

DEFINABILITY IN SUBSTRUCTURE ORDERINGS, IV: FINITE LATTICES

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ABSTRACT. Let \mathcal{L} be the ordered set of isomorphism types of finite lattices, where the ordering is by embeddability. We study first-order definability in this ordered set. Our main result is that for every finite lattice \mathbf{L} , the set $\{\ell, \ell^{\text{opp}}\}$ is definable, where ℓ and ℓ^{opp} are the isomorphism types of \mathbf{L} and its opposite (\mathbf{L} turned upside down). We shall show that the only non-identity automorphism of \mathcal{L} is the map $\ell \mapsto \ell^{\text{opp}}$.

1. INTRODUCTION AND FIRST CONCEPTS

This paper is the last in a series of four exploring definability in substructure orderings. The paper [2] dealt with finite semilattices, [3] deals with finite ordered sets, [4] treats finite distributive lattices, and here we deal with finite lattices. The set \mathcal{L} of isomorphism types of finite lattices is denumerable. This set becomes a poset under the order induced by the substructure relation—we put $l_0 \leq l_1$, where l_i is the type of the finite lattice \mathbf{L}_i , iff \mathbf{L}_0 is isomorphic to a sublattice of \mathbf{L}_1 . In this way we obtain a poset $\langle \mathcal{L}, \leq \rangle$. We explore the scope of first-order definitions in the structure $\langle \mathcal{L}, \leq \rangle$.

Every lattice has its opposite. For a lattice $\mathbf{A} = \langle A, \wedge, \vee \rangle$ we denote by \mathbf{A}^{opp} the lattice $\langle A, \vee, \wedge \rangle$. The map $\mathbf{A} \mapsto \mathbf{A}^{\text{opp}}$ induces an automorphism, $\ell \mapsto \ell^{\text{opp}}$, of the ordered set $\langle \mathcal{L}, \leq \rangle$. Our goal here is to show that this is the only non-identity automorphism of $\langle \mathcal{L}, \leq \rangle$, and that $\{\ell, \ell^{\text{opp}}\}$ is first-order definable in this structure, for every $\ell \in \mathcal{L}$. For this purpose, it proves to be convenient to fix a constant $p_1 \in \mathcal{L}$. With the proper choice of $p_1 \neq p_1^{\text{opp}}$, we shall be able to prove that $\{\ell\}$ is first-order definable in the pointed ordered set $\langle \mathcal{L}, \leq, p_1 \rangle$ for every $\ell \in \mathcal{L}$.

Our principal object of investigation will actually be the quasi-ordered set LATT whose members are all the lattices \mathbf{A} whose members constitute a finite set of non-negative integers. The quasi-ordering is the substructure ordering, so that $\mathbf{A} \leq \mathbf{B}$ means that \mathbf{A} is isomorphic to a sublattice of \mathbf{B} . Denote by \mathbf{P}_1 a lattice belonging to LATT with elements a_0, a_1, a_2, a_3, a_4 and covers $a_0 < a_1 < a_2 < a_4$ and $a_1 < a_3 < a_4$, and by p_1 the isomorphism type

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of \mathbf{P}_1 . We define LATT' to be the pointed quasi-ordered set $\langle \text{LATT}, \leq, \mathbf{P}_1 \rangle$, and \mathcal{L}' to be the pointed ordered set $\langle \mathcal{L}, \leq, p_1 \rangle$.

When we say that a subset of LATT or a relation over LATT is definable in LATT' , we shall mean definable by a formula in the first-order language with two non-logical symbols, \leq and \mathbf{P}_1 , and without the equality symbol. To denote that two lattices are isomorphic, we write $\mathbf{A} \cong \mathbf{B}$. The relation $\{(\mathbf{A}, \mathbf{B}) : \mathbf{A} \cong \mathbf{B}\}$ is definable in LATT' —since $\mathbf{A} \cong \mathbf{B}$ iff $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A}$ for finite \mathbf{A} and \mathbf{B} —and it is easily proved (say by induction on the complexity of formulas) that for every formula $\varphi(x_0, \dots, x_{n-1})$ in this language and for $\mathbf{A}_0, \mathbf{B}_0, \dots, \mathbf{A}_{n-1}, \mathbf{B}_{n-1} \in \text{LATT}$ with $\mathbf{A}_i \cong \mathbf{B}_i$ for $i < n$ we have $\text{LATT}' \models \varphi(\mathbf{A}_0, \dots, \mathbf{A}_{n-1})$ if and only if $\text{LATT}' \models \varphi(\mathbf{B}_0, \dots, \mathbf{B}_{n-1})$. Thus with our convention about the language (omitting equality) first-order definability in LATT' is only “up to isomorphism”. In particular, $\{\mathbf{P}_1\}$ is not definable, although $\{\mathbf{A} : \mathbf{A} \cong \mathbf{P}_1\}$ is definable. However, we write that “ \mathbf{P}_1 is a definable member of LATT' ”, meaning that it is definable up to isomorphism; and we shall generally use this language with respect to all definable elements, definable subsets and definable relations over LATT' .

The relation of isomorphism, definable in LATT' , is an equivalence relation over LATT that gives rise to the pointed ordered set \mathcal{L}' of isomorphism types. Via the map sending $\mathbf{A} \in \text{LATT}$ to $\mathbf{A}/\cong \in \mathcal{L}$, definable relations over LATT' become definable relations over \mathcal{L}' , and conversely. Thus working over LATT' is simply a convenient means to give a more concrete feel to the study of definability over \mathcal{L}' .

We now introduce some very basic concepts for our study. We use $\mathbf{A} < \mathbf{B}$ to denote that $\mathbf{A} \leq \mathbf{B}$ and the two lattices are not isomorphic. The least and largest elements of \mathbf{A} will be denoted $0_{\mathbf{A}}$ and $1_{\mathbf{A}}$. For every $n \geq 0$, we denote by \mathbf{C}_n the chain of height n (so that $|\mathbf{C}_n| = n+1$). The height $\text{ht}(\mathbf{A})$ of a finite lattice \mathbf{A} is the greatest n for which $\mathbf{C}_n \leq \mathbf{A}$. For every $n \geq 1$ we denote by \mathbf{M}_n the only lattice of height 2 with n atoms. Thus we have $\mathbf{C}_2 \cong \mathbf{M}_1$.

By a *cut-point* in a lattice \mathbf{A} we mean an element $c \in A$ that is comparable to all elements of \mathbf{A} .

For two finite lattices \mathbf{A}, \mathbf{B} we denote by $\mathbf{A} \oplus \mathbf{B}$ the lattice \mathbf{C} that has a cut-point c such that the interval $I[0_{\mathbf{C}}, c]$ is isomorphic to \mathbf{A} and the interval $I[c, 1_{\mathbf{C}}]$ is isomorphic to \mathbf{B} . For two non-trivial finite lattices \mathbf{A}, \mathbf{B} we denote by $\mathbf{A} + \mathbf{B}$ the lattice with the underlying set the disjoint union of the universes A and B of the lattices, but with $0_{\mathbf{A}}$ identified with $0_{\mathbf{B}}$ and $1_{\mathbf{A}}$ identified with $1_{\mathbf{B}}$, such that \mathbf{A} and \mathbf{B} are sublattices and there are no order relations between elements of $A - \{0_{\mathbf{A}}, 1_{\mathbf{A}}\}$ and $B - \{0_{\mathbf{B}}, 1_{\mathbf{B}}\}$. (Observe that $\mathbf{A} + \mathbf{C}_1 \cong \mathbf{A}$ and $|\mathbf{A} + \mathbf{C}_2| = |\mathbf{A}| + 1$.)

2. DEFINABILITY OF CHAINS, FLAT LATTICES AND SOME SMALL LATTICES

An element \mathbf{A} of LATT is said to be covered by an element \mathbf{B} of LATT if $\mathbf{A} < \mathbf{B}$ and there is no $\mathbf{C} \in \text{LATT}$ with $\mathbf{A} < \mathbf{C} < \mathbf{B}$. We write $\mathbf{A} \prec \mathbf{B}$ and

also say that \mathbf{B} is a cover of \mathbf{A} , or that \mathbf{A} is a subcover of \mathbf{B} . Clearly, if $\mathbf{A} < \mathbf{B}$ and $|\mathbf{B}| = |\mathbf{A}| + 1$ then $\mathbf{A} \prec \mathbf{B}$.

2.1. Lemma. *Let $n \geq 0$. The only covers of \mathbf{C}_n in LATT are \mathbf{C}_{n+1} and the lattices $\mathbf{C}_k \oplus (\mathbf{C}_l + \mathbf{C}_2) \oplus \mathbf{C}_m$ for $k + l + m = n$ with $l \geq 2$.*

Proof. Clearly, all these lattices are covers of \mathbf{C}_n . Let \mathbf{L} be a cover of \mathbf{C}_n . Clearly, \mathbf{L} is of height either n or $n + 1$, and if it is of height $n + 1$ then $\mathbf{L} \cong \mathbf{C}_{n+1}$. Let \mathbf{L} be of height n and let $a_0 < a_1 < \dots < a_n$ be a chain in \mathbf{L} (necessarily, this is a maximal chain). There is an index $i \leq n - 2$ such that a_i has a cover b different from a_{i+1} . The sublattice $\{a_0, \dots, a_i\} \cup \uparrow a_{i+1}$ is above \mathbf{C}_n and does not contain the element b ; consequently, it coincides with $\{a_0, a_1, \dots, a_n\}$. Thus $a_{i+1} \vee b = a_j$ for some $j \geq i + 2$. Now $\{a_0, \dots, a_n, b\}$ is a sublattice of \mathbf{L} isomorphic to $\mathbf{C}_i \oplus (\mathbf{C}_{j-i} + \mathbf{C}_2) \oplus \mathbf{C}_{n-j}$. \square

2.2. Lemma. *The lattice \mathbf{M}_2 has four covers: \mathbf{M}_3 , $\mathbf{N}_5 = \mathbf{C}_3 + \mathbf{C}_2$, $\mathbf{P} = \mathbf{M}_2 \oplus \mathbf{C}_1$ and $\mathbf{P}^{\text{opp}} = \mathbf{C}_1 \oplus \mathbf{M}_2$. The lattice \mathbf{M}_3 has infinitely many covers.*

Proof. The first statement is easy. A slight modification of a construction from [1] gives a lattice \mathbf{L}_n with $12 + 2n$ elements, for any $n \geq 2$, such that \mathbf{L}_n contains a single copy of \mathbf{M}_3 as a sublattice, and this sublattice is maximal in \mathbf{L}_n . For $n = 6$ the lattice \mathbf{L}_n is pictured in Fig. 1. \square

Fig. 1

2.3. Theorem. *The set of finite chains is definable. The set $\{\mathbf{M}_n : n \geq 1\}$ is definable. Every finite chain and every lattice \mathbf{M}_n is a definable element of LATT' .*

Proof. It is easy to see that an element \mathbf{L} of LATT' has the property that the principal ideal generated by \mathbf{L} is a chain—i.e., for all $\mathbf{A} \leq \mathbf{L}$ and $\mathbf{B} \leq \mathbf{L}$ either $\mathbf{A} \leq \mathbf{B}$ or $\mathbf{B} \leq \mathbf{A}$ —if and only if \mathbf{L} is either a chain or \mathbf{M}_n for some n . Thus the set $U = \{\mathbf{C}_n : n \geq 0\} \cup \{\mathbf{M}_n : n \geq 1\}$ is definable. (According to our convention, this language just means to assert that $\{\mathbf{A} \in \text{LATT} : \mathbf{A} \cong \mathbf{U} \text{ for some } \mathbf{U} \in U\}$ is definable.) Also, it follows that the set $\{\mathbf{C}_4, \mathbf{M}_3\}$ is definable: \mathbf{C}_4 and \mathbf{M}_3 are the only two lattices $\mathbf{Q} \in U$ such that $|\{\mathbf{R} \in \text{LATT}' : \mathbf{R} < \mathbf{Q}\}| \cong 4$. By 2.1 and 2.2, \mathbf{C}_4 has only seven covers, while \mathbf{M}_3 has more than seven covers in LATT' . Thus both \mathbf{C}_4 and \mathbf{M}_3 are definable elements. A finite lattice is a chain if and only if it belongs to $\{\mathbf{C}_n : n \geq 0\} \cup \{\mathbf{M}_n : n \geq 1\}$ and is comparable with \mathbf{C}_4 . \square

It also follows that the mapping $\mathbf{L} \mapsto \mathbf{C}$, where $\mathbf{L} \in \text{LATT}'$ and \mathbf{C} is the chain of height $\text{ht}(\mathbf{L})$, is definable.

2.4. Lemma. *The lattices \mathbf{N}_5 , \mathbf{P}_1 , $\mathbf{P}_0 = \mathbf{P}_1^{\text{opp}}$ and $\mathbf{N}_6 = \mathbf{C}_3 + \mathbf{C}_3$ are definable.*

Proof. It follows from 2.2 that the set consisting of the first three lattices is definable. Out of these three lattices, \mathbf{N}_5 is the only one that has a cover of height 3 that is not above any of the remaining two lattices, and also is not above \mathbf{M}_3 ; it has only one such cover and this cover is isomorphic to \mathbf{N}_6 . Thus \mathbf{N}_5 , and also \mathbf{N}_6 , are definable elements. Consequently, the set $\{\mathbf{P}_0, \mathbf{P}_1\}$ is definable. But \mathbf{P}_1 is definable in LATT' by definition, so that \mathbf{P}_0 is definable as well. \square

Since the opposite of \mathbf{P}_1 is definable, it follows that whenever a relation is definable in LATT' then also its opposite is definable in LATT' .

Since \mathbf{N}_5 and \mathbf{M}_3 are definable, the set of finite modular lattices and the set of finite distributive lattices are definable.

3. DEFINABILITY OF THE RELATION $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$

3.1. Lemma. *Let $n \geq 2$ and $k \geq 0$. Then $\mathbf{C}_k \oplus \mathbf{M}_n$ is the least lattice \mathbf{L} with the following properties:*

- (1) \mathbf{L} is modular;
- (2) $\mathbf{M}_n \leq \mathbf{L}$;
- (3) $\text{ht}(\mathbf{L}) = k + 2$;
- (4) $\mathbf{M}_2 \oplus \mathbf{C}_1 \not\leq \mathbf{L}$.

Proof. Let \mathbf{L} have these properties. There exists a sublattice \mathbf{M} of \mathbf{L} isomorphic to \mathbf{M}_n . By (4), $1_{\mathbf{M}} = 1_{\mathbf{L}}$. Put $o = 0_{\mathbf{M}}$ and denote by a_1, \dots, a_n the atoms of \mathbf{M} . For every $i = 1, \dots, n$ there is a cover b_i of o in \mathbf{L} with $b_i \leq a_i$. For $i \neq j$ we have $b_i \vee b_j = 1_{\mathbf{L}}$ by (4). Since \mathbf{L} is modular, it follows that $b_i = a_i$. Thus $o \prec a_i$ in \mathbf{L} for all i . But then, again since \mathbf{L} is modular, also $a_i \prec 1_{\mathbf{L}}$ in \mathbf{L} for all i . Thus o is of height k in \mathbf{L} . But then $\mathbf{C}_k \oplus \mathbf{M}_n \leq \mathbf{L}$. \square

3.2. Lemma. *Let $n \geq 2$ and $k, l \geq 0$. Then $\mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l$ is the least lattice \mathbf{L} with the following properties:*

- (1) \mathbf{L} is modular;
- (2) $\mathbf{M}_n \leq \mathbf{L}$;
- (3) $\text{ht}(\mathbf{L}) = k + 2 + l$;
- (4) $\mathbf{C}_k \oplus \mathbf{M}_n \leq \mathbf{L}$;
- (5) $\mathbf{M}_n \oplus \mathbf{C}_l \leq \mathbf{L}$;
- (6) $\mathbf{C}_{k+1} \oplus \mathbf{M}_2 \not\leq \mathbf{L}$;
- (7) $\mathbf{M}_2 \oplus \mathbf{C}_{l+1} \not\leq \mathbf{L}$.

Consequently, the mapping $\langle \mathbf{C}_k, \mathbf{M}_n, \mathbf{C}_l \rangle \mapsto \mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l$ is definable and every $\mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l$ is a definable element of LATT' .

Proof. Let \mathbf{L} have these properties. There exists a sublattice M of \mathbf{L} isomorphic to \mathbf{M}_n . Denote by o the least, by I the greatest and by a_1, \dots, a_n the remaining elements of M . By (4) we can assume that $\text{ht}(o) \geq k$. By (6), $\text{ht}(o) = k$. For every $i = 1, \dots, n$ there is a sub-cover, b_i of I in \mathbf{L} with $b_i \geq a_i$. If $i \neq j$ and $b_i \wedge b_j > o$ then we obtain a contradiction by (3)

and (6), since \mathbf{L} is modular. Thus it follows (by modularity) that the interval $I[o, I]$ has height two, and consequently, by (3) and modularity, the interval $I[I, 1_{\mathbf{L}}]$ has height l . But this means that $\mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l \leq \mathbf{L}$. That concludes our proof.

For the definability of the mapping we need to apply Lemma 3.1 and its opposite. \square

3.3. Lemma. *The following ternary relation R on LATT' is definable: $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ if and only if $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are chains and $\text{ht}(\mathbf{C}) = \text{ht}(\mathbf{A}) + \text{ht}(\mathbf{B})$.*

Proof. It follows from 3.2. \square

$$\mathbf{M}_3^1 \quad \mathbf{M}_3^2 \quad \mathbf{M}_3^3 \quad \mathbf{M}_3^4 \quad \mathbf{M}_3^5 \quad \mathbf{M}_3^6 \quad \mathbf{M}_3^7$$

Fig. 2

For $n \geq 3$ denote by \mathbf{M}_n^1 the lattice $\mathbf{C}_1 \oplus \mathbf{M}_n$ with one element added, this element being an atom of \mathbf{L} below one of the atoms of \mathbf{M}_n . Denote by \mathbf{M}_n^2 the opposite of \mathbf{M}_n^1 . Put $\mathbf{M}_n^3 = (\mathbf{C}_1 \oplus \mathbf{M}_n) + \mathbf{C}_2$, $\mathbf{M}_n^4 = (\mathbf{M}_n \oplus \mathbf{C}_1) + \mathbf{C}_2$, $\mathbf{M}_n^5 = (\mathbf{C}_1 \oplus \mathbf{M}_n \oplus \mathbf{C}_1) + \mathbf{C}_2$, $\mathbf{M}_n^6 = \mathbf{M}_n \oplus \mathbf{M}_2$ and $\mathbf{M}_n^7 = \mathbf{M}_n \oplus \mathbf{C}_1 \oplus \mathbf{M}_2$. These lattices are pictured in Fig. 2 for $n = 3$.

3.4. Lemma. *Let $n \geq 3$. Then \mathbf{M}_n^1 and \mathbf{M}_n^3 are the only covers of $\mathbf{C}_1 \oplus \mathbf{M}_n$ of height 3 that are not above \mathbf{M}_{n+1} ; the first is modular, the second is not. Consequently, the mappings $\mathbf{M}_n \mapsto \mathbf{M}_n^1$ and $\mathbf{M}_n \mapsto \mathbf{M}_n^3$ are definable. Similarly, the mappings $\mathbf{M}_n \mapsto \mathbf{M}_n^2$ and $\mathbf{M}_n \mapsto \mathbf{M}_n^4$ are definable.*

Proof. Let \mathbf{L} be a cover of $\mathbf{C}_1 \oplus \mathbf{M}_n$ such that $\text{ht}(\mathbf{L}) = 3$ and $\mathbf{M}_{n+1} \not\leq \mathbf{L}$. Then \mathbf{L} has a proper sublattice $K = \{0_{\mathbf{L}}, o, a_1, \dots, a_n, 1_{\mathbf{L}}\}$ isomorphic to $\mathbf{C}_1 \oplus \mathbf{M}_n$.

Suppose that there is an element $b \in L - K$ comparable with o . Clearly, $b > o$. But then b is of height 2 and \mathbf{L} has a sublattice $(K - \{0_{\mathbf{L}}\}) \cup \{b\} \cong \mathbf{M}_{n+1}$, a contradiction.

Thus all elements of $L - K$ are incomparable with o . Consider first the case when there is an element $c \in L - K$ that is below at least one of the coatoms a_i of \mathbf{L} . Clearly, the index i is unique. Without loss of generality, $c < a_1$ and $c \parallel a_i$ for all $i > 1$. Clearly, c is an atom of \mathbf{L} . Thus $c \wedge a_i = 0_{\mathbf{L}}$ and $c \vee a_i = 1_{\mathbf{L}}$ for $i > 1$. It follows that $K \cup \{c\}$ is a sublattice of \mathbf{L} isomorphic to \mathbf{M}_n^1 .

It remains to consider the case when $c \parallel a_i$ for all i and all $c \in L - K$. Take one element $c \in L - K$. Since a_i are coatoms, we have $c \vee a_i = 1_{\mathbf{L}}$. If $c \wedge a_i > 0_{\mathbf{L}}$ for some i then $c \wedge a_i$ is an element of $L - K$ below a_i , a contradiction. Thus $c \wedge a_i = 0_{\mathbf{L}}$ for all i and $K \cup \{c\}$ is a sublattice of \mathbf{L} isomorphic to \mathbf{M}_n^3 . \square

3.5. Lemma. *Let $n \geq 3$. Then \mathbf{M}_n^5 is the only cover of $\mathbf{C}_1 \oplus \mathbf{M}_n \oplus \mathbf{C}_1$ of height 4 that is not above any of the lattices $\mathbf{M}_n^1, \mathbf{M}_n^2, \mathbf{M}_n^3, \mathbf{M}_n^4$ and \mathbf{M}_{n+1} . Consequently, the mapping $\mathbf{M}_n \mapsto \mathbf{M}_n^5$ is definable.*

Proof. Let \mathbf{L} be such a cover. Denote by K a copy of $\mathbf{C}_1 \oplus \mathbf{M}_n \oplus \mathbf{C}_1$ in \mathbf{L} ; denote by o the only atom and by I the only coatom in this copy. Clearly, $\downarrow I \cup \{1_{\mathbf{L}}\}$ is a sublattice of \mathbf{L} ; if it contains an element not in K then $\downarrow I$ is a sublattice properly extending $\mathbf{C}_1 \oplus \mathbf{M}_n$, so that (according to 3.4) it contains either \mathbf{M}_n^1 or \mathbf{M}_n^3 , a contradiction. Thus $\downarrow I = \mathbf{M}_n \cup \{0_{\mathbf{L}}\}$ and similarly $\uparrow o = \mathbf{M}_n \cup \{1_{\mathbf{L}}\}$. Thus for any element $c \in L - K$, $K \cup \{c\}$ is a sublattice of \mathbf{L} isomorphic to \mathbf{M}_n^5 . \square

3.6. Lemma. *Let $n \geq 3$. \mathbf{M}_n^6 is the only cover of $\mathbf{M}_n \oplus \mathbf{C}_2$ of height 4 that is modular and is not above any of the lattices \mathbf{M}_{n+1} and \mathbf{M}_n^2 . Also, \mathbf{M}_n^7 is the only cover of $\mathbf{M}_n \oplus \mathbf{C}_3$ of height 5 that is modular and is not above any of the lattices \mathbf{M}_{n+1} , \mathbf{M}_n^2 and \mathbf{M}_n^6 . Consequently, the mappings $\mathbf{M}_n \mapsto \mathbf{M}_n^6$ and $\mathbf{M}_n \mapsto \mathbf{M}_n^7$ are definable.*

Proof. It is easy. \square

3.7. Lemma. *Let \mathbf{B} be a finite lattice of height $n + m$ where $n \geq 0$ and $m > 0$. Then $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_m$ for a lattice \mathbf{A} of height n if and only if for every positive integer l there exist a positive integer $k > l$ and a finite lattice \mathbf{C} such that the following conditions are satisfied:*

- (1) $\mathbf{B} < \mathbf{C}$;
- (2) $\text{ht}(\mathbf{C}) = n + m + 1$;
- (3) $\mathbf{C}_n \oplus \mathbf{M}_k \oplus \mathbf{C}_{m-1} \leq \mathbf{C}$;
- (4) \mathbf{C} is not above any of the lattices $\mathbf{M}_{k+1}, \mathbf{M}_k^1, \mathbf{M}_k^2, \mathbf{M}_k^3, \mathbf{M}_k^4, \mathbf{M}_k^5, \mathbf{M}_k^6, \mathbf{M}_k^7$;
- (5) If $n = 0$ then \mathbf{B} is a chain. If $n > 0$ then $\mathbf{C}_{n-1} \oplus \mathbf{M}_2 \not\leq \mathbf{B}$.

Proof. If $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_m$ then the lattice $\mathbf{A} = \mathbf{A} \oplus \mathbf{M}_k \oplus \mathbf{C}_{m-1}$ has all these properties for every sufficiently large k . Conversely, let k and \mathbf{C} exist for every l . By (2) and (3) there exists a sublattice M of \mathbf{C} isomorphic to \mathbf{M}_k with the least element o , the largest element I and the remaining elements a_1, \dots, a_k such that o is of height n , I is of height $n + 2$ in \mathbf{C} and $\uparrow I$ is of height $m - 1$; all covers in M are covers in \mathbf{C} . Since $\mathbf{M}_k^6 \not\leq \mathbf{C}$ and $\mathbf{M}_k^7 \not\leq \mathbf{C}$, $\uparrow I$ is a chain; denote it by D ; it must be a chain of height $m - 1$.

Since $\mathbf{M}_{k+1} \not\leq \mathbf{C}$, $\mathbf{M}_k^2 \not\leq \mathbf{C}$ and $\mathbf{M}_k^4 \not\leq \mathbf{C}$, it is easy to see that every element of $\uparrow o$ belongs to $M \cup D$. Since $\mathbf{M}_k^1 \not\leq \mathbf{C}$, $\mathbf{M}_k^3 \not\leq \mathbf{C}$ and $\mathbf{M}_k^5 \not\leq \mathbf{C}$, it is easy to see that o is a cut-point in \mathbf{C} . Thus $C = \downarrow o \oplus M \oplus D$. Since \mathbf{B} is a sublattice of \mathbf{C} of height $\text{ht}(\mathbf{C}) - 1$, it follows from (5) that $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_m$ for some \mathbf{C} . \square

3.8. Lemma. *The set of finite lattices with precisely one coatom (or precisely one atom, respectively) is definable. Consequently, also the set of finite lattices with at least two coatoms (or atoms, respectively) is definable.*

Proof. It follows from 3.7, since a finite lattice \mathbf{B} has precisely one coatom if and only if $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_1$ for a finite lattice \mathbf{A} . \square

3.9. Lemma. *The following ternary relation R on LATT is definable in LATT' :*

$(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ if and only if \mathbf{B} is a chain and $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$.

Proof. Let \mathbf{B} be a chain. If \mathbf{A} has at least two coatoms (this being definable by 3.8), then $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$ if and only if $\mathbf{C} \cong \mathbf{A}' \oplus \mathbf{B}$ for some \mathbf{A}' of the same height as \mathbf{A} (this being definable by 3.7) and \mathbf{A} is up to isomorphism the largest element of LATT' below \mathbf{C} with at least two coatoms. If \mathbf{A} is arbitrary (and not a chain) then $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{B}_0$ for a lattice \mathbf{A}_0 with at least two coatoms and a chain \mathbf{B}_0 ; we have $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$ if and only if $\mathbf{C} \cong \mathbf{A}_0 \oplus (\mathbf{B}_0 \oplus \mathbf{B})$. (We need to apply 3.3.) \square

3.10. Lemma. *Let \mathbf{A} be a finite lattice of height m ; let $2 \leq n < m$. Then \mathbf{A} has a cut-point at height n if and only if there exists $\mathbf{B}, \mathbf{C} \in \text{LATT}'$ with the following properties:*

- (1) $\mathbf{A} \leq \mathbf{B}$;
- (2) $\text{ht}(\mathbf{B}) = m$;
- (3) $\text{ht}(\mathbf{C}) = n$;
- (4) $\mathbf{C} \oplus \mathbf{C}_{m-n} \leq \mathbf{B}$;
- (5) whenever $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \leq \mathbf{B}$ then $\mathbf{C}'' \cong \mathbf{C} \oplus \mathbf{C}_2$.

Proof. Let \mathbf{A} have a cut-point at height n , so that $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ where $\text{ht}(\mathbf{A}_1) = n$ and $\text{ht}(\mathbf{A}_2) = m - n$; the element $1_{\mathbf{A}_1} = 0_{\mathbf{A}_2}$ is the cut-point. Let $k > 0$ be sufficiently large so that $\mathbf{C} = \mathbf{M}_l + \mathbf{A}_1 \not\leq \mathbf{A}_2$. Put $\mathbf{B} = \mathbf{C} \oplus \mathbf{A}_2$. Since $n \geq 2$, then with this choice of \mathbf{B} and \mathbf{C} we have $\mathbf{A} \leq \mathbf{B}$, and in fact conditions (1) through (4) are obviously true.

To check that (5) is satisfied, suppose that $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \leq \mathbf{B}$. We claim that the only sublattice of \mathbf{B} isomorphic to \mathbf{C} is C , equal to the interval $I[0_{\mathbf{B}}, 0_{\mathbf{A}_2}]$ in \mathbf{B} . Indeed, let E be a sublattice of \mathbf{B} isomorphic to \mathbf{C} . Then $E \not\leq \mathbf{A}_2$. It easily follows that the 0, 1-sublattice of E isomorphic to \mathbf{M}_k is contained in C , and thus $E \subseteq C$, yielding that $E = C$ by cardinality. Thus we have sublattices $\mathbf{C} \subset \mathbf{C}' \subset \mathbf{C}'' \subseteq \mathbf{B}$ with $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}''$ and we are to show that $\mathbf{C}'' \cong \mathbf{C} \oplus \mathbf{C}_2$.

Now every element of $B \setminus C$ is greater than all elements of C . Thus clearly, $\mathbf{C}' = C \cup \{p\}$ for an element $p \in A_2 - \{0_{\mathbf{A}_2}\}$. The sublattice \mathbf{C}'' contains an element $q \notin \mathbf{C}'$. If p, q are incomparable then $C \cup \{p, p \vee q\}$ is a sublattice, so that $\mathbf{C}'' = C \cup \{p, p \vee q\}$ and $q \notin \mathbf{C}''$, a contradiction. Thus p, q are comparable and $\mathbf{C}'' = C \cup \{p, q\} \cong \mathbf{C} \oplus \mathbf{C}_2$. Condition (5) is therefore satisfied.

To prove the reverse implication, suppose that $2 \leq n < m$, $\text{ht}(\mathbf{A}) = m$, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfy (1) through (5). By (4) we can assume that $\mathbf{C} \oplus \mathbf{C}_{m-n}$ is a sublattice of \mathbf{B} .

Suppose that C is a proper sublattice of the interval $I[0_{\mathbf{B}}, 1_{\mathbf{C}}]$ in \mathbf{B} . Then there is a lattice $\mathbf{C}' \succ \mathbf{C}$ with $\mathbf{C}' \subseteq \mathbf{B}$ and, in fact, $C \subset \mathbf{C}' \subseteq I[0_{\mathbf{B}}, 1_{\mathbf{C}}]$.

Clearly, $\text{ht}(\mathbf{C}') = n$. Let $\mathbf{C}'' = \mathbf{C}' \cup \{p\}$ with $p \in B$ and $p > 1_{\mathbf{C}}$. This gives a lattice \mathbf{C}'' with $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \leq \mathbf{B}$ and $\text{ht}(\mathbf{C}'') = n + 1$. Clearly, $\mathbf{C}'' \not\cong \mathbf{C} \oplus \mathbf{C}_2$. This contradicts (5), so it follows that C is identical to the interval $I[0_{\mathbf{B}}, 1_{\mathbf{C}}]$ in \mathbf{B} .

We claim that $1_{\mathbf{C}}$ is a cut-point of \mathbf{B} . Suppose not, so that there exists an element $a \in B$ incomparable with $1_{\mathbf{C}}$. Choose a maximal among all elements of B that are incomparable with $1_{\mathbf{C}}$. Then it is easily checked that $\mathbf{C}'' = \mathbf{C} \cup \{a, a \vee 1_{\mathbf{C}}\}$ is a sublattice of \mathbf{B} , as is $\mathbf{C}' = \mathbf{C} \cup \{a \vee 1_{\mathbf{C}}\}$. Clearly, this gives $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \leq \mathbf{B}$, and here $\text{ht}(\mathbf{C}'') = n + 1$ so that $\mathbf{C}'' \not\cong \mathbf{C} \oplus \mathbf{C}_2$. This contradicts (5).

It follows that \mathbf{B} has a unique element of height n , namely $1_{\mathbf{C}}$. Now we have $\mathbf{A} \cong \mathbf{A}' \subseteq \mathbf{B}$, $\text{ht}(\mathbf{A}') = \text{ht}(\mathbf{A}) = \text{ht}(\mathbf{B}) = m$. These conditions imply that an element of height n in \mathbf{A}' must be of height n in \mathbf{B} . So \mathbf{A}' contains the cut-point $1_{\mathbf{C}}$, which is therefore a cut-point of height n in \mathbf{A}' . This concludes our proof. \square

3.11. Lemma. *The following binary relation R is definable in LATT' : $(\mathbf{A}, \mathbf{B}) \in R$ if and only if $\mathbf{B} \cong \mathbf{C}_n$ for an n such that \mathbf{A} has a cut-point at height n .*

Proof. The case $2 \leq n < \text{ht}(\mathbf{A})$ follows from 3.10. The case $n = 1$ has been handled in Lemma 3.8. The other cases are trivial. \square

3.12. Theorem. *The following ternary relation R is definable in LATT' : $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ if and only if $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$.*

Proof. Put $n = \text{ht}(\mathbf{A})$ and $m = \text{ht}(\mathbf{B})$. We have $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$ if and only if \mathbf{C} has a cut-point at height n , \mathbf{A} is, up to isomorphism, the largest element of LATT' with $\text{ht}(\mathbf{A}) = n$ and $\mathbf{A} \oplus \mathbf{C}_m \leq \mathbf{C}$, and \mathbf{B} is, up to isomorphism, the largest element of LATT' with $\text{ht}(\mathbf{B}) = m$ and $\mathbf{C}_n \oplus \mathbf{B} \leq \mathbf{C}$. \square

4. DEFINABILITY OF PRINCIPAL IDEALS AND INTERVALS

4.1. Definition. *Let \mathbf{A} be a finite lattice and $a \in A - \{1_{\mathbf{A}}\}$. We define a lattice \mathbf{K} with $K = A \cup \{i, b\}$ in such a way that \mathbf{A} is a sublattice of \mathbf{K} , $i = 1_{\mathbf{K}}$, $x \leq b$ for $x \in A$ if and only if $x \leq a$, and $x \geq b$ for $x \in A$ never happens. This lattice \mathbf{K} will be denoted by $\mathbf{V}_a(\mathbf{A})$. By a V -extension of \mathbf{A} we mean a lattice isomorphic to $\mathbf{V}_a(\mathbf{A})$ for some $a \in A - \{1_{\mathbf{A}}\}$. We say that \mathbf{B} is a V -extension of \mathbf{A} with bottom \mathbf{C} if for some $a \in A - \{1_{\mathbf{A}}\}$, $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A})$ and \mathbf{C} is isomorphic with the interval $I[0_{\mathbf{A}}, a]$ in \mathbf{A} .*

4.2. Lemma. *Let $\mathbf{A}, \mathbf{B} \in \text{LATT}'$. Then \mathbf{B} is a V -extension of \mathbf{A} if and only if \mathbf{B} is a cover of $\mathbf{A} \oplus \mathbf{C}_1$ such that \mathbf{B} has more than one coatom. Consequently, the binary relation 'is a V -extension of' on LATT' is definable.*

Proof. The direct implication is clear. Let \mathbf{B} be a cover of $\mathbf{D} = \mathbf{A} \oplus \mathbf{C}_1$ with more than one coatom. We can assume that \mathbf{D} is a proper sublattice of \mathbf{B} . If \mathbf{B} contains some elements not in \mathbf{D} and below $1_{\mathbf{A}}$ then the sublattice $\mathbf{D} \cup \{x \in B : x \leq 1_{\mathbf{A}}\}$ of \mathbf{B} equals \mathbf{B} and \mathbf{B} has only one coatom, a

contradiction. Thus all elements of \mathbf{B} below $1_{\mathbf{A}}$ belong to \mathbf{A} . If some element e of $B - D$ is larger than $1_{\mathbf{D}}$ then $B = D \cup \{e\}$ has only one coatom, a contradiction. Thus $1_{\mathbf{B}} = 1_{\mathbf{D}}$. If there is an element $e \in B$ with $1_{\mathbf{A}} < e < 1_{\mathbf{B}}$, we get a contradiction in the same way. Thus $1_{\mathbf{A}}$ is a coatom of \mathbf{B} . There exists a coatom b of \mathbf{B} other than $1_{\mathbf{A}}$. Put $a = 1_{\mathbf{A}} \wedge b$, so that $a \in A - \{1_{\mathbf{A}}\}$. It is easy to see that $D \cup \{b\}$ is a sublattice of \mathbf{B} . Thus $B = D \cup \{b\}$ and $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A})$. \square

Let \mathbf{A} be a finite lattice. Denote by \mathbf{A}^+ the lattice $\mathbf{A} \oplus \mathbf{M}_n$ where n is the least number such that $n \geq 3$ and $\mathbf{M}_n \not\leq \mathbf{A}$. By results we have already proved, the mapping $\mathbf{A} \mapsto \mathbf{A}^+$ is definable.

4.3. Lemma. *The following ternary relation R on LATT is definable in LATT' :*

$(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ iff for some $a \in A$, $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A}^+)$ and $\mathbf{C} \cong I[0_{\mathbf{A}}, a]$.

Proof. We have $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ if and only if \mathbf{B} is a V-extension of $\mathbf{A}^+ = \mathbf{A} \oplus \mathbf{M}_n$, $\mathbf{A} \oplus \mathbf{M}_n^2 \not\cong \mathbf{B}$ and one of the following two cases takes place:

- (i) $\mathbf{A} \oplus \mathbf{M}_n^4 \cong \mathbf{B}$ and $\mathbf{C} \cong \mathbf{A}$;
- (ii) $\mathbf{A} \oplus \mathbf{M}_n^4 \not\cong \mathbf{B}$ and \mathbf{C} is up to isomorphism the largest element of LATT' such that $\mathbf{C} \oplus \mathbf{M}_n^5 \leq \mathbf{B}$. \square

4.4. Theorem. *The binary relations ‘is isomorphic to a principal ideal of’, ‘is isomorphic to a principal filter of’ and ‘is isomorphic to an interval of’ on LATT' are definable.*

Proof. It follows from 4.3. \square

5. INDIVIDUAL DEFINABILITY AND AUTOMORPHISMS

Let \mathbf{A} be a finite lattice and $s = \langle a_1, \dots, a_k \rangle$ be a nonempty simple sequence of elements of \mathbf{A} . (The sequence is called simple if $a_i \neq a_j$ whenever $i \neq j$.) We define $\mathbf{H}_s(\mathbf{A}) = \mathbf{V}_{a_k} \dots \mathbf{V}_{a_1}(\mathbf{A}^+)$. The lattices $\mathbf{H}_s(\mathbf{A})$ will be called simple sequential extensions of \mathbf{A} .

Observe that $\mathbf{H}_s(\mathbf{A})$ has only one sublattice isomorphic to \mathbf{M}_n (where n is such that $\mathbf{A}^+ = \mathbf{A} \oplus \mathbf{M}_n$).

Clearly, $\mathbf{H}_s(\mathbf{A})$ with $s = \langle a_1, \dots, a_k \rangle$ is a maximal simple sequential extension of \mathbf{A} if and only if $k = |\mathbf{A}|$.

For example, a maximal simple sequential extension of the pentagon is pictured in Fig. 4.

Fig. 4

5.1. Lemma. *Let \mathbf{A} be a finite lattice and let $1 \leq k \leq |\mathbf{A}|$; let $\mathbf{A}^+ = \mathbf{A} \oplus \mathbf{M}_n$. A finite lattice \mathbf{B} is isomorphic to a simple sequential extension of \mathbf{A} by a simple sequence of k elements if and only if the following conditions are satisfied:*

- (1) $\mathbf{B} \geq \mathbf{A}^+ \oplus \mathbf{C}_k$;
- (2) $\text{ht}(\mathbf{B}) = \text{ht}(\mathbf{A}) + 2 + k$;
- (3) $\mathbf{M}_n^2 \not\leq \mathbf{B}$;
- (4) $\mathbf{M}_n^6 \not\leq \mathbf{B}$ and $\mathbf{M}_n^7 \not\leq \mathbf{B}$;
- (5) every principal ideal of \mathbf{B} of any height $h > \text{ht}(\mathbf{A}) + 2$ is a V -extension of a principal ideal of \mathbf{B} of height $h - 1$, with the bottom of height at most $\text{ht}(\mathbf{A})$ (bottom as defined above for V -extensions);
- (6) whenever \mathbf{D} is a V -extension of either \mathbf{M}_n^4 or \mathbf{M}_n^5 with bottom isomorphic to \mathbf{C}_0 then $\mathbf{D} \not\leq \mathbf{B}$.

Proof. The direct implication is easy. Let the six conditions be satisfied. We can suppose that $\mathbf{A}^+ \oplus \mathbf{C}_k = \mathbf{A} \oplus \mathbf{M}_n \oplus \mathbf{C}_k$ is a sublattice of \mathbf{B} . It follows from (2) and (4) that $I[1_{\mathbf{A}^+}, 1_{\mathbf{B}}]$ is a chain of height k ; denote its elements by $1_{\mathbf{A}^+} = b_0 < b_1 < \dots < b_k = 1_{\mathbf{B}}$.

Let $1 \leq i \leq k$. Since $I[0_{\mathbf{B}}, b_i]$ is a principal ideal of \mathbf{B} of height at least $\text{ht}(\mathbf{A}) + 2 + i > \text{ht}(\mathbf{A}) + 2$, by (5) this ideal has precisely two coatoms, only one of which is at height $\geq \text{ht}(\mathbf{A}) + 2$; this coatom must be the element b_{i-1} . Denote by c_i the other coatom. We have $I[0_{\mathbf{B}}, b_i] = I[0_{\mathbf{B}}, b_{i-1}] \cup \{b_i, c_i\}$. Since $(b_0)_{\mathbf{B}} = A^+$, we get $B = A^+ \cup \{b_1, \dots, b_k\} \cup \{c_1, \dots, c_k\}$. Clearly, $c_i \neq b_j$ for all i, j and c_1, \dots, c_k are pairwise different.

For $1 \leq i \leq k$ denote by a_i the only lower cover of c_i . It follows from (3) and (4) that $a_i \notin \{b_1, \dots, b_k\}$ and if $a_i \in A^+$ then $a_i \in A$.

Suppose that $a_i = c_j$ for some i and j . Let i be the least index such that $a_i = c_j$ for some j . Clearly, $i < j$. By the minimality of i , $a_i \in A$. Then $\mathbf{M}_n \cup \{a_i, b_i, b_j, c_i, c_j\}$ is a sublattice of B and this sublattice is a V -extension of either \mathbf{M}_n^4 or \mathbf{M}_n^5 , a contradiction with (6).

Thus $a_i \notin \{c_1, \dots, c_k\}$ and it follows that $a_i \in A$ for all i .

If $a_i = a_j$ for some $i \neq j$, we also get a contradiction by (6). Thus $s = \langle a_1, \dots, a_k \rangle$ is a simple sequence of elements of \mathbf{A} and $\mathbf{B} \cong \mathbf{H}_s(\mathbf{A})$. \square

5.2. Lemma. *The following binary relation R on LATT' is definable: $(\mathbf{A}, \mathbf{B}) \in R$ iff $\mathbf{A}, \mathbf{B} \in \text{LATT}'$ and \mathbf{B} is a chain of height $|\mathbf{A}|$.*

Proof. It follows from 5.1: for a chain \mathbf{C}_m we have $(\mathbf{A}, \mathbf{C}_m) \in R$ if and only if $\mathbf{A} \oplus \mathbf{C}_2 \oplus \mathbf{C}_m$ is of the same height as any maximal simple sequential extension of \mathbf{A} . \square

A finite lattice will be called *wide* if it is not a chain and contains an element that is both an atom and a coatom.

5.3. Lemma. *The set of wide elements of LATT' is definable.*

Proof. Suppose that $\mathbf{A} \in \text{LATT}$ is not a chain and n_0 is the least integer n such that $n \geq 3$ and $\mathbf{M}_n \not\leq \mathbf{A}$. Then \mathbf{A} is wide iff \mathbf{A}^+ has a V -extension $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A}^+)$ where $a \in A$ and $I[0_{\mathbf{A}}, a] \cong \mathbf{C}_1$ with the property: whenever $\mathbf{D} \in \text{LATT}$ is such that $\mathbf{M}_{n_0}^5 \leq \mathbf{D} \leq \mathbf{B}$ and $\mathbf{D} \geq \mathbf{E} \oplus \mathbf{M}_{n_0}^5$ holds only for a one-element lattice \mathbf{E} , then $\mathbf{D} \cong \mathbf{M}_{n_0}^5$.

To see that this is equivalent to a first-order definition, see Lemma 4.3. To see that the formulated properties capture the concept of wide lattice, suppose first that \mathbf{A} is wide. Choosing a to be an element of \mathbf{A} that is both maximal and minimal in \mathbf{A} , it should be clear that the V-extension $\mathbf{V}_a(\mathbf{A}^+)$ fulfills the conditions.

Next, suppose that $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A}^+)$ fulfills the conditions. Then obviously, the element a is an atom of \mathbf{A} . If there exists $b \in A$ with $a < b < 1_{\mathbf{A}}$, then we have a sublattice \mathbf{D} of \mathbf{B} isomorphic to $(\mathbf{C}_2 \oplus \mathbf{M}_{n_0} \oplus \mathbf{C}_1) + \mathbf{C}_2$, with least element a . Here $\mathbf{D} > \mathbf{M}_{n_0}^5$ and $\mathbf{D} \geq \mathbf{E} \oplus \mathbf{M}_{n_0}^5$ holds only for the one-element lattice \mathbf{E} . \square

Let $\mathbf{B} = \mathbf{H}_s(\mathbf{A})$ for a finite lattice \mathbf{A} and a maximal simple sequence $s = \langle a_1, \dots, a_k \rangle$ of elements of \mathbf{A} (so that $k = |\mathbf{A}|$). Denote by $b_0 \prec \dots \prec b_k$ the elements of the chain $I[1_{\mathbf{A}^+}, 1_{\mathbf{B}}]$ in \mathbf{B} . For every $i = 0, \dots, k$, b_i is the only element of \mathbf{A} of height $\text{ht}(A) + 2 + i$ and $I[0_{\mathbf{B}}, b_i]$ is the only ideal of \mathbf{B} of that height. For $i = 1, \dots, k$ denote by c_i the only coatom of $I[0_{\mathbf{B}}, b_i]$ different from b_{i-1} , so that $a_i = b_{i-1} \wedge c_i$. It is easy to see that the filter $I[a_i, 1_{\mathbf{B}}]$ of \mathbf{B} is not isomorphic to any other filter of \mathbf{B} . This filter is the only filter D of \mathbf{B} with the property that its only ideal containing \mathbf{M}_n and of height $\text{ht}(D) + i - k$ (i.e., the interval $I[a_i, b_i]$ of \mathbf{B}) is a wide lattice. D will be called the i -th essential filter of \mathbf{B} . We get:

5.4. Lemma. *The following quaternary relation R on LATT' is definable: $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in R$ iff $\mathbf{B} \cong \mathbf{H}_s(\mathbf{A})$ for a maximal simple sequence $s = \langle a_1, \dots, a_k \rangle$ of elements of \mathbf{A} , \mathbf{C} is a chain of height i with $1 \leq i \leq k$ and \mathbf{D} is isomorphic to the i -th essential filter of \mathbf{B} .*

5.5. Lemma. *The following quaternary relation R on LATT' is definable: $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in R$ iff $\mathbf{B} \cong \mathbf{H}_s(\mathbf{A})$ for a maximal simple sequence $s = \langle a_1, \dots, a_k \rangle$ of elements of \mathbf{A} , \mathbf{C} is a chain of height i , \mathbf{D} is a chain of height j , $1 \leq i, j \leq k$ and $a_i \leq a_j$ in \mathbf{A} .*

Proof. We have $a_i \leq a_j$ if and only if the j -th essential filter of \mathbf{B} is a filter of the i -th essential filter of \mathbf{B} . \square

5.6. Theorem. *Every element of LATT' is definable.*

Proof. It follows from 5.5. \square

5.7. Theorem. *The ordered set of isomorphism types of finite lattices with respect to embeddability has only two automorphisms: the identity and the opposite map. The isomorphism type of any finite lattice is definable in this ordered set up to the two automorphisms.*

Proof. It follows from 5.6. \square

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