FEW SUBPOWERS, CONGRUENCE DISTRIBUTIVITY AND NEAR-UNANIMITY

PETAR MARKOVIĆ AND RALPH MCKENZIE

Abstract. We prove that for any variety $\mathcal{V}$, the existence of an edge-term (defined in [1]) and Jónsson terms is equivalent to the existence of a near-unanimity term. This gives some evidence that the edge terms could be one of the conditions necessary for characterizing dualizability in congruence modular setting.

1. Introduction and notation

Definition 1.1. [1] We say a variety $\mathcal{V}$ has a Γ-special cube term t when t is an idempotent term $t(x_1, \dots, x_m)$ in the language of $\mathcal{V}$ which for some subset $\Gamma = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subseteq \{x, y\}^n - \{x\}^n$ satisfies the identities $\varepsilon_1, \dots, \varepsilon_n$ of the form

$$\varepsilon_i : t(\mathbf{a}_1(i), \dots, \mathbf{a}_m(i)) \approx x.$$

When $\Gamma = \{x, y\}^n - \{x\}^n$, we call such a term just a (n-dimensional) cube term. When Γ contains just all the vectors in $\{x, y\}^n - \{x\}^n$ which have one value equal to y and one more vector which is equal to y at the first two positions and to x elsewhere, this is called an (n-dimensional) edge term. Finally, when Γ contains just all the vectors in $\{x, y\}^n - \{x\}^n$ which have one value equal to y, this is the well-known near-unanimity term. We will call the identities ε_i the *defining identities*, or *defining equations* of the term t.

Another way to look at Γ-special cube terms is as the idempotent Mal'cev conditions which satisfy absorption identities in two variables which fail in the variety of sets. From [1], we know that the existence of any Γ-special cube term implies the existence of an edge term of the arity $n + 1$ (of course, for $\Gamma \subseteq \{x, y\}^n - \{x\}^n$).

For the purpose of this paper we need a couple of technical definitions:

Definition 1.2. Let $\Gamma = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subseteq \{x, y\}^n - \{x\}^n$. We define a matrix $M(\Gamma)$ with n rows and $|\Gamma|$ columns to be $M(\Gamma)(i, j) = 0$ when $\mathbf{a}_j(i) = x$ and $M(\Gamma)(i, j) = 1$ when $\mathbf{a}_j(i) = y$.

Definition 1.3. Let Γ be as above. We say that Γ satisfies the divisibility property when

(1) there exists a linear combination of rows of $M(\Gamma)$ with integer coefficients which is equal in $\mathbf{Z}^{|\Gamma|}$ to a row of all k, where k is a positive integer and
(2) for each prime divisor p of k there exists a linear combination of rows of $M(\Gamma)$ with positive integer coefficients such that each entry in the resulting row (in $\mathbf{Z}^{|\Gamma|}$) is congruent to 1 modulo p.

2. Main result

Theorem 2.1. *The following are equivalent:*

(1) $\mathcal{V}$ *is congruence distributive and contains an edge term* t.
(2) $\mathcal{V}$ *contains a* Γ*-special cube term* s *such that its defining identities fail in any nontrivial module.*
(3) $\mathcal{V}$ *contains a* Γ*-special cube term* s *such that its defining identities fail in any nontrivial vector space over the field of rationals* $\mathbf{Q}$ *and in any nontrivial vector space over the field* $\mathbf{Z}_p$ *for all primes* p.
(4) $\mathcal{V}$ *contains a* Γ*-special cube term* s *such that* Γ *satisfies the divisibility property.*
(5) $\mathcal{V}$ *contains a near-unanimity term.*

Proof. (1) $\Rightarrow$ (2) : Let $t = t(x_0, x_1, \ldots, x_n)$, the nontrivial (i.e. not equal to projections) Jónsson terms are $p_1, \ldots, p_k$ and define the sequence of terms $s_0 = t$, and given $s_{i-1} = s_{i-1}(z_1, \ldots, z_m)$, then define

$$s_i = s_{i-1}(p_i(z_1, z_2, z_3), \ldots, p_i(z_{3m-2}, z_{3m-1}, z_{3m})).$$

Call the term $s' := s_k$. It has $l = (n+1)3^k$ many variables. Now define s to be obtained by a substitution from s', by replacing each variable of s' with a new instance of s', so that each pair of new instances has disjoint sets of variables.

Now, from the fact that we get t as identification of blocks of variables of s' and therefore of s as well, it follows that s' and s certainly satisfy the defining equations of t. Therefore, there exist systems of absorbing identities in two variables which fail in the variety of sets both for s' and for s, so s and s' are Γ-special cube terms. Moreover, each of the p_i can also be obtained by identifying variables of s, even of s', and therefore there are absorption identities of s which express the fact that for all i, $p_i(x, y, x) = x$, that $p_1(x, x, y) = x$ and that either $p_k(x, y, y) = y$ or $p_k(x, x, y) = y$ (depending on the parity of k). We only need to show that $p_i(x, y, y) = p_{i+1}(x, y, y)$ for all even i and $p_i(x, x, y) = p_{i+1}(x, x, y)$ for all odd i are also consequence of absorption identities of s.

To do this, assume that there is a module **M** in which s is a term which satisfies all the absorption identities in two variables which are consequences of the edge equations for t and Jónsson equations for the p_is. To prove the equation $p_1(x, y, y) = p_2(x, y, y)$ from these absorption equations, we denote by

$$q_j(x, y, z) = s'(p_1(x, y, z), \dots, p_1(x, y, z), p_2(x, y, z), \dots, p_2(x, y, z))$$

(the first j instances of variables of s' are replaced by $p_1(x, y, z)$, while the remaining ones are replaced by $p_2(x, y, z)$). Each q_j is a substitution instance of s'. We are going to prove that in $\mathbf{M} \models q_j(x, y, y) \approx q_{j+1}(x, y, y)$. Notice that, since s' is a Γ-special cube term, there exists an absorption equation in two variables

$$\varepsilon : s'(z_1, \dots, z_l) \approx x$$

(each z_i is either x or y) such that $z_{j+1} = y$. Therefore, by substituting all y in ε by $p_2(x, y, y)$ we obtain the same result x (by the Substitution Rule of equational logic), and this is an absorption equation ε_1 of s which must hold in **M**. But, if we substitute now the occurrence of $p_2(x, y, y)$ which is in place of z_{j+1} by $p_1(x, y, y)$, we get again x, as this is an absorption equation ε_2 of s which is a consequence of ε_1 and of $p_1(x, y, y) \approx p_2(x, y, y)$. So, in particular, the two terms on the left hand side of these two equations are equal in **M**, and since **M** is an Abelian algebra, the term condition yields that $q_j(x, y, y) = q_{j+1}(x, y, y)$ in **M**. Then, inductively, $\mathbf{M} \models p_1(x, y, y) \approx p_2(x, y, y)$. Analogously, we can show that all the other remaining Jónsson equations hold in **M**. Therefore, **M** has Jónsson terms, so **M** must be a trivial module.

(2) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (4) : Notice that the term $t(x_1, \dots, x_m)$ in any nontrivial vector space over a field **F** must have the form

$$t = \sum_{i=1}^{m} b_i x_i,$$

where b_i are elements of the field. To say that the equation ε_i (from the definition 1.1) holds in this vector space is equivalent to saying that the sum of all coefficients b_j such that $\mathbf{a}_j(i) = y$ is equal to 0, while the sum of the remaining b_j is equal to 1. Let $Y_i = \{j | \mathbf{a}_j(i) = y\}$. Therefore, the term t will satisfy all of the equations ε_i in this vector space iff the system of equations

$$\begin{aligned} \sum_{j=1}^{m} b_j &= 1 \\ \sum_{j\in Y_1} b_j &= 0 \\ &\vdots \\ \sum_{j\in Y_n} b_j &= 0 \end{aligned}$$

has a solution in $\mathbf{F}^{n+1}$.

We claim that such a solution of the above system of equations will exist iff the row vector $\mathbf{1} := \langle 1, 1, \ldots, 1\rangle \in \mathbf{F}^m$ is not in the subspace of $\mathbf{F}^m$ generated by the row vectors of $M(\Gamma)$. If a solution exists, then the row vector $\mathbf{1}$ obviously can't be in the subspace of $\mathbf{F}^m$ generated by the row vectors of $M(\Gamma)$, as this would imply that $0 = 1$ in $\mathbf{F}$. On the other hand, if the row $\mathbf{1}$ is not the subspace of $\mathbf{F}^m$ generated by the row vectors of $M(\Gamma)$, and the rank of $M(\Gamma)$ is r, then the matrix of the above system of equations has the rank $r + 1$. Consider the expanded matrix of the system (with the added column of results). It also has the rank $r + 1$, because if we take a square submatrix M' which contains the column of results, it will be regular iff the first row (the row $\mathbf{1}$) is included in M' and the submatrix of M' obtained by deleting the first row and last column is regular. But, this is a square submatrix of $M(\Gamma)$, so it must have dimension at most r. Therefore, by Kronecker-Capelli's theorem, the system has a solution.

The converse of this last claim implies that when the equations of a Γ-special term t fail in a nontrivial vector space over $\mathbf{Q}$, then there exist rational numbers $q_1, \ldots, q_n$ such that

$$\sum_{i=1}^{n} q_i\chi_{Y_1}(i) = \sum_{i=1}^{n} q_i\chi_{Y_2}(i) = \cdots = \sum_{i=1}^{n} q_i\chi_{Y_m}(i) = 1,$$

where χ_{Y_j} is the characteristic function of the set Y_j. Let k be the least common denominator of all the rationals q_j and let $m_j = q_j k$. Then the last system of equations implies that there are integers m_j such that

$$\sum_{i=1}^{n} m_i\chi_{Y_1}(i) = \sum_{i=1}^{n} m_i\chi_{Y_2}(i) = \cdots = \sum_{i=1}^{n} m_i\chi_{Y_m}(i) = k,$$

the first condition of the divisibility property.

The second condition is obtained by an analogous argument, from $\mathbf{F} = Z_p$, for each prime divisor p of k.

(4) $\Rightarrow$ (5) : Let $\mathcal{V}$ be the variety such that there is a Γ-special term t which satisfies the divisibility property. We may assume $\mathcal{V}$ is idempotent, otherwise just take the idempotent reduct of $\mathcal{V}$-free algebra in countable set of free generators and generate a variety $\mathcal{V}'$. If we prove $\mathcal{V}'$ has a near-unanimity term, then so does $\mathcal{V}$.

Let $\mathbf{F} = \mathbf{F}(x, y)$ be the free algebra in $\mathcal{V}$ freely generated by $\{x, y\}$. Let $\mathbf{y}_i$, $i \in \omega$ be the elements of $\mathbf{F}^\omega$ such that $\mathbf{y}_i(i) = y$ and $\mathbf{y}_i(j) = x$, for $j \neq i$ and let $\mathbf{G} \leq \mathbf{F}^\omega$ be the subalgebra generated by $\{\mathbf{y}_i : i \in \omega\}$. We are going to prove a series of claims about $\mathbf{G}$.

Claim 1. For every $\mathbf{a} \in \mathbf{G}$ and all but finitely many $i \in \omega$, $\mathbf{a}(i) = x$. Also, for every permutation $\pi \in \mathrm{Sym}(\omega)$, $\mathbf{a}^\pi \in \mathbf{G}$, where $\mathbf{a}^\pi$ is defined by $\mathbf{a}^\pi(i) = \mathbf{a}(\pi(i))$ ($\mathbf{G}$ is a totally symmetric subpower of $\mathbf{F}$).

The first sentence should be obvious. For the second, note that if $\mathbf{a} = p(\mathbf{y}_1, \dots, \mathbf{y}_r))$ for some term p, then $p(\mathbf{y}_{\pi(1)}, \dots, \mathbf{y}_{\pi(r)}) = \mathbf{a}^{\pi^{-1}}$. $\square$

For elements of $\mathbf{H} \leq \mathbf{F}^\omega$ consisting of all tuples which are almost exerywhere equal to x (and which contains $\mathbf{G}$), we adopt a notation $\mathbf{a} = a_1^{j_1} a_2^{j_2} \dots a_l^{j_l}$ to mean that $\mathbf{a}(i) = a_r$ when $\sum_{s=1}^{r-1} j_s < i \leq \sum_{s=1}^{r} j_s$, while $\mathbf{a}(i) = x$ when $\sum_{s=1}^{l} j_s < i$. Now we can define 'concatenation', an operation on $\mathbf{H}$ by:

$$\mathbf{ab} = a_1^{j_1} a_2^{j_2} \dots a_l^{j_l} b_1^{k_1} b_2^{k_2} \dots b_l^{k_r}$$

when $\mathbf{a} = a_1^{j_1} a_2^{j_2} \dots a_l^{j_l}$ and $\mathbf{b} = b_1^{k_1} b_2^{k_2} \dots b_l^{k_r}$. Although $\mathbf{G}$ is not closed under 'concatenation', we will use it as a notational shortcut.

Without loss of generality, assume that the first row of $M(\Gamma)$ is of the form $\langle 0, \dots, 0, 1, \dots, 1\rangle$, where the first s elements are 0, and the remaining $m - s$ are 1. We define $u_i \in F$, for $i \leq s$, to be $t(x, \dots, x, y, x, \dots, x)$ (y is at the ith position in t). By the definition of Γ-special cube terms, there must exist at least one defining equation ε of t such that on the right-hand side of ε at the ith position of t there is y (otherwise the ith member of Γ would be in $\{x\}^n$). We define $v_i \in F$ to be the result of t applied to a tuple of xs and ys equal to the tuple on the right hand side of ε, except at the ith position, where it is equal to x.

Claim 2. For each tuple $\mathbf{a} \in G$, all $i \leq s$ and all j, the tuple $\mathbf{a} u_i^j v_i^j$ is also in G.

We apply the term t to a tuple of $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$s, so that $\mathbf{y}_1$s are in the positions where xs are in ε, $\mathbf{y}_2$ is in the ith position and $\mathbf{y}_3$s are in the remaining positions of t. The resulting element of $\mathbf{G}$ is, obviously,

yu_iv_i. Because of the total symmetricity of $\mathbf{G}$, the tuples of the form $\mathbf{y}_jx^ru_iv_i \in G$. Now, $\mathbf{a} = p(\mathbf{y}_1, \dots, \mathbf{y}_r)$ for some term p and $r \in \omega$. Therefore,

$$p(\mathbf{y}_1x^{r-1}u_iv_i, \dots, \mathbf{y}_jx^{r-j}u_iv_i, \dots, \mathbf{y}_ru_iv_i) = \mathbf{a}u_iv_i.$$

Now we can inductively prove that if $\mathbf{a}u_i^jv_i^j \in G$, then $\mathbf{a}u_i^jv_i^ju_iv_i \in G$, and then because of total symmetricity of $\mathbf{G}$, we get $\mathbf{a}u_i^{j+1}v_i^{j+1} \in G$. $\square$

Now we fix $k \in \omega$ to be the number from the first condition of the divisibility property of t.

Claim 3. If $\mathbf{a}u_i^k \in G$, or $\mathbf{a}u_i^k \in G$, then $\mathbf{a} \in G$. Also, if $\mathbf{a} \in G$, then $\mathbf{a}u_i^k \in G$ and $\mathbf{a}u_i^k \in G$.

To prove the first sentence of this claim, let us first rephrase the first condition of the divisibility property: We know that there exist two finite sequences of the defining equations of t, S^+ and S^- (equations can be repeated in each sequence), such that the sum of occurrences of variable y at any coordinate of t in S^+ is by k greater than the sum of occurrences of y at the same coordinate in S^-. Let there be n_j many ys at the jth coordinate of t in the sequence S^- (hence, clearly, there are $n_j + k$ many ys at the jth coordinate of t in the sequence S^+).

Assume $\mathbf{a}u_i^k \in G$. Let the length of the word representing $\mathbf{a}$ be α, and let β and γ be the lengths of the sequences of equations S^+ and S^-, respectively. By the Claim 2, elements $\mathbf{a}'_j$ of the form $\mathbf{a}u_i^{n_j+k}v_i^{n_j} \in G$, for all $j \leq m$. We can also use the total symmetricity of $\mathbf{G}$ to insert any number of letters x between the letters of the word representing $\mathbf{a}'_j$ and still obtain a word representing an element of G. Therefore, $\mathbf{a}_j \in G$, where we define $\mathbf{a}_j$ by:

- for $l \leq \alpha$, $\mathbf{a}_j(l) = \mathbf{a}(l)$,
- for $\alpha < l \leq \alpha + \beta$, $\mathbf{a}_j(l) = u_i$ if the $(l - \alpha)$th equation of S^+ has y at the jth coordinate of t and $\mathbf{a}_j(l) = x$ otherwise,
- for $\alpha + \beta < l \leq \alpha + \beta + \gamma$, $\mathbf{a}_j(l) = v_i$ if the $(l - \alpha - \beta)$th equation of S^- has y at the jth coordinate of t and $\mathbf{a}_j(l) = x$ otherwise, and
- for $\alpha + \beta + \gamma < l$, $\mathbf{a}_j(l) = x$.

We claim that $t(\mathbf{a}_1, \dots, \mathbf{a}_n) = \mathbf{a}$. Clearly, by the idempotence of t, $t(\mathbf{a}_1, \dots, \mathbf{a}_n)(l) = \mathbf{a}(l)$ for $l \leq \alpha$ or $\alpha + \beta + \gamma < l$. For the remaining coordinates l we use the substitutions $y \mapsto u_i$ and $y \mapsto v_i$ and the fact that S^+ and S^- consist of the defining equations of t to obtain $t(\mathbf{a}_1, \dots, \mathbf{a}_n)(l) = x = \mathbf{a}(l)$. To finish the proof of the first sentence of this Claim, just notice that we can interchange u_i and v_i in the above proof.

The second sentence of the Claim follows from the first one and the fact that if $\mathbf{a} \in G$, then by the Claim 2, $\mathbf{a}u_i^k v_i^k \in G$ and $\mathbf{a}u_i^k v_i^k \in G$. $\square$

We denote by $P(l)$ the property of a positive integer l that for all $\mathbf{a} \in H$, $\mathbf{a} \in G$ iff $\mathbf{a}u_i^l \in G$. The above Claim proves $P(k)$.

Claim 4. For any positive integer l such that $l|k$ and any prime $q|l$, if $P(l)$, then $P(\frac{l}{q})$.

To prove this Claim we use the second part of the divisibility property, for the prime q. We restate this condition similarly as in the previous Claim, and stete that there exists a finite sequence of defining identities of t (we again are allowing repetition of identities), S_1, such that for each position j of the term t, the variable y occurs $k_j q+1$ many times at the jth position of t, for some $k_j \geq 0$. Consider the sequence S of the defining identities of t which is equal to $\frac{l}{q}$ many copies of the sequence S_1 concatencted. The sequence S has the property that the variable y occurs at the jth position of t $k_j l+\frac{l}{q}$ many times.

Now assume that $\mathbf{a}u_i{}^{\frac{l}{q}} \in G$. Then for each $j \leq m$, $\mathbf{a}'_j = \mathbf{a}u_i^{k_j l+\frac{l}{q}} \in G$, by $P(l)$. Aslo assume that the length of the word representing $\mathbf{a}$ is α and the length of the sequence S is β. Analogously as in the proof of the previous Claim, we use the total symmetricity of $\mathbf{G}$ to 'insert' letters x in the appropriate places in the word for $\mathbf{a}'_j$ to get that $\mathbf{a}_j \in G$, where $\mathbf{a}_j$ is defined by:

- for $l \leq \alpha$, $\mathbf{a}_j(l) = \mathbf{a}(l)$,
- for $\alpha < l \leq \alpha + \beta$, $\mathbf{a}_j(l) = u_i$ if the $(l-\alpha)$th equation of S has y at the jth coordinate of t and $\mathbf{a}_j(l) = x$ otherwise,
- for $\alpha + \beta < l$, $\mathbf{a}_j(l) = x$.

We claim that $t(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{a}$. Clearly, by the idempotence of t, $t(\mathbf{a}_1, \ldots, \mathbf{a}_n)(l) = \mathbf{a}(l)$ for $l \leq \alpha$ or $\alpha + \beta < l$. For the remaining coordinates l we use the substitutions $y \mapsto u_i$ and the fact that S consists of the defining equations of t to obtain $t(\mathbf{a}_1, \ldots, \mathbf{a}_n)(l) = x = \mathbf{a}(l)$.

To finish the proof of this Claim just notice that if $\mathbf{a} \in G$, then $\mathbf{a}u_i^l \in G$ by $P(l)$ and then by the other direction of this Claim applied $q-1$ times, $\mathbf{a}u_i{}^{\frac{l}{q}} \in G$. $\square$

From the last Claim, it is obvious that $P(1)$ holds. Notice that

$$t(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s, \mathbf{y}_{s+1}, \mathbf{y}_{s+1}, \ldots, \mathbf{y}_{s+1}) = u_1 u_2 \ldots u_s \in G.$$

As we have proved that we can 'erase' u_i (and $i \leq s$ was any fixed index), we deduce that the constant tuple $\mathbf{x} \in \{x\}^\omega$ is also in G. This means that there exists a term p such that $p(\mathbf{y}_1, \dots, \mathbf{y}_l) = \mathbf{x}$, and that means that the equations of the form $p(x, \dots, x, y, x, \dots, x) = x$ (y is in any position) hold in $\mathbf{F}$. As $\mathbf{F}(x, y)$ is the $\mathcal{V}$-free algebra, this means that p is an l-ary near-unanimity term.

(5) $\Rightarrow$ (1) is a consequence of the fact that a near-unanimity term implies both the edge term (see [1]) and congruence distributivity. $\square$

We are able to prove that (1) $\Rightarrow$ (5) directly, as well. This other proof, on the other hand fails to give us the useful criterion (4) which recognizes exactly which Γ-special cube terms imply congruence distributivity, and is less aesthetically pleasing. However, it is shorter, and also gives the best possible arity of the near-unanimity term (given the results of [1]), so we supply it below:

Theorem 2.2. *A variety $\mathcal{V}$ is congruence distributive and has a $k+1$-ary edge term t iff $\mathcal{V}$ has a k-ary near-unanimity term s ($k \geq 3$).*

Proof. Assume that $\mathbf{F} = \mathbf{F}(x, y)$ is the free algebra in $\mathcal{V}$ freely generated by $\{x, y\}$. Let $\mathbf{y}_i$, $1 \leq i \leq k$ be the elements of $\mathbf{F}^k$ such that $\mathbf{y}_i(i) = y$ and $\mathbf{y}_i(j) = x$, for $j \neq i$ and let $\mathbf{G} \leq \mathbf{F}^k$ be the subalgebra generated by $\{\mathbf{y}_i : 1 \leq i \leq k\}$. As in the proof of the Theorem 2.1, (4) $\Rightarrow$ (5), we are going to try to prove that $\mathbf{x} \in G$, where $\mathbf{x}(i) = x$ for all $1 \leq i \leq k$.

Similarly to the Claim 1 of the proof of Theorem 2.1, (4) $\Rightarrow$ (5), we can easily obtain that $\mathbf{G}$ is a totally symmetric subpower of $\mathbf{F}$. We also introduce the notation of elements of $\mathbf{G}$ as words on the alphabet $\mathbf{F}$ which have length k.

We suppose that the edge term $t(x_0, x_1, \dots, x_k)$ of $\mathcal{V}$ has the matrix $M(\Gamma)$ such that only the first column has two values 1, at the first two coordinates, and the ith column has value 1 only at the $i-1$st coordinate. We also suppose that the variety $\mathcal{V}$ has Jónsson terms $p_0, p_1, \dots, p_n$ such that $p_0(x, y, z) \approx x$, $p_n(x, y, z) \approx z$, $p_i(x, y, x) \approx x$ for all $i \leq n$, $p_i(x, x, y) \approx p_{i+1}(x, x, y)$ for all even $i < n$ and $p_i(x, y, y) \approx p_{i+1}(x, y, y)$ for all odd $i < n$.

We define now the elements $a_i, b_i, c_i \in \mathbf{F}$ for $1 \leq i < n$ to be $a_i = p_i(y, x, x)$, $b_i = p_i(y, y, x)$ and $c_i = p_i(x, x, y)$.

We first prove that for each $1 \leq i < n$, $a_i c_i x^{k-2}$ and $b_i c_i x^{k-2}$ are both in G. To see this, just notice that $a_i c_i x^{k-2} = p_i(\mathbf{y}_1, \mathbf{y}_3, \mathbf{y}_2)$ and that $b_i c_i x^{k-2} = p_i(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_2)$. Next notice that $b_1 x^{k-1} = y x^{k-1} = \mathbf{y}_1 \in G$.

Claim 5. For each i, $1 \leq i < n$, $a_i x^{k-1} \in G$ iff $b_i x^{k-1} \in G$.

We prove one implication only, as the other one is analogous. So, assume that $a_i x^{k-1} \in G$. The total symetricity of $\mathbf{G}$ implies that for each $j < k-1$, the elements $\mathbf{b}_j = b_i x^j c_i x^{k-2-j}$ are in G. Now we calculate that

$$b_i x^{k-1} = t(a_i c_i x^{k-2}, a_i x^{k-1}, \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{k-2}).$$

Indeed,

$$t(a_i c_i x^{k-2}, a_i x^{k-1}, \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{k-2})(1) = t(a_i, a_i, b_i, b_i, \dots, b_i) = b_i,$$

$$t(a_i c_i x^{k-2}, a_i x^{k-1}, \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{k-2})(2) = t(c_i, x, c_i, x, x, \dots, x) = x$$

and

$$t(a_i c_i x^{k-2}, a_i x^{k-1}, \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{k-2})(j) = t(x, x, \dots, x, c_i, x, \dots, x) = x,$$

for $3 \leq j \leq k-2$. □

Now we proceed to prove by induction on i that for each $0 < i < n$, both $a_i x^{k-1}$ and $b_i x^{k-1}$ are in G. For $i = 1$ we already have $b_1 x^{k-1} \in G$, and by the Claim 5, this means that $a_1 x^{k-1} \in G$, as well. Assume that both $a_i x^{k-1}$ and $b_i x^{k-1}$ are in G. By the Jónsson equations, either $a_i = a_{i+1}$ or $b_i = b_{i+1}$. Therefore, at least one of the elements $a_{i+1} x^{k-1}$ and $b_{i+1} x^{k-1}$ is in G. But, by the Claim 5, this means that both of them are in G.

Note that since one of the elements a_{n-1}, b_{n-1} is equal to x by Jónsson equations, this means that $\mathbf{x} \in G$. Now we finish the proof as in the proof of Theorem 2.1, (4) ⇒ (5). Namely, as $\mathbf{x} \in G$, then there must exist a term p such that $p(\mathbf{y}_1, \dots, \mathbf{y}_k) = \mathbf{x}$. But looking at coordinates of this equation, they imply that in $\mathbf{F}$ all equations of the form $p(x, \dots, x, y, x, \dots, x) = x$ hold, and since $\mathbf{F}$ is the $\mathcal{V}$-free algebra, then p satisfies the near-unanimity equations in $\mathcal{V}$. □

3. Acknowledgements

The authors wish to thank Keith Kearnes for suggesting that the result proved here might be true, Miklós Maróti for suggesting that a direct syntactical proof could also exist and Igor Dolinka for helpful discussions.

References

[1] J. Berman, P. Idziak, P. Marković, R. McKenzie, M. Valeriote and R. Willard, *Varieties with few subalgebras of powers*, to appear.