h_i) \in R_{n+1,S'}(\mathbf{B^P})$. In particular, $\ell_{n+1} = \psi''(f)$ and, $\ell_0 = \langle \delta_\psi(a_0) \rangle = \psi(f_0)$. Note that $g_1 \in R(\psi^{-1})^P$ and $\ell_1 = \langle \tau_{1,\psi}(a_1) \rangle$ by Lemma 7.2. Thus the pairwise comparable sequence $\ell_1, \ldots, \ell_{n+1} = \psi''(f)$ demonstrates that $\psi''(f) \in R_{n,K'}(\mathbf{B^P})$.

Thus we see that ψ'' is an isomorphism of $R_{n,K}(\mathbf{A^P})$ with a subset of $R_{n,K'}(\mathbf{B^P})$ over the map $a \mapsto \tau_{1,\psi}(a)$, $a \in K$. By the same token, we have the isomorphism $g \mapsto \psi^{-1}(\tau' \circ \psi^{-1}(g))$, of $R_{n,K'}(\mathbf{B^P})$ with a subset of $R_{n,K}(\mathbf{A^P})$ over the map $b \mapsto \tau_{1,\psi}^{-1}(b)$, $b \in K'$. This map is obviously the inverse of ψ''. So it follows that ψ'' is an $(K, K', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphism over $\tau_{1,\psi}$.

Finally, suppose that $a \in R(\psi)$. Then $a \in R_{n+1,S}(\mathbf{A})$, $\langle a \rangle \in R_{n,K}(\mathbf{A^P})$ and $\psi(\langle a \rangle) = \langle \delta_\psi(a) \rangle$, and $\delta_\psi(a) = \tau'(a)$, so it is clear that $\delta_{\psi''}(a) = \delta_\psi(a)$. •

Theorem 7.1 *Let $\mathbf{A}, \mathbf{B}, \mathbf{P}$ be finite ordered sets with $\mathbf{P}$ connected and directly indecomposable. Suppose that $n > 0$, $\emptyset \neq S \subseteq A$, $\emptyset \neq S' \subseteq B$ and ψ is an $(S, S', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphism over δ. There exists an isomorphism $\tau : R_{n,S}(\mathbf{A}) \cong R_{n,S'}(\mathbf{B})$ which extends $\tau_{1,\psi}$.*

PROOF. We prove this by induction on n. For $n = 1$, apply Lemma 7.2 with $n = 1$ and $\tau = \tau_{1,\psi}$.

Suppose that $n \geq 1$ and the theorem is true for all $(S, S', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphisms ψ over δ, irrespective of S, S', δ. Let ψ be a $(S, S', \mathbf{A^P}, \mathbf{B^P}, n+1)$-isomorphism over δ.

Let T, T', K, K' be as in Lemmas 7.3, 7.4 and 7.5, so that ψ', or ψ restricted to $R_{n,T}(\mathbf{A^P})$, is a $(T, T', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphism over $\delta_\psi|_T$, and $\delta_{\psi'} = \delta_\psi$.

By the induction assumption, we have an isomorphism $\tau' : R_{n,T}(\mathbf{A}) \cong R_{n,T'}(\mathbf{B})$ which extends $\tau_{1,\psi'}$ and hence extends δ_ψ. By Lemma 7.5, we have a $(K, K', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphism ψ'' over $\tau_{1,\psi}$ with

$$\tau_{1,\psi''} \supseteq \delta_{\psi''} \supseteq \tau_{1,\psi} \supseteq \delta_\psi \, .$$

By the induction assumption again, we have an isomorphism $\tau : R_{n,K}(\mathbf{A}) \cong R_{n,K'}(\mathbf{B})$ which extends $\tau_{1,\psi''}$. Finally, it is trivial that $R_{n,K}(\mathbf{A}) = R_{n+1,S}(\mathbf{A})$ and $R_{n,K'}(\mathbf{B}) = R_{n+1,S'}(\mathbf{B})$, so our proof is finished. •

Definition 7.4 Let ψ be a $(S, S', \mathbf{A^P}, \mathbf{B^P}, n)$-isomorphism over δ, where $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are finite and $\mathbf{P}$ is connected and directly indecomposable. Our proof of Theorem 7.1 leads to a recursive definition of $\tau_\psi : R_{n,S}(\mathbf{A}) \cong R_{n,S'}(\mathbf{B})$ extending $\tau_{1,\psi}$. In this definition, τ_ψ actually depends not only upon ψ but upon the choice of n, S, S'; however, to keep a manageable system of notation, we denote it simply by τ_ψ.

Namely, if $n = 0$, then $\tau_\psi = \delta_\psi$. If $n = 1$, then $\tau_\psi = \tau_{1,\psi}$. Now suppose that $n = k + 1 > 1$. Then we put $K = R_{1,S}(\mathbf{A})$, $T = R(\psi) \cap K$, we put $K' = R_{1,S'}(\mathbf{B})$, $T' = R(\psi^{-1}) \cap K'$ and take

$$\psi_0 : R_{k,T}(\mathbf{A^P}) \cong R_{k,T'}(\mathbf{B^P})$$

to be the restriction of ψ and we take

$$\psi_1 : R_{k,K}(\mathbf{A^P}) \cong R_{k,K'}(\mathbf{B^P})$$

as defined by $\psi_1(f) = \psi(\tau_{\psi_0}^{-1} \circ \psi(f))$. Finally, we put

$$\tau_\psi = \tau_{\psi_1} \, .$$

Problem 7.1 Is it the case that under the assumptions of Theorem 7.1, ψ must have the zig-zag property, that is, for every $a \in R_{n,S}(\mathbf{A})$, $\psi(\langle a \rangle) \subseteq R(\psi^{-1})^P$ and $\psi(\delta_{\psi^{-1}} \circ \psi(\langle a \rangle))$ is constant?

Problem 7.2 Is it the true that every isomorphism $\mathbf{A^P} \cong \mathbf{B^P}$ with $\mathbf{A}, \mathbf{B}, \mathbf{P}$ finite and connected and $\mathbf{P}$ directly indecomposable must have the zig-zag property?

8 Cancellation of connected exponents

Theorem 8.1 *Suppose that* $\mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$*, or* $\mathcal{C}(\mathbf{A}^{\mathbf{P}}) \cong \mathcal{C}(\mathbf{B}^{\mathbf{P}})$*, where* $\mathbf{A}, \mathbf{B}, \mathbf{P}$ *are finite and* $\mathbf{P}$ *is connected. Then* $\mathbf{A} \cong \mathbf{B}$.

PROOF. First we consider that $\mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$ implies $\mathbf{A} \cong \mathbf{B}$. The proof is by double induction, first on $|\mathbf{P}|$, then on $|\mathbf{A}|$. We assume that $\phi : \mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$, and that $\mathbf{A}, \mathbf{B}, \mathbf{P}$ are finite, and $\mathbf{P}$ is connected. We also assume that whenever $\mathbf{C}^{\mathbf{Q}} \cong \mathbf{D}^{\mathbf{Q}}$, with $\mathbf{C}$ and $\mathbf{D}$ finite and $\mathbf{Q}$ connected, and either $|\mathbf{Q}| < |\mathbf{P}|$, or else $|\mathbf{Q}| = |\mathbf{P}|$ and $|\mathbf{C}| < |\mathbf{A}|$, then $\mathbf{C} \cong \mathbf{D}$.

If $\mathbf{P}$ is not directly indecomposable, say $\mathbf{P} \cong \mathbf{R} \times \mathbf{S}$ with $|\mathbf{R}| > 1$ and $|\mathbf{S}| > 1$, then $\mathbf{R}$ and $\mathbf{S}$ are connected and

$$(\mathbf{A}^{\mathbf{R}})^{\mathbf{S}} \cong \mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}} \cong (\mathbf{B}^{\mathbf{R}})^{\mathbf{S}}.$$

Using the induction assumption, we obtain $\mathbf{A}^{\mathbf{R}} \cong \mathbf{B}^{\mathbf{R}}$, and using it again, we conclude that $\mathbf{A} \cong \mathbf{B}$. Thus we can, and do, assume that $\mathbf{P}$ is directly indecomposable.

By Lemma 6.2, $R(\phi)$ is non-empty. We choose $a \in R(\phi)$ and put $S = \{a\}$, and where $\phi(\langle a \rangle) = \langle b \rangle$, we put $S' = \{b\}$. We choose $N > 0$ such that $R_{N,S}(\mathbf{A}) = R_{N+1,S}(\mathbf{A})$ and $R_{N,S'}(\mathbf{B}) = R_{N+1,S'}(\mathbf{B})$. Obviously, $R_{N,S}(\mathbf{A})$ and $R_{N,S'}(\mathbf{B})$ are the connected components of $\mathbf{A}$ and $\mathbf{B}$ containing the respective points a, b.

Our isomorphism ϕ induces by restriction a $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{P}}, N)$-isomorphism ψ over the map $a \mapsto b$. Since $\mathbf{P}$ is directly indecomposable and connected, Theorem 7.1 supplies us with an isomorphism τ of $R_{N,S}(\mathbf{A})$ with $R_{N,S'}(\mathbf{B})$.

Now we have $\mathbf{A} = \mathbf{U} + \mathbf{A}'$, $\mathbf{B} = \mathbf{V} + \mathbf{B}'$ where $\mathbf{U}$, $\mathbf{V}$ are the ordered subsets based on $R_{N,\{a\}}(\mathbf{A})$, $R_{N,\{b\}}(\mathbf{B})$ and $+$ denotes unordered sum. Since $\mathbf{P}$ is connected, $\mathbf{A}^{\mathbf{P}} = \mathbf{U}^{\mathbf{P}} + \mathbf{A}'^{\mathbf{P}}$ and $\mathbf{B}^{\mathbf{P}} = \mathbf{V}^{\mathbf{P}} + \mathbf{B}'^{\mathbf{P}}$. We have proved that $\mathbf{U} \cong \mathbf{V}$, and all these isomorphisms yield

$$\mathbf{U}^{\mathbf{P}} + \mathbf{A}'^{\mathbf{P}} \cong \mathbf{U}^{\mathbf{P}} + \mathbf{B}'^{\mathbf{P}}.$$

In this equation, we can cancel $\mathbf{U}^{\mathbf{P}}$ (by Corollary 4.1) and find that $\mathbf{A}'^{\mathbf{P}} \cong \mathbf{B}'^{\mathbf{P}}$. By the induction assumption, we have $\mathbf{A}' \cong \mathbf{B}'$. Then

$$\mathbf{A} = \mathbf{U} + \mathbf{A}' \cong \mathbf{V} + \mathbf{B}' = \mathbf{B}.$$

Our proof of Theorem 8.1 is complete.

The proof that $\mathbf{A} \cong \mathbf{B}$ under the assumption that $\mathcal{C}(\mathbf{A}^{\mathbf{P}}) \cong \mathcal{C}(\mathbf{B}^{\mathbf{P}})$ does not essentially differ from the proof under the assumption $\mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{P}}$, and is left to the reader. •

9 The extended algebras of finite ordered sets

Definition 9.1 We put

$$T = \{\mathbf{A} : \mathbf{A} \text{ is a finite, non-void ordered set}\} / \cong,$$

the set of isomorphism types of finite ordered sets. On this denumerable set, we have operations $x+y$ and $x \cdot y$ induced by G. Birkhoff's operations $\mathbf{A}+\mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$, and we have two possible exponentiation operations—the one induced by G. Birkhoff's exponentiation $\mathbf{A}^{\mathbf{B}}$, and the other induced by the operation $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$. The algebra $\langle T, 1, x+y, xy, x^y \rangle$ in which x^y is G. Birkhoff's operation will be denoted $\mathcal{T}$. The algebra $\langle T, 1, x+y, xy, x^y \rangle$ in which x^y is $\mathcal{C}(\mathbf{A}^{\mathbf{B}})/\cong$ when $x = \mathbf{A}/\cong$ and $y = \mathbf{B}/\cong$, will be denoted by $\mathcal{T}^c$. The addition and multiplication operations and the constant 1, denoting the type of a one-element ordered set, are the same in both algebras.

Definition 9.2 A finite ordered set $\mathbf{A}$ is *absolutely $\mathcal{C}$-indecomposable* if and only if $\mathbf{A}$ is connected and directly indecomposable, and whenever $\mathbf{A} \cong \mathcal{C}(\mathbf{B}^{\mathbf{P}})$ then $\mathbf{A} \cong \mathbf{B}$ and $\mathbf{P} \cong \mathbf{1}$.

To finish the proof of Birkhoff's conjecture, one must remove the hypothesis that $\mathbf{P}$ is connected from Theorem 8.1. The path to that goal taken in [21] involved obtaining a complete description of the algebra $\mathcal{T}^c$. Recall from §4 that with exponentiation removed, the algebra of order types can be satisfactorily represented as isomorphic to the semi-ring $\mathbf{Z}^+[\bar{x}]$ consisting of non-zero polynomials with positive integer coefficients over the indeterminates $x_0, x_1, \ldots$ In this representation, the connected and directly indecomposable ordered sets correspond to the indeterminates. The algebra $\mathcal{T}^c$, on the other hand, is generated by its subset $\mathcal{P}$ of types of absolutely indecomposable finite ordered sets. The word problem for this algebra with respect to this generating set was solved in [21], using two results which we formulate below. The solution showed, in particular, that every permutation of $\mathcal{P}$ extends to an automorphism of $\mathcal{T}^c$. (The situation for $\mathcal{T}$ is different, due to the fact that $\mathbf{A}^{\mathbf{B}}$ can fail to be $+$-indecomposable, even when $\mathbf{A}$ and $\mathbf{B}$ are three-ways indecomposable—see Example 4.1. It seems hopeless to seek a solution of the word problem for the algebra $\mathcal{T}$. It is interesting to observe that $\mathcal{T}^c$ can be represented in various ways, as isomorphic to a subalgebra of $\mathcal{T}$. For example, $\mathcal{T}^c$ is isomorphic to the subalgebra of $\mathcal{T}$ generated by those types $\tau \in T$ such that $\tau = \mathbf{E}/\cong$ where $\mathbf{E}$ is some absolutely $\mathcal{C}$-indecomposable ordered set with a least element—within this subalgebra, the two exponentiation operations coincide.)

The effective description of $\mathcal{T}^c$ reduces the exponential cancellation property for the operation $\mathcal{C}(\mathbf{A}^{\mathbf{P}})$ to an unusual statement about matrix arithmetic over $\mathbf{Z}^+[\bar{x}]$. This statement was formulated and proved in [21], yielding the result that $\mathcal{T}^c$ has exponential cancellation: $x^z = y^z \to x = y$. In the inductive argument to prove G. Birkhoff's conjecture that $\mathcal{T}$ has exponential cancellation, as given in [21], it proved to be sufficient at every stage to deal with appropriately chosen sub-posets of the form $\mathcal{C}(\mathbf{U}^{\mathbf{V}})$. That proof reduced the desired result to a second statement about matrix arithmetic over $\mathbf{Z}^+[\bar{x}]$. These two statements are the content of Lemma 7.1 in [21].

The algebra $\mathcal{T}^c$ is a very interesting object. I will not reproduce the remainder of the proof of G. Birkhoff's conjecture in this paper. Rather, I state the two results needed to describe the algebra $\mathcal{T}^c$, provide a new and simpler proof of one of them, and then explore this algebra. In the next and final section of this paper, I explore the groups $\mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$ using the zig-zag theorems from §5.

Theorem 9.1 *Suppose that* $\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{Q}$ *are finite and connected and* $\mathbf{P}$ *and* $\mathbf{Q}$ *are directly indecomposable and non-isomorphic. If* $\mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{Q}}$ *or if* $\mathcal{C}(\mathbf{A}^{\mathbf{P}}) \cong \mathcal{C}(\mathbf{B}^{\mathbf{Q}})$ *then there is a connected ordered set* $\mathbf{E}$ *with* $\mathbf{A} \cong \mathcal{C}(\mathbf{E}^{\mathbf{Q}})$ *and* $\mathbf{B} \cong \mathcal{C}(\mathbf{E}^{\mathbf{P}})$.

Theorem 9.2 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite and connected and* $\mathcal{C}(\mathbf{A}^{\mathbf{B}}) \cong \prod_{i\in I} \mathbf{C}_i$. *There exist ordered sets* $\mathbf{A}_i$ *with* $\mathbf{C}_i \cong \mathcal{C}(\mathbf{A}_i^{\mathbf{B}})$ *for* $i \in I$, *such that* $\mathbf{A} \cong \prod_{i\in I} \mathbf{A}_i$.

The above are Theorems 5.2 and 5.3 in [21]. The proof of Theorem 9.1 will be omitted here. It parallels the proof of Theorem 7.1, uses $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{B}^{\mathbf{Q}}, n)$-isomorphisms induced by an isomorphism $\mathbf{A}^{\mathbf{P}} \cong \mathbf{B}^{\mathbf{Q}}$ (or $\mathcal{C}(\mathbf{A}^{\mathbf{P}}) \cong \mathcal{C}(\mathbf{B}^{\mathbf{Q}})$), and begins with an application of Theorem 5.2. It is a good exercise for the reader to attempt to construct this proof.

Proof of Theorem 9.2. It will suffice to handle the case $I = \{0, 1\}$. Let $\psi : \mathcal{C}(\mathbf{A}^{\mathbf{B}}) \cong \mathbf{C}_0 \times \mathbf{C}_1$. Putting $E = \psi(\Delta_A)$, our first objective is to prove that E is a rectangle, i.e., $E = A_0 \times A_1$ for some $A_i \subseteq C_i$. For $c = (c_0, c_1) \in C_0 \times C_1$, we write $p_i(c) = c_i$ $(i \in \{0, 1\})$.

Claim 1: Suppose that $a \in A, f \in \mathcal{C}(\mathbf{A}^{\mathbf{B}})$ and $\langle a \rangle \leq f$ or $f \leq \langle a \rangle$. If $i \in \{0, 1\}$ and $p_i(\psi(f)) = p_i(\psi(\langle a \rangle)$ then for all $b \in B$, $p_i(\psi(\langle f(b) \rangle)) = p_i(\psi(\langle a \rangle)$.

To prove the claim, we can take $i = 0$ without losing generality. Let $b \in B$, put $(c_0, c_1) = \psi(\langle a \rangle)$ and suppose that $\psi(f) = (c_0, q)$ and $f(b) = a'$. Without loss of generality, we assume that $\langle a \rangle \leq f$. For $x \in B$ put $h_x = \langle a, a'[z \geq x] \rangle$, and write (u_x, v_x) for $\psi(h_x)$. Note that if $y \leq y'$ then $h_{y'} \leq h_y$. We claim that if $y < y'$ and $u_{y'} = c_0$ then also $u_y = c_0$. To prove this, define $h = \psi^{-1}(u_y, c_1)$, $h' = \psi^{-1}(c_0, v_y)$. Now $u_{y'} = c_0$ and $v_{y'} \leq v_y$ implies $h_{y'} \leq h' \leq h_y$. Also, $(u_y, c_1) \wedge (c_0, v_y) = (c_0, c_1)$ and $(u_y, c_1) \vee (c_0, v_y) = (u_y, v_y)$ in $\mathbf{C}_0 \times \mathbf{C}_1$, so $h \wedge h' = \langle a \rangle$ and $h \vee h' = h_y$ in $\mathcal{C}(\mathbf{A}^{\mathbf{B}})$. Since $h_{y'} \leq h' \leq h_y$, then $h'(y') = a'$, giving $h(y') = a$ (from the fact that $h \wedge h' = \langle a \rangle$); and we get also $h(y) = a$ since $y < y'$, thus implying that $h'(y) = a'$ (since $\langle a \rangle \leq h, h' \leq h_y$ and $h \vee h' = h_y$). Now $h'(y) = a'$ forces $h' = h_y$. Thus $u_y = c_0$ as claimed.

Now notice that since $f(b) = a'$ then $\langle a \rangle \leq h_b \leq f$, implying that $u_b = c_0$. Then the fact just proved, together with the connectivity of $\mathbf{B}$, yields that $u_x = c_0$ for all x. Since $\langle a' \rangle = \bigvee_{x \in B} h_x$, then $p_0\psi(\langle a' \rangle) = c_0$. This finishes our proof of Claim 1.

Claim 2: Suppose that $a_0, a_1 \leq a$ in A and $\psi(\langle a_i \rangle) = (u_i, v_i)$ $(i \in \{0, 1\})$. Then $(u_0, v_1), (u_1, v_0) \in E$.

To see it, put $\psi(\langle a \rangle) = (u, v)$. Let $\psi(f) = (u_0, v)$. Then $\langle a_0 \rangle \leq f \leq \langle a \rangle$ and $p_0\psi(f) = p_0\psi(\langle a_0 \rangle)$ and $p_1\psi(f) = p_1\psi(\langle a \rangle)$. Two applications of Claim 1 yield that $\psi(\langle f(b) \rangle) = (u_0, v)$ for every $b \in B$; i.e., f is constant. In the same way, we deduce that where $\psi(g) = (u, v_1)$, g is constant. Now since $\mathbf{C}_0 \times \mathbf{C}_1 \models (u_0, v) \wedge (u, v_1) = (u_0, v_1)$, it follows that $\mathcal{C}(\mathbf{A}^{\mathbf{B}}) \models f \wedge g =$

$\psi^{-1}(u_0, v_1)$. It follows that $\psi^{-1}(u_0, v_1)$ is constant. By the same token, $\psi^{-1}(u_1, v_0)$ is constant.

Claim 3: $E \ (= \psi(\Delta_A)) = A_0 \times A_1$ for some $A_i \subseteq C_i$.

We must prove that for any $a_0, a_1 \in A$, $(p_0\psi(\langle a_0 \rangle), p_1\psi(\langle a_1 \rangle)) \in E$. For some n, there are $e_0 = a_0, e_1, \ldots, e_n = a_1$ and there are m_i $(0 \leq i < n)$ so that $e_i, e_{i+1} \leq m_i$. Put $\psi(\langle e_i \rangle) = (u_i, v_i)$ and $\psi(\langle m_i \rangle) = (r_i, s_i)$. By Claim 2, $(u_i, v_{i+1}) \in E$ for $i < n$ and $(r_i, s_{i+1}) \in E$ for $i < n-1$. Moreover, $(u_i, v_{i+1}), (u_{i+1}, v_{i+2}) \leq (r_i, s_{i+1})$ for $i < n-1$. Then in the same way, $(u_i, v_{i+2}) \in E$ for $i < n-1$ and $(r_i, s_{i+2}) \in E$ for $i < n-2$. Continuing in this way, we find, eventually, that $(u_0, v_n) \in E$, which is the desired result.

Writing $\mathbf{A}_i$ for the ordered sub-poset of $\mathbf{C}_i$ based on the set A_i, we now have an isomorphism $\phi : \mathbf{A} \cong \mathbf{A}_0 \times \mathbf{A}_1$ defined by $\phi(a) = \psi(\langle a \rangle)$. This gives a collateral isomorphism $\bar{\phi} : \mathcal{C}(\mathbf{A}^{\mathbf{B}}) \cong \mathcal{C}(\mathbf{A}_0^{\mathbf{B}}) \times \mathcal{C}(\mathbf{A}_1^{\mathbf{B}})$. We compare the isomorphisms $\psi, \bar{\phi}$.

By Proposition 3.1, there are ordered sets $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ and isomorphisms

$$\begin{aligned} \alpha : \mathbf{X} \times \mathbf{Y} &\cong \mathbf{C}_0\,, \\ \beta : \mathbf{Z} \times \mathbf{W} &\cong \mathbf{C}_1\,, \\ \delta : \mathbf{X} \times \mathbf{Z} &\cong \mathcal{C}(\mathbf{A}_0^{\mathbf{B}})\,, \\ \gamma : \mathbf{Y} \times \mathbf{W} &\cong \mathcal{C}(\mathbf{A}_1^{\mathbf{B}}) \end{aligned}$$

so that for all $(x, y, z, w) \in X \times Y \times Z \times W$,

$$\psi^{-1}(\alpha(x, y), \beta(z, w)) = \bar{\phi}^{-1}(\delta(x, z), \gamma(y, w))\,.$$

Claim 5: $|Y| = |Z| = 1$.

To prove this, let $a \in A$ and $\psi(\langle a \rangle) = (p, q) = (\alpha(x, y), \beta(z, w))$. Since $\mathbf{Y}$ is connected, to see that $|Y| = 1$, it will suffice to show that whenever $y' \in Y$ is comparable to y then $y' = y$. So let y' be comparable to y. Now we have $\psi(f) = (p', q) = (\alpha(x, y'), \beta(z, w))$ for some $f \in \mathcal{C}(\mathbf{A}^{\mathbf{B}})$. Thus

$$\psi(\langle a \rangle) = (p, q) = (\alpha(x, y), \beta(z, w))\,,$$

$$\psi(f) = (p', q) = (\alpha(x, y'), \beta(z, w))\,,$$

$$\bar{\phi}(\langle a \rangle) = (\langle p \rangle, \langle q \rangle) = (\delta(x, z), \gamma(y, w))\,,$$

$$\bar{\phi}(f) = (\langle p \rangle, g) = (\delta(x,z), \gamma(y',w))\,.$$

Now $\langle a \rangle$ is comparable to f and $p_1\psi(\langle a \rangle) = p_1\psi(f) = q$; hence, by Claim 1, $p_1\psi(\langle f(b) \rangle) = q$ for all $b \in B$. But this means that $g = \langle q \rangle$, implying that $f = \langle a \rangle$, and $y = y'$.

Since $|Y| = |Z| = 1$, then $\mathbf{C}_0 \cong \mathbf{X} \cong \mathcal{C}(\mathbf{A}_0^{\mathbf{B}})$ and $\mathbf{C}_1 \cong \mathbf{W} \cong \mathcal{C}(\mathbf{A}_1^{\mathbf{B}})$ and the isomorphisms $\psi, \bar{\phi}$ are naturally isomorphic. •

Theorem 9.3 *A finite ordered set* $\mathbf{A}$ *is connected and directly indecomposable (i.e., is* $+$*- and* $\times$ *-indecomposable) iff it can be represented as isomorphic to* $\mathcal{C}(\mathbf{E}^{\mathbf{P}})$ *for some absolutely* $\mathcal{C}$*-indecomposable* $\mathbf{E}$ *and some connected* $\mathbf{P}$*. For given* $\mathbf{A}$*, the absolutely* $\mathcal{C}$*-indecomposable ordered set* $\mathbf{E}$ *and the connected ordered set* $\mathbf{P}$ *in this representation are unique up to isomorphism.*

Proof. We begin by observing that Theorem 9.2 implies that $\mathcal{C}(\mathbf{E}^{\mathbf{P}})$ is directly indecomposable when $\mathbf{P}$ is connected and $\mathbf{E}$ is directly indecomposable and connected; and that $\mathcal{C}(\mathbf{E}^{\mathbf{P}})$ is by definition connected when $\mathbf{E}$ is connected.

In the other direction, when $|\mathbf{A}| > 1$ and $|\mathbf{P}| > 1$ and $\mathbf{A} \cong \mathcal{C}(\mathbf{E}^{\mathbf{P}})$ is finite, then the cardinality of $\mathbf{A}$ exceeds that of $\mathbf{E}$; hence (for $+$- and $\times$-indecomposable finite $\mathbf{A}$) if $\mathbf{A} \cong \mathcal{C}(\mathbf{E}^{\mathbf{P}})$ then ($\mathbf{P}$ is connected and $\mathbf{E}$ is $+$- and $\times$-indecomposable, and) if $|\mathbf{E}|$ is minimal for such a representation, then $\mathbf{E}$ must be absolutely $\mathcal{C}$-indecomposable.

It remains to prove the uniqueness of the representation. We prove that $\mathcal{C}(\mathbf{E}^{\mathbf{C}}) \cong \mathcal{C}(\mathbf{F}^{\mathbf{D}})$ with all structures finite, $\mathbf{E}$ and $\mathbf{F}$ absolutely $\mathcal{C}$-indecomposable, and $\mathbf{C}$ and $\mathbf{D}$ connected, always implies $\mathbf{E} \cong \mathbf{F}$ and $\mathbf{C} \cong \mathbf{D}$. If this fails, there is a counterexample $\mathcal{C}(\mathbf{E}^{\mathbf{C}}) \cong \mathcal{C}(\mathbf{F}^{\mathbf{D}})$ with $\mathcal{C}(\mathbf{E}^{\mathbf{C}})$ of minimum size. Suppose first that $\mathbf{C}$ and $\mathbf{D}$ have a common directly indecomposable factor $\mathbf{P}$, say $\mathbf{C} \cong \mathbf{P} \times \mathbf{R}$, $\mathbf{D} \cong \mathbf{P} \times \mathbf{S}$. Now $\mathcal{C}(\mathcal{C}(\mathbf{E}^{\mathbf{R}})^{\mathbf{P}}) \cong \mathcal{C}(\mathcal{C}(\mathbf{F}^{\mathbf{S}})^{\mathbf{P}})$, so by Theorem 8.1, we conclude that $\mathcal{C}(\mathbf{E}^{\mathbf{R}}) \cong \mathcal{C}(\mathbf{F}^{\mathbf{S}})$. Our minimality assumption now implies that $\mathbf{E} \cong \mathbf{F}$ and $\mathbf{R} \cong \mathbf{S}$, giving a clear contradiction.

Next, suppose that $\mathbf{C}$ and $\mathbf{D}$ have no common indecomposable factors. If either were a one-element structure, then the absolute $\mathcal{C}$-indecomposability of $\mathbf{E}$ and $\mathbf{F}$ would imply that $\mathbf{E} \cong \mathbf{F}$ and $\mathbf{C} \cong \mathbf{D}$. Thus we can write $\mathbf{C} \cong \mathbf{P} \times \mathbf{R}$ and $\mathbf{D} \cong \mathbf{Q} \times \mathbf{S}$ where $\mathbf{P}$ and $\mathbf{Q}$ are directly indecomposable and, of course, non-isomorphic. We can now apply Theorem 5.2 to the isomorphism $\mathcal{C}(\mathcal{C}(\mathbf{E}^{\mathbf{R}})^{\mathbf{P}}) \cong \mathcal{C}(\mathcal{C}(\mathbf{F}^{\mathbf{S}})^{\mathbf{Q}})$. This produces a connected ordered set

M with $\mathcal{C}(\mathbf{E}^{\mathbf{R}}) \cong \mathcal{C}(\mathbf{M}^{\mathbf{Q}})$ and $\mathcal{C}(\mathbf{F}^{\mathbf{S}}) \cong \mathcal{C}(\mathbf{M}^{\mathbf{P}})$. Write $\mathbf{M} \cong \mathcal{C}(\mathbf{G}^{\mathbf{L}})$ with **G** absolutely $\mathcal{C}$-indecomposable and **L** connected. Now the minimality of our chosen example implies that the isomorphism $\mathcal{C}(\mathbf{E}^{\mathbf{R}}) \cong \mathcal{C}(\mathbf{G}^{\mathbf{LQ}})$ gives $\mathbf{E} \cong \mathbf{G}$ and $\mathbf{R} \cong \mathbf{L} \times \mathbf{Q}$. But now, **Q** proves to be a divisor of both **C** and **D**. This contradiction establishes the theorem. •

We can now consider the word-problem for $\mathcal{T}^c$. This algebra satisfies A. Tarski's "high school equations" (see Proposition 4.1 and the remark that follows it):

(1) $1x = x$, $x^1 = x$, $1^x = 1$,

(2) $x + y = y + x$, $xy = yx$

(3) $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$,

(4) $x(y + z) = xy + xz$,

(5) $x^{yz} = (x^y)^z$,

(6) $x^{y+z} = x^y + x^z$,

(7) $(xy)^z = x^z y^z$,

and the rule:

(8') if $c = p^t$ with $p \in \mathcal{P}$ and $t \in T$, then for all $\{x, y\} \subseteq T$, $(x+y)^c = x^c + y^c$.

It follows from Theorem 9.1, and J. Hashimoto's results on isomorphic products of connected ordered sets, and from the elementary fact that every ordered set can be written in essentially only one way as a sum of connected ordered sets, that every finite ordered set **A** has an essentially unique representation as a sum of ordered sets each of which is isomorphic to **1** or is itself expressed as a product of ordered sets of the form $\mathcal{C}(\mathbf{E}^{\mathbf{P}})$ with **E** absolutely $\mathcal{C}$-indecomposable, **P** connected (and $|\mathbf{P}| < |\mathbf{A}|$). This means that any member m of $\mathcal{T}^c$, written as a term without variables in the operations $1, x + y, xy, x^y$ applied to the generators $\mathcal{P}$, can be re-written using the equations (1)–(7) and the rule (8') as a sum of terms each of which is either 1, or is itself a term in which 1 and + do not occur and in which every subterm of the form c^d has $c \in \mathcal{P}$. This expression for m is unique up to commutativity and associativity of addition and multiplication. I call it the "normal form" for m.

The equality of two members of $\mathcal{T}^c$, expressed as generated by $\mathcal{P}$, can be tested by constructing their normal forms and checking whether these are commutative-associative equivalent. The operations of $\mathcal{T}^c$, considered as acting on normal forms, can easily be computed. Thus $\mathcal{T}^c$ is a "recursive algebra of exponential polynomials".

One approach to understanding these normal forms for elements of $\mathcal{T}^c$ is to identify them with multi-sets whose individual entries are $\mathcal{P}$-labelled forests (unions of $\mathcal{P}$-labelled trees), including possibly empty forests (representing a one-element connected component). In this approach, the $+$-and $\times$-indecomposable elements correspond to $\mathcal{P}$-labelled trees. In the $\mathcal{P}$-labelled tree for $\mathcal{C}(\mathbf{E}^{\mathbf{P}})/\cong$ where $\mathbf{E}/\cong \in \mathcal{P}$ and $\mathbf{P}$ is connected, the label at the root is $\mathbf{E}/\cong$ and the labelled subtrees above the root are identified with the $\times$-indecomposable factors of $\mathbf{P}/\cong$.

During my seminar presentation of these ideas, Nikolaos Galatos pointed out that there is a way to view $\mathcal{T}^c$ as a free algebra over a set of equations. Let $\mathcal{T}_2^c = \langle T, P, \mu, 1, x+y, xy, x^y \rangle$ be the two-sorted algebra with universes T and $P = \mathcal{P}$, with all operations of $\mathcal{T}^c$ acting in the universe T, and with the operation $\mu : P \to T$ being the identity function $\mu(p) = p$ for $p \in \mathcal{P}$. The rule (8') then becomes the multi-sorted equation

(8) $(x+y)^{\mu(p)^t} = x^{\mu(p)^t} + y^{\mu(p)^t}$ for $p \in P$ and $x, y, t \in T$.

Then it is easy to check that $\mathcal{T}_2^c$ is the free two-sorted algebra freely generated by the set P included in the second universe, with respect to the equations (1)–(8).

10 The automorphism group of $\mathbf{A}^{\mathbf{P}}$

An ordered set $\mathbf{A}$ will be said to be *exponentially indecomposable* iff $\mathbf{A} \cong \mathbf{B}^{\mathbf{X}}$ implies $\mathbf{X} \cong \mathbf{1}$. $\mathbf{A}$ is called *bounded* provided that it has both 0 and 1. Below is the chief result of B. Jónsson [18].

Theorem 10.1 *Suppose* $\mathbf{Q}$ *is a bounded, directly indecomposable poset that satisfies the descending chain condition, and suppose* $\mathbf{Q} \cong \mathbf{A}^{\mathbf{B}}$*, where* $\mathbf{A}$ *is exponentially indecomposable. Then*

$$\mathrm{Aut}(\mathbf{Q}) \cong \mathrm{Aut}(\mathbf{A}) \times \mathrm{Aut}(\mathbf{B}) .$$

B. Jónsson actually shows under these assumptions, that every automorphism of $\mathbf{Q}$ takes the form $f \mapsto \alpha \circ f \circ \gamma$ where $\alpha \in \mathrm{Aut}(\mathbf{A})$ and $\gamma \in \mathrm{Aut}(\mathbf{B})$.

Theorem 10.1 fails if its assumptions are relaxed to require only that $\mathbf{Q}$ has 0, but not 1. For example, let $\mathbf{A}$ be the ordered set underlying the five-element non-modular lattice—$\mathbf{A} \cong \mathbf{1} \oplus (\mathbf{1} + \mathbf{2}) \oplus \mathbf{1}$—and let $\mathbf{B}$ be obtained by removing the top element of $\mathbf{A}$. Theorem 10.1 implies that $\mathrm{Aut}(\mathbf{A}^{\mathbf{2}})$ is a one-element group. On the other hand, we have $\mathrm{Aut}(\mathbf{B}^{\mathbf{2}}) \cong \mathbf{Z}_2$. (It may be instructive to draw the Hasse diagrams of $\mathbf{A}^{\mathbf{2}}$ and $\mathbf{B}^{\mathbf{2}}$.)

Nevertheless, our zig-zag theorems provide some tools for investigating the automorphism group of $\mathbf{A}^{\mathbf{P}}$ when $\mathbf{A}$ is L-dense with 0 and $\mathbf{P}$ is connected. These groups seem quite interesting. Theorem 10.2 below was an early result. (It is Theorem 11.2 in B. Jónsson, R. McKenzie [17].)

An ordered set (or any structure) is called *rigid* provided that it possesses only one automorphism (the identity automorphism). A group G is said to be an *extension of* N *by* H if N is a normal subgroup of G, and H is a group isomorphic with G/N.

Theorem 10.2 *Suppose that* $\mathbf{A}$ *is* L*-dense with* 0*, and rigid. Suppose also that* $\mathbf{P}$ *is connected and directly indecomposable. If the logarithm,* $\mathbf{L}(\mathbf{A})$*, has precisely* N *connected components, then* $\mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$ *is isomorphic to an extension of a subgroup of* $(\mathrm{Aut}(\mathbf{P}))^N$ *by an elementary Abelian* 2*-group.*

In this section, we attempt to make progress toward understanding the group $\mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$ where $\mathbf{A}$ is a finite ordered set with 0 and $\mathbf{P}$ is a finite, connected, directly indecomposable ordered set. This is at present a largely unexplored topic. We also have some modest results for the case where $\mathbf{A}$ is connected but does not have 0 or 1. Where possible, the results will be stated and proved so as to apply, like Theorem 10.2, to certain infinite ordered sets as well as finite ones.

Definition 10.1 Where $G = \mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$ (the automorphism group), we define

$$D_1 = \{\phi \in G : \quad \phi(\langle a \rangle) = \langle a \rangle \quad \text{for all } a \in A\} \quad \text{and}$$

$$D = \{\phi \in G : \quad \phi(\Delta_A) = \Delta_A\} = \{\phi \in G : \quad R(\phi) = A = R(\phi^{-1})\}\,.$$

We define K to be the normal subgroup of G generated by D, and K_1 to be the normal subgroup of G generated by D_1. For $\tau \in \mathrm{Aut}(\mathbf{A})$, we use $\mathring{\tau}$ to denote the automorphism of $\mathbf{A}^{\mathbf{P}}$ such that $\mathring{\tau}(f) = \tau \circ f$ for all $f \in \mathrm{M}(\mathbf{P}, \mathbf{A})$. Finally, we put $D_0 = \{\mathring{\tau}\colon \tau \in \mathrm{Aut}(\mathbf{A})\}$.

In the following sequence of results, we freely use the concepts and results of §5.

Theorem 10.3 *Suppose that* $\mathbf{A}$ *and* $\mathbf{P}$ *are ordered sets and* G, D, D_0, D_1, K, K_1 *are defined as above.*

(i) D, D_0, D_1, K *and* K_1 *are subgroups of* G.

(ii) D_1 *is a normal subgroup of* D, $D_0 \cap D_1 = \{\mathrm{id}\}$ *and* $D_0 D_1 = D$. *Thus* D *is isomorphic with a semidirect product of* D_0 *and* D_1.

(iii) *If* $\mathbf{A}$ *is L-dense with* 0, *and* $\mathbf{P}$ *is connected and directly indecomposable, then* $D_1 \subseteq K_1 \subseteq D$.

(iv) $D_0 \cong \mathrm{Aut}(\mathbf{A})$ *and, if* $\mathbf{A}$ *is finite with* 0, *and* $\mathbf{P}$ *is connected and directly indecomposable, and if* N *is the number of connected components of* $L(\mathbf{A})$, *then* D_1 *can be embedded into* $\mathrm{Aut}(\mathbf{P})^N$.

Proof. (i) is clear. For (ii), it is clear that D_1 is normal in D and $D_0 \cap D_1 = \{\mathrm{id}\}$. To see that $D = D_0 D_1$, let $\varphi \in D$. Then $\delta_\varphi \in \mathrm{Aut}(\mathbf{A})$ and for $a \in A$, we have $\varphi(\langle a \rangle) = \langle \delta_\varphi(a) \rangle$, and thus where $\tau = \mathring{\delta}_\varphi^{-1} \varphi$ we have $\tau(\langle a \rangle) = \langle a \rangle$. So $\varphi = \mathring{\delta}_\varphi \tau$ with $\mathring{\delta}_\varphi \in D_0$ and $\tau \in D_1$.

To prove (iii), it suffices to show that every conjugate in G of a member of D_1 belongs to D. Let $\varphi \in D_1$ and $\alpha \in G$, and set $\beta = \alpha^{-1}\varphi\alpha$. Then let $a \in A$ and put $q = \beta(\langle a \rangle)$. By Theorem 5.1, both α and $\varphi\alpha$, and their inverses, have the zig-zag property. Also, since $\varphi \in D_1$, then $R(\varphi\alpha) = R(\alpha)$,

$\delta_{\varphi\alpha} = \delta_\alpha$, and $R((\varphi\alpha)^{-1}) = R(\alpha^{-1})$. Since $\varphi\alpha$ has the zig-zag property, $\alpha(q) = \varphi\alpha(\langle a\rangle) \in R(\alpha^{-1})^P$ and we can calculate

$$\varphi\alpha(\delta_\alpha^{-1} \circ \alpha(q)) = \varphi\alpha(\delta_{\varphi\alpha}^{-1} \circ \varphi\alpha(\langle a\rangle)) = \langle \tau_{\varphi\alpha}(a)\rangle\,.$$

With $\varphi \in D_1$, this means that $\alpha(\delta_\alpha^{-1} \circ \alpha(q)) = \langle \tau_{\varphi\alpha}(a)\rangle$. But this equation implies that

$$q = \langle \tau_\alpha^{-1}\tau_{\varphi\alpha}(a)\rangle\,;$$

i.e., $q \in \Delta_A$, $a \in R(\beta)$, and $\delta_\beta(a) = \tau_\alpha^{-1}\tau_{\varphi\alpha}(a)$. Since the same argument will work with φ replaced by φ^{-1}, we can conclude that $\beta \in D$.

In (iv), it is obvious that $D_0 \cong \mathrm{Aut}(\mathbf{A})$. From the next lemma, the restriction homomorphism $r : G \to \mathrm{Aut}(\mathbf{L}(\mathbf{A}^{\mathbf{P}}))$ is injective on D_1 and the image $r(D_1)$ is naturally isomorphic to a subgroup of $\mathrm{Aut}(\mathbf{P})^N$. •

Lemma 10.1 *Let* $\mathbf{A}$ *be* L*-dense with* 0*, and* $\mathbf{P}$ *be connected and directly indecomposable. Let* G, D, D_1 *be as in Definition 10.1, and let* $\varphi \in G$*.*

(i) $\varphi \in D$ *iff for every connected component* A_i *of* $L(\mathbf{A})$ *there is a connected component* A_i' *of* $L(\mathbf{A})$ *and an isomorphism* p_i *of* A_i *with* A_i' *(with respect to the order inherited from* $\mathbf{A}$*), and an automorphism* q_i *of* $\mathbf{P}$ *so that for all* $a \in A_i$ *and* $x \in P$*,* $\varphi(\langle 0, a\rangle_x) = \langle 0, p_i(a)\rangle_{q_i(x)}$*.*

(ii) $\varphi \in D_1$ *implies that for each connected component* A_i *of* $L(\mathbf{A})$ *there is an automorphism* q_i *of* $\mathbf{P}$ *so that for all* $a \in A_i$ *and* $x \in P$*,* $\varphi(\langle 0, a\rangle_x) = \langle 0, a\rangle_{q_i(x)}$*.*

(iii) *Suppose that* $\mathbf{A}^{\mathbf{P}}$ *has the descending chain condition. If* $\varphi \in D_1$ *then* φ *is the identity element of* G *iff* φ *acts as the identity on* $L(\mathbf{A}^{\mathbf{P}})$*.*

Theorem 10.4 *Let* $\mathbf{A}$ *be* L*-dense with* 0 *and* $\mathbf{P}$ *be a connected and directly indecomposable ordered set. For each* $\varphi \in G = \mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$*, if* $\tau = \tau_\varphi^{-1}$*, then* $(\overset{\circ}{\tau}\,\varphi)^2 \in D_1$*. It follows that* $\mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$ *is an extension of* K *by a Boolean group.*

PROOF OF THEOREM 10.4. Since φ and φ^{-1} have the zig-zag property, we have that τ_φ is an automorphism of $\mathbf{A}$, and for all $a \in A$ holds $\varphi(\langle a\rangle) \in R(\varphi^{-1})^P$ and

$$\varphi(\delta_\varphi^{-1} \circ \varphi(\langle a\rangle)) = \langle \tau_\varphi(a)\rangle .$$

It is easily checked that τ_φ agrees with δ_φ on $R(\varphi)$. This means that the displayed equation can be rewritten as

$$\varphi \stackrel{\circ}{\tau} \varphi(\langle a\rangle) = \langle \tau_\varphi(a)\rangle \quad \text{or, equivalently}$$

$$\stackrel{\circ}{\tau} \varphi \stackrel{\circ}{\tau} \varphi(\langle a\rangle) = \langle a\rangle .$$

This holds for all $a \in A$. It follows that $\stackrel{\circ}{\tau} \varphi \stackrel{\circ}{\tau} \varphi = \sigma \in D_1$. Then

$$\varphi^2 = \stackrel{\circ}{\tau}{}^{-1} \sigma(\varphi^{-1} \stackrel{\circ}{\tau}{}^{-1} \varphi) \in K$$

since $\stackrel{\circ}{\tau} \in D_0 \subseteq K$ and $\sigma \in D_1 \subseteq K$. We have proved that the quotient group of G by K (which is the normal subgroup generated by $D_0 \cup D_1$) is a Boolean group. •

We remark that in case $|D_0| = 1$, $K = K_1 = D_1 = D$ and G is an extension of D_1 by a Boolean group (by Theorem 10.3). This is essentially how Theorem 10.2 was proved.

PROOF OF LEMMA 10.1. We return to Lemmas 5.3–5.9, taking $\mathbf{B} = \mathbf{A}$ and $\mathbf{C} = \mathbf{D} = \mathbf{P}$ there. Thus we have $L(\mathbf{A})$ (the logarithm of $\mathbf{A}$) $= \sum_{t \in T} A_t = \sum_{t \in T} B_t$ where the A_t (and also the B_t) are a one-one list of the connected components of $L(\mathbf{A})$. We have $\psi : \mathbf{L}(\mathbf{A}) \times \mathbf{P}^\partial \cong \mathbf{L}(\mathbf{A}) \times \mathbf{P}^\partial$ satisfying $\psi(A_t \times P) = B_t \times P$ for all $t \in T$, and for $(a, p) \in L(\mathbf{A}) \times P$, $\psi(a, p) = (b, q)$ iff $\varphi(\langle 0, a\rangle_p) = \langle 0, b\rangle_q$. Moreover, we have for each t, posets $\mathbf{X}_t, \mathbf{Y}_t, \mathbf{Z}_t, \mathbf{W}_t$ and isomorphisms

$$\begin{aligned} \alpha_t(\mathbf{X}_t \times \mathbf{Y}_t) &\cong \mathbf{A}_t , \\ \gamma_t(\mathbf{Z}_t \times \mathbf{W}_t) &\cong \mathbf{P}^\partial , \\ \beta_t(\mathbf{X}_t \times \mathbf{Z}_t) &\cong \mathbf{B}_t , \\ \delta_t(\mathbf{Y}_t \times \mathbf{W}_t) &\cong \mathbf{P}^\partial \end{aligned}$$

such that for all $(x, y, z, w) \in X_t \times Y_t \times Z_t \times W_t$,

$$\psi(\alpha_t(x, y), \gamma_t(z, w)) = (\beta_t(x, z), \delta_t(y, w)) , \quad \text{i.e.,}$$

$$\varphi(\langle 0, \alpha_t(x,y)\rangle_{\gamma_t(z,w)}) = \langle 0, \beta_t(x,z)\rangle_{\delta_t(y,w)} .$$

For (i), we simply note that the conclusion in (i) is easily seen to be equivalent to: for all $t \in T$, $|Y_t| = |Z_t| = 1$, which by Lemma 5.3 is equivalent to $R(\varphi) = A = R(\varphi^{-1})$, i.e., to $\varphi \in D$.

The conclusion in (ii) is equivalent to: for all $t \in T$, $A_t = B_t$, and $Y_t = \{y_t\}$ and $Z_t = \{z_t\}$ where for all $x \in X_t$, $\alpha_t(x, y_t) = \beta_t(x, z_t)$. The displayed equation above makes it clear that this is forced if $\varphi \in D_1$. We remark that this condition may hold even if $\varphi \in D \setminus D_1$. For example, let $\mathbf{A} = 1 \oplus 2 \oplus 2$ and $\mathbf{P} = \mathbf{2}$. $L(\mathbf{A})$ consists of the two atoms of $\mathbf{A}$. Let $\varphi = \overset{\circ}{\tau}$ where τ is the automorphism of $\mathbf{A}$ which exchanges the two maximal elements of $\mathbf{A}$ and fixes the remaining three points.

For (iii), suppose that $\varphi \in D_1$ and φ is the identity on $L(\mathbf{A}^\mathbf{P})$. First, I claim that for all $a \in A$ and $p \in P$, $\varphi(\langle 0, a\rangle_p) = \langle 0, a\rangle_p$. Write $f = \langle 0, a\rangle_p$ and $g = \varphi(f)$. Assume that $g \neq f$. Since $\mathbf{A}^\mathbf{P}$ has the descending chain condition, we cannot have $g \geq f$ (else $g > \varphi^{-1}(g) > \varphi^{-2}(g) > \cdots$). Hence it follows that $g(p) \not\geq a$. However, $g \leq \varphi(\langle a\rangle) = \langle a\rangle$. Thus we must have $g(p) = a' < a$. We can choose $u \in L(\mathbf{A})$, $u \leq a$, $u \not\leq a'$ (by Lemma 5.1). Then $h = \langle 0, u\rangle_p \leq f$ and $h \not\leq g$. But $h \in L(\mathbf{A}^\mathbf{P})$ by Lemma 5.2. We thus have $h = \varphi(h) \leq \varphi(f) = g$—contradiction.

Now for any $f \in \mathrm{M}(\mathbf{P}, \mathbf{A})$, we have

$$f = \bigvee_{p \in P} \langle 0, f(p)\rangle_p \ \text{ in } \mathbf{A}^\mathbf{P}$$

implying that

$$\varphi(f) = \bigvee_{p \in P} \varphi(\langle 0, f(p)\rangle_p) = \bigvee_{p \in P} \langle 0, f(p)\rangle_p = f .$$

This ends the proof of Lemma 10.1. •

Theorem 10.4 suffers from the defect that I have been unable to prove anything about the normal closure in $\mathrm{Aut}(\mathbf{A}^\mathbf{P})$ of the subgroup D_0. It may be very large, for all I know. The next theorem, in which it is no longer assumed that $\mathbf{A}$ has 0 or 1, also suffers from this defect, and has a weaker conclusion that Theorem 10.4.

Theorem 10.5 *Let* $\mathbf{A}$ *and* $\mathbf{P}$ *be finite, connected ordered sets and assume that* $\mathbf{P}$ *is directly indecomposable. Where* D_0 *and* D_1 *are the subgroups of* $G = \mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$ *defined in Definition 10.1, the quotient group of* G *over the normal subgroup* K *generated by* $D = D_0 D_1$ *is a group of order* 2^a *for some integer* a.

PROOF. Let $\phi \in G$. We wish to show that $\phi^{2^n} \in K$ for some positive n. Put $S = R(\phi)$ and $S' = R(\phi^{-1})$. By Lemma 6.2, $S \neq \emptyset$. For $n \geq 0$, put $S_n = R_{n,S}(\mathbf{A})$ (so that $S_0 = S$). Choose $N > 0$ with $S_N = A$. Now ϕ induces an $(S, S', \mathbf{A}^{\mathbf{P}}, \mathbf{A}^{\mathbf{P}}, N)$-isomorphism over δ_φ and so, by Theorem 7.1, there is $\alpha_0 \in \mathrm{Aut}(\mathbf{A})$ so that α_0 extends $\tau_{1,\phi}$ (which is an isomorphism of $R_{1,S}(\mathbf{A})$ with $R_{1,S'}(\mathbf{A})$). Now put $\phi_0 = \phi$ and $\phi_1 = (\mathring{\alpha}_0^{-1}\phi)^2$. Since ϕ restricted to $R_{1,S}(\mathbf{A}^{\mathbf{P}})$ has the zig-zag property, it follows from the proof of Theorem 10.4 that $R(\phi_1) \supseteq S_1$ and δ_{ϕ_1} is the identity on S_1.

Now, inductively, suppose that $\phi_0, \ldots, \phi_n$ have been defined so that δ_{ϕ_n} is the identity on S_n, and $R(\phi_n) \supseteq S_n$. Then apply Theorem 7.1 once more to obtain $\alpha_n \in \mathrm{Aut}(\mathbf{A})$ extending τ_{1,ϕ_n} on $R_{1,S_n}(\mathbf{A}) = S_{n+1}$. Put $\phi_{n+1} = (\mathring{\alpha}_n^{-1}\phi_n)^2$. Then again, $R(\phi_{n+1}) \supseteq S_{n+1}$ and $\delta_{\phi_{n+1}}$ is the identity on S_{n+1}.

Since $S_N = A$, it follows that $\phi_N \in D_1$. Since every $\mathring{\alpha}_i \in D_0$, it should be clear from the construction that $\phi^{2^N} \in \langle D_0 \rangle D_1 \subseteq K$. •

Theorem 10.6 *Let* $\mathbf{A}$ *and* $\mathbf{P}$ *be finite connected ordered sets. Assume that every non-trivial direct divisor of* $\mathbf{P}$ *is of cardinality larger than* $|\mathbf{A}|$ *(or, more generally, assume that for every* $a \in A$, $\mathbf{P}$ *has no common non-trivial direct divisors with any connected component of* $\mathbf{L}((a\uparrow)^\partial)$ *or of* $\mathbf{L}((a\downarrow)^\partial)$*). Then* $R(\phi) = A$ *for every* $\phi \in Aut(\mathbf{A}^{\mathbf{P}})$*, hence (with the notation of Definition 10.1),* $G = D = D_0 D_1$.

PROOF. Let $\phi \in \mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$. By Lemma 6.2, $S = R(\phi) \neq \emptyset$. Suppose that $a \in S$ and $a' \in A$ is comparable to a. It will suffice to show that $a' \in S$. Without losing generality, let $a \leq a'$. Define $\mathbf{Q}$ to be the sub-poset of $\mathbf{A}$ based on $a\uparrow$. Since $a \in S$, then we have $\phi(\mathbf{Q}^{\mathbf{P}}) = \mathbf{R}^{\mathbf{P}}$ where $\phi(\langle a \rangle) = \langle b \rangle$ and $\mathbf{R}$ is the sub-poset based on $b\uparrow$. Since $\mathbf{Q}, \mathbf{R}$ have 0, the results of section §5 apply to this isomorphism. Our assumption implies that, in that analysis, $|Y_t| = |Z_t| = 1$ for all t. This means that $\phi(\Delta_Q) = \Delta_R$, i.e., $Q \subseteq R(\phi)$. We have shown that $a' \in S$, concluding the proof. •

For the final theorem, we suppose that $\mathbf{A}$ is a L-dense ordered set with 0, and that $\mathbf{P}$ is a connected ordered set, and $\mathbf{P} = \mathbf{C} \times \mathbf{D}$ where $\mathbf{C} \ncong \mathbf{D}$ and $\mathbf{C}$ and $\mathbf{D}$ are directly indecomposable. We define $G = \mathrm{Aut}(\mathbf{A}^{\mathbf{P}})$. For $\alpha \in \mathrm{Aut}(\mathbf{A}^{\mathbf{C}})$ (or $\beta \in \mathrm{Aut}(\mathbf{A}^{\mathbf{D}})$), we write α_p for the member of G which becomes $\mathring{\alpha}$ through the natural identification of $\mathbf{A}^{\mathbf{P}}$ with $(\mathbf{A}^{\mathbf{C}})^{\mathbf{D}}$ (or β_p for the member of G which becomes $\mathring{\beta}$ through the natural identification of $\mathbf{A}^{\mathbf{P}}$ with $(\mathbf{A}^{\mathbf{D}})^{\mathbf{C}}$). We define

$$G_c = \{\alpha_p : \alpha \in \mathrm{Aut}(\mathbf{A}^{\mathbf{C}})\} \text{ and } G_d = \{\beta_p : \beta \in \mathrm{Aut}(\mathbf{A}^{\mathbf{D}})\}.$$

Theorem 10.7 *Under the above assumptions, we have that*

$$G = G_c G_d$$

and $G_c \cong \mathrm{Aut}(\mathbf{A}^{\mathbf{C}})$, $G_d \cong \mathrm{Aut}(\mathbf{A}^{\mathbf{D}})$, $G_c \cap G_d = D_0$, $|G||\mathrm{Aut}(\mathbf{A})| = |G_c| \cdot |G_d|$.

PROOF. To see that $G = G_c G_d$, choose $\phi \in G$. In order to apply Theorem 5.2, write ϕ^{CD} for the isomorphism $(\mathbf{A}^{\mathbf{D}})^{\mathbf{C}} \cong (\mathbf{A}^{\mathbf{C}})^{\mathbf{D}}$ that results from rewriting the domain of ϕ as $(\mathbf{A}^{\mathbf{D}})^{\mathbf{C}}$ and the range as $(\mathbf{A}^{\mathbf{C}})^{\mathbf{D}}$. By Theorem 5.2, we have an isomorphism $\psi : \mathbf{A}^{\mathbf{D}} \cong \mathbf{E}^{\mathbf{D}}$ where $\mathbf{E} = \langle R(\phi^{CD}), \leq \rangle$, so that for $f \in \mathrm{M}(\mathbf{D}, \mathbf{A})$,

$$\begin{aligned} \phi(\langle f(y) : (x,y) \in P \rangle) = g \qquad & \text{where for } d \in D, \\ \langle g(x,d) : (x,y) \in P \rangle \;=\; & \phi(\langle \psi(f)(d)(y) : (x,y) \in P \rangle). \end{aligned} \tag{5}$$

By Theorem 5.1, we can choose an isomorphism $\lambda : \mathbf{E} \cong \mathbf{A}$.

From equation (3) in the proof of Theorem 5.2, we have that for $f \in \mathrm{M}(\mathbf{C}, \mathrm{M}(\mathbf{D}, \mathbf{A}))$, $\bar{f} = \phi^{CD}(f)$,

$$\phi^{CD}(\langle f(c) \rangle)[d] = \delta_{\phi^{CD}}((\phi^{CD})^{-1}(\langle \bar{f}(d) \rangle)[c]).$$

Expressed in terms of ϕ, this means that where $f \in \mathrm{M}(\mathbf{P}, \mathbf{A})$, $\bar{f} = \phi(f)$, $(c,d) \in P$, we have

$$\begin{aligned} &\langle \phi(\langle f(c,y) : (x,y) \in P \rangle)[z,d] : (z,w) \in P \rangle = \\ &\quad \phi(\langle \phi^{-1}(\langle \bar{f}(x,d) : (x,y) \in P \rangle)[c,w] : (z,w) \in P \rangle). \end{aligned} \tag{6}$$

We define α, β mapping $\mathrm{M}(\mathbf{P},\mathbf{A})$ into itself. For $f \in \mathrm{M}(\mathbf{P},\mathbf{A})$, $\bar{f} = \phi(f)$, $(c,d) \in C \times D$, we put

$$\begin{aligned} \alpha(f)[c,d] &= \phi(\langle \lambda^{-1}(f(x,d))(y) : (x,y) \in P \rangle)[c,d]\,, \\ \beta(f)[c,d] &= \lambda(\langle \phi^{-1}(\langle \bar{f}(x,d) : (x,y) \in P \rangle)[c,z] : z \in D \rangle)\,. \end{aligned} \tag{7}$$

The claim is that $\alpha \in G_c$, $\beta \in G_d$ and $\alpha \circ \beta = \phi$. It is easy to see that α, β are monotone self-maps of $\mathbf{A}^{\mathbf{P}}$. To see that they are automorphisms, we introduce their inverses.

$$\begin{aligned} \beta'(f) &= \phi^{-1}\left[\langle \phi(\langle \lambda^{-1}(g(z,y))(w) : (z,w) \in P \rangle)(x,y) : (x,y) \in P \rangle\right] \\ \alpha'(f)[c,d] &= \lambda(\langle \phi^{-1}(\langle f(x,d) : (x,y) \in P \rangle)[c,y] : y \in D \rangle)\,. \end{aligned} \tag{8}$$

These are also obviously monotone self-maps of $\mathbf{A}^{\mathbf{P}}$.

The proof that $G = G_c G_d$ is finished by showing that $\alpha \circ \alpha' = \alpha' \circ \alpha = \mathrm{id}$, $\beta \circ \beta' = \beta' \circ \beta = \mathrm{id}$, $\phi = \alpha \circ \beta$, $\alpha \in G_c$ and $\beta \in G_d$. These calculations, using equations (4)–(7), are left to the reader.

Now to see that $G_c \cap G_d = D_0$, let $\phi \in G_c \cap G_d$. It is clear that $\phi \in D$. Thus there is $\alpha \in D_0$ so that $\psi = \alpha\phi \in D_1$. We have $\psi \in D_1 \cap G_c \cap G_d$, and we must show that $\psi = \mathrm{id}$. It will suffice to show that for all $p = (c,d) \in P$ and $a \in A$, we have $\psi(\langle 0,a \rangle_p) = \langle 0,a \rangle$. So let $g = \psi(\langle 0,a \rangle_p$. Note that $g \leq \psi(\langle a \rangle) = \langle a \rangle$. Now for any $p' = (c',d') \in P$, consider $g(p')$. Since $\psi \in G_c$, if $c' \not\geq c$ then $g(p') = 0$. Likewise, if $d' \not\geq d$ then $g(p') = 0$. Thus we conclude that $g(p') = 0$ when $p' \not\geq p$. This, together with the fact that $g \leq \langle a \rangle$, gives that $g \leq \langle 0,a \rangle_p$. The same argument, applied to ψ^{-1} gives that $\psi^{-1}(\langle 0,a \rangle_p) \leq \langle 0,a \rangle_p$; equivalently, $\langle 0,a \rangle_p \leq \psi(\langle 0,a \rangle_p) = g$. So in fact, we have $\psi(\langle 0,a \rangle_p) = \langle 0,a \rangle_p$, as required.

Finally, it is an elementary fact in group theory that for subgroups G_c, G_d of a group G, $|G_c G_d| = (|G_c||G_d|)\,/|G_c \cap G_d|$. •

References

[1] H. Bauer, Garben und Automorphismen geordneter Mengen, Dissertation, Technische Hochschule Darmstadt, 1982.

[2] C. Bergman, R. McKenzie, Sz. Nagy, *How to cancel a linearly ordered exponent,* Colloquium Mathematica Societatis János Bolyai 21 (1982), 87–94.

[3] G. Birkhoff, *Extended arithmetic,* Duke Mathematical Journal 3 (1937), 311–316.

[4] G. Birkhoff, *Generalized arithmetic,* Duke Mathematical Journal 9 (1942), 283–302.

[5] G. Birkhoff, Lattice Theory, Second Edition, Colloquium Publications, Vol. **25**, American Mathematical Society, Providence, R.I.

[6] C. C. Chang, B. Jónsson, A. Tarski, *Refinement properties for relational structures,* Fundamenta Mathematicae **55** (1964), 249–281.

[7] B. A. Davey, D. Duffus, R. W. Quackenbush, I. Rival, *Exponents of finite simple lattices,* J. London Math. Soc. **17** (1978), 203–221.

[8] M. M. Day, *Arithmetic of ordered systems,* Trans. Amer. Math. Soc. **58** (1945), 1–43.

[9] D. Duffus, I. Rival, *A logarithmic property for exponents of partially ordered sets,* Canadian Journal of Mathematics **30** (1978), 797–807.

[10] D. Duffus, B. Jónsson, I. Rival, *Structure results for function lattices,* Canadian Journal of Mathematics **30** (1978), 392–400.

[11] D. Duffus, R. Wille, *A theorem on partially ordered sets of order-preserving mappings,* Proceedings American Mathematical Society **76** (1979), 14–16.

[12] D. Duffus, *Powers of ordered sets,* Order **1** (1982), 83–92.

[13] J. D. Farley, *The automorphism group of a function lattice: a problem of Jónsson and McKenzie,* Algebra Universalis **36** (1996), 8–45.

[14] J. Hashimoto, *On the product decomposition of partially ordered sets,* Mathematica Japonica **1** (1948), 120–123.

[15] J. Hashimoto, T. Nakayama, *On a problem of G. Birkhoff,* Proceedings of the American Mathematical Society **1** (1950), 141–142.

[16] J. Hashimoto, *On direct product decomposition of partially ordered sets,* Annals of Mathematics (2) 54 (1951), 315–318.

[17] B. Jónsson, R. McKenzie, *Powers of partially ordered sets: cancellation and refinement properties,* Mathematica Scandinavica 51 (1982), 87–120.

[18] B. Jónsson, *Powers of partially ordered sets: the automorphism group,* Mathematica Scandinavica 51 (1982), 121–141.

[19] L. Lovasz, *Operations with structures,* Acta Mathematica Academiae Scientiarum Hungaricae **18** (1967), 337–339.

[20] R. McKenzie, *Arithmetic of finite ordered sets: cancellation of exponents, I,* Order **16** (1999), 313–333.

[21] R. McKenzie, *Arithmetic of finite ordered sets: cancellation of exponents, II,* Order **17** (2000), 309–332.

[22] M. Novotný, *Über gewisse Eigenschaften von Kardinaloperationen,* Spisy Přírod. Fak. Univ. Brno (1960), 465–484.