Outline

1. Introduction
   - Shifts and subshifts
   - The chain relation

2. Characterisation of the chain relation
   - Linking graph
   - Theorem about chain relation
   - Corollaries
We are interested in the structure of biinfinite words $A^\mathbb{Z}$.

We can equip $A^\mathbb{Z}$ with a metric $\rho$; the distance $\rho(x, y)$ of $x \neq y$ is equal to $2^{-n}$ where $n$ is the absolute value of the first index where $x$ differs from $y$. 
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Labelled graph

- **Labelled graph** is an oriented multidigraph whose edges are labelled by letters from $A$.
- $x \in \Sigma$ iff $x$ has a *presentation* in $G$: There exists a biinfinite walk in $G$ labelled by letters from $x$.
- Without loss of generality assume that $G$ is *essential*, that is every vertex has at least one outgoing and at least one ingoing edge.
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An $\varepsilon$-chain from the word $x$ to the word $y$ is sequence of words $x^0, x^1, \ldots, x^n \in \Sigma$ such that $x^0 = x, x^n = y$ and $\rho(\sigma(x^i), x^{i+1}) < \varepsilon$.

The words $x, y \in \Sigma$ are in the chain relation $\mathcal{C}$ if for every $\varepsilon > 0$ there exists an $\varepsilon$-chain (of nonzero length) from $x$ to $y$. 
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$\varepsilon$-chain in picture

$x = x^0 \rightarrow x^i \rightarrow x^{i+1} \rightarrow \ldots \rightarrow x^n = y \rightarrow \sigma(x^i)$

Alexandr Kazda
The chain relation in sofic subshifts
ε-chain and jumps

Take the subshift \( \{ a^{-\infty} ba^{\infty}, a^{\infty} \} \). Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

\[
\begin{align*}
\ldots & aab \quad aa \ldots \\
\ldots & aabaa \ldots aa \quad aa \ldots \\
& \quad \text{Hop!} \\
\ldots & aa \ldots a \quad a \ldots aabaa \ldots \\
& \quad \ldots aa \quad baa \ldots
\end{align*}
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We have managed to shift the word to the right.
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How to describe the chain relation in a general sofic subshift $\Sigma$?

- The main idea: We can jump between some vertices of $G$.
- Call such pairs of vertices *linked*.
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Call two vertices $u, v$ of a labelled graph $G$ linked if for any length $n$ there exists a word $w$ of length $n$ that has presentations beginning in both $u, v$ and not leaving the components of $u, v$.

By joining all pairs of linked vertices we obtain the linking graph $G/\sim$.

We have a natural projection of $G$ onto $G/\sim$. 
Linked vertices

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We have a natural projection of $G$ onto $G/\approx$. 
Linked vertices in picture

\begin{align*}
  \text{a} & \rightarrow \text{b} \\
  \text{a} & \rightarrow \text{c} \\
  \text{c} & \rightarrow \text{e} \\
  \text{e} & \rightarrow \text{e}
\end{align*}
Linked vertices in picture

The chain relation in sofic subshifts
Linking graph in picture
Components of $G/\approx$

- The components of the graph $G/\approx$ can be partially ordered by the relation $K \leq L$ meaning “there exists a walk from $K$ to $L$”.
- For $x$ infinite word define $\alpha(x)$ and $\omega(x)$ as the components of $G/\approx$ where the image of a presentation of $x$ begins and ends.
- The components $\alpha(x), \omega(x)$ are well-defined: They always exist and do not depend on the choice of presentation of $x$. 
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The chain relation in sofic subshifts
The components of $G/\sim$ in picture

$K_1$ is incomparable with $K_2$, $K_3$ and $K_2 \leq K_3$.

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Theorem about chain relation

Let $\Sigma$ be a sofic subshift, $G$ its labelled graph. Let $x, y \in \Sigma$. Then $(x, y) \in C$ iff $\omega(x) \leq \alpha(y)$ or $y = \sigma^n(x)$ for some $n > 0$. 

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The chain relation in sofic subshifts
Other properties of subshifts

- A subshift is *chain-transitive* if every two its words are in the chain relation.
- A subshift is *chain-mixing* if for every two words $x, y \in \Sigma$ and $\varepsilon > 0$ exists a $k$ such that for all $n > k$ there exists an $\varepsilon$-chain from $x$ to $y$ of length $n$.
- Chain transitivity is often used when describing dynamic systems, the chain mixing property can be useful for finding attractors of cellular automata.
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Characterising chain transitive and chain mixing subshifts

**Theorem**

Let $\Sigma$ be a sofic subshift, $G$ its essential graph. Then $\Sigma$ is chain transitive iff $G/\approx$ is (strongly) connected.

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Let $\Sigma$ be a sofic subshift, $G$ its essential graph. Then $\Sigma$ is chain mixing iff $G/\approx$ is (strongly) connected and aperiodic.
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Intermezzo: Aperiodic graphs

A graph is *periodic* iff $V(G)$ can be partitioned into $n > 1$ disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ such that every edge $e \in E(G)$ leads from some $v \in V_k$ to some $u \in V_{k+1}$ for a suitable $k$.

A graph is *aperiodic* if it is not periodic.
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A graph is aperiodic if it is not periodic.
Informally, an attractor of a dynamic system is a closed set that attracts all trajectories from its neighbourhood.

Theorem

The attractors of the dynamic system \((\Sigma, \sigma)\) are precisely all the subshifts described by the preimages of nonempty terminal subgraphs of \(G/\approx\).
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A subgraph $H$ of $G/\approx$ is terminal iff there are no edges leading from $V(H)$ to $V(G) \setminus V(H)$.
Intermezzo: Terminal subgraphs

A subgraph $H$ of $G/\sim$ is terminal iff there are no edges leading from $V(H)$ to $V(G) \setminus V(H)$. 
Using the linking graph we can describe the chain relation in sofic subshifts.

Using this knowledge we can explicitly describe:
- Chain transitivity
- The chain-mixing property
- The attractors of the subshift dynamic system

All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.
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The End

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