The connecting homomorphism for K-theory of generalized free products

Ross Staffeldt
Department of Mathematical Sciences, New Mexico State University
Las Cruces, NM 88003, USA

April 16, 2007

Abstract

We interpret the connecting homomorphism of the the long exact sequence of algebraic K-groups associated to a generalized free product diagram of rings satisfying certain conditions at a categorical level. Diagrams of groups rings arising from Seifert-van Kampen situations where the homomorphisms are injective provide examples. The techniques follow Waldhausen's approach to algebraic K-theory of categories with cofibrations and weak equivalences.

1 Introduction

We start with a diagram

$$A \xleftarrow{\alpha} C \xrightarrow{\beta} B$$

of discrete rings satisfying freeness and purity conditions. That is, we require for purity that $\alpha : C \longrightarrow A$ be injective and that there be a splitting $A = \alpha(C) \oplus A'$ of C-bimodules. Though the splitting is not part of the data and only its existence is required, it is convenient to refer to a fixed complement A' of $\alpha(C)$ in A. The freeness condition we will impose is that A' shall be free as a left C-module. We impose the same conditions on $\beta : C \longrightarrow B$.

Let

$$R = A *_C B$$

be the free product of A and B amalgamated along the common subring C. As a C-bimodule, then

$$R = C \oplus A' \oplus B' \oplus A' \otimes_C B' \oplus B' \otimes_C A' \oplus A' \otimes_C B' \otimes_C A' \oplus B' \otimes_C A' \otimes_C B' \oplus \cdots$$

and the problem is to describe the K-theory of R in terms of the K-theories of A, B, and C.

In his paper from the *Annals of Mathematics* in 1978, Waldhausen derived an exact sequence

$$\longrightarrow K_{n+1}(R) \xrightarrow{\partial_*} K_n(C) \oplus \mathrm{Nil}_n(C; A', B') \longrightarrow K_n(B) \oplus K_n(C) \longrightarrow K_n(R) \xrightarrow{\partial_*}$$

and proved that the term $\mathrm{Nil}_n(C; A', B')$ splits off $K_{n+1}(R)$.

In 1995 Roland Schwänzl and I published a paper in the *Transactions* essentially to rederive these results using the techniques of the algebraic K-theory of spaces, which had come along in the mid-1980's. Our derivation was more straightforward.

In 2005 Andrew Ranicki asked me if the methods of our paper could be used to shed light on the homomorphism

$$\partial_* \colon K_{n+1}(R) \longrightarrow K_n(C) \oplus \mathrm{Nil}_n(C; , A', B').$$

Today I report on what I have developed to answer his questions.

2 The context of our 1995 paper

Let A be a ring with 1. A right module over A is a simplicial abelian group M, together with a unital and associative action of A from the right. This action may be codified in terms of a homomorphism of abelian groups $M \otimes A \longrightarrow M$ where the tensor product is taken degreewise, such that certain diagrams commute.

$$\mathcal{M}(A) = \text{the category of right } A\text{-modules and their } A\text{-linear maps}$$

A simplicial set Y gives rise to a module $[Y]A$ if we let

$$([Y]A)_n = [Y_n]A,$$

the free right A-module generated by Y_n.

Example 2.1. Notice that if $X = \Delta^q/\partial\Delta^q$ is a standard model for the simplicial q-sphere, then $[X]A$ is isomorphic to a wedge of simplicial Eilenberg-MacLane spaces $K(A,q)\vee K(A,0)$.

To attach an n-cell to an A-module M means to form a pushout of the kind

$$M \longleftarrow [\partial\Delta^n]A \longrightarrow [\Delta^n]A.$$

A module N is obtained from M by attaching cells if it can be constructed from M by means of this process together with taking a direct limit.

We will also be using the following notations. A pair of simplicial sets (Y, Y') gives rise to a pair of modules denoted by $[Y, Y']A$. More generally, if M is any A-module, then we define

$$[Y, Y']M = [Y, Y']\mathbf{Z} \otimes M,$$

where $\mathbf{Z}$ denotes the integers and where we take the tensor product degreewise.

The category of modules we are interested in is

$$\mathcal{M}_f(A) = \text{the full subcategory of } \mathcal{M}(A) \text{ consisting of finite modules}$$

That is, the objects are the modules obtainable from the zero module by attaching finitely many cells.

We take K-theory to be defined in terms of the $S_\bullet$ construction, which from a category with cofibrations and weak equivalences produces its K-theory space. In our module category $\mathcal{M}_f(A)$ the cofibrations $M \rightarrowtail N$ are the maps arising in the cell attaching process, and the weak equivalences are the h-maps, namely, those maps which become homotopy equivalences after realization. However, simplicial abelian groups are Kan sets, so it would suffice to use simplicial homotopy equivalences.

Using all the rings, we have categories denoted by $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$, respectively. Each of these categories then supports a notion of cofibration, and a notion of weak equivalence, namely, the subcategory of h-maps. I suppress explaining how the h-maps interact with the cofibrations, although it is important for details of the constructions.

Then the K-theory of $\mathcal{M}_f(A)$ with respect to these notions of cofibration and weak equivalence (generically indicated by the presence of an h somewhere in the symbol) is defined as

$$K(\mathcal{M}_f(A); h) = \Omega|hS_\bullet\mathcal{M}_f(A)|.$$

According to Waldhausen, the relation with Quillen's plus construction definition of K-theory is

$$\Omega|hS_\bullet\mathcal{M}_f(A)| \simeq K'_0(A) \times BGL(A)^+,$$

where $K_0'(A)$ is the subgroup of the usual Grothendieck group of isomorphism classes of projective modules $K_0(A)$ generated by the free modules. Therefore, we will abbreviate

$$K(\mathcal{M}_f(A); h) = K(A)$$

and use similar notations for the other rings B, C, and R.

2.1 Mayer-Vietoris presentations

The framework for the 1995 paper used a notion of Mayer-Vietoris presentation generalizing the notion used by Waldhausen in his 1978 *Annals* paper. In our category of Mayer-Vietoris presentations $\mathcal{MV}$, an object is a sextuple

$$M_- = (M, M_A, M_B, M_C, \iota, \kappa)$$

where M is an object of $\mathcal{M}_f(R)$, and M_A, M_B, and M_C are similarly objects of $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, and $\mathcal{M}_f(C)$, respectively. Also, ι and κ are maps of R-modules such that

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

is a fibration sequence of simplicial R-modules and

$$\kappa(M_A) \subset M_C \otimes_C A \text{ and } \kappa(M_B) \subset M_C \otimes_C B.$$

These are the same conditions imposed by Waldhausen, but now taken degreewise. A map of Mayer-Vietoris presentations is a quadruple of maps $f_- = (f, f_A, f_B, f_C)$ in the respective module categories, such that the resulting ladder diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"'] & M_A \otimes_A \oplus M_B \otimes_B R \arrow[r] \arrow[d, "f_A\otimes R"', "\oplus f_B \otimes R"] & M_C \otimes_C R \arrow[r] \arrow[d, "f_C \otimes R"'] & 0 \\
0 \arrow[r] & N \arrow[r] & N_A \otimes_A \oplus N_B \otimes_B R \arrow[r] & N_C \otimes_C R \arrow[r] & 0
\end{tikzcd}$$

commutes.

By means of the forgetful functors to the respective module categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$, one defines cofibrations in $\mathcal{MV}$.

Definition 2.1 (Weak equivalences of Mayer-Vietoris presentations). A map of Mayer-Vietoris presentations (f, f_A, f_B, f_C) is a *coarse weak equivalence*

if the map f is a homotopy equivalence.

The subcategory of these maps will be denoted by $w\mathcal{MV}$, and such a map will be called a w-map for short.

A map of Mayer-Vietoris presentations is a *fine weak equivalence* if the maps f_A, f_B, and f_C are homotopy equivalences.

We denote the subcategory of $\mathcal{MV}$ where the arrows are the v-equivalences by $v\mathcal{MV}$, and call these arrows v-maps for short.

Notice that it follows from the five lemma that a v-map is also a w-map, so that $v\mathcal{MV} \subset w\mathcal{MV}$.

In 1995 our starting point was the following square

$$\begin{tikzcd} vS_\bullet\mathcal{MV}^w \arrow[r, "g"] \arrow[d] & wS_\bullet\mathcal{MV}^w \arrow[d] \\ vS_\bullet\mathcal{MV} \arrow[r, "h"] & wS_\bullet\mathcal{MV} \end{tikzcd} \tag{2.1}$$

where $\mathcal{MV}^w$ denotes the subcategory of $\mathcal{MV}$ consisting of Mayer-Vietoris presentations in which the R-module is contractible. By Waldhausen's fibration theorem, the square is homotopy cartesian after geometric realization, and the upper right corner realizes to a contractible space.

After taking loop spaces of the realizations, the non-trival terms are by definition the K-theory of the category of Mayer-Vietoris presentations of contractible R-modules with respect to v-maps; the K-theory of the category of Mayer-Vietoris presentations with respect to the v-equivalences (the fine equivalences); and the K-theory of the category of Mayer-Vietoris presentations with respect to the w-equivalences (the coarse equivalences).

We proved three interpretative theorems.

Theorem 2.1. *The forgetful functors from $\mathcal{MV}$ to the module categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, and $\mathcal{M}_f(C)$ induce a homotopy equivalence*

$$u_*\colon vS_\bullet\mathcal{MV} \longrightarrow hS_\bullet\mathcal{M}_f(A) \times hS_\bullet\mathcal{M}_f(B) \times hS_\bullet\mathcal{M}_f(C).$$

In other words,

$$u_*\colon K(\mathcal{MV}; v\mathcal{MV}) \xrightarrow{\simeq} K(A) \times K(B) \times K(C).$$

□

The proof used the additivity theorem.

Theorem 2.2. *The forgetful functor*

$$u_R\colon \mathcal{MV} \longrightarrow \mathcal{M}_f(R)$$

induces a homotopy equivalence

$$wS_\bullet\mathcal{MV} \longrightarrow hS_\bullet\mathcal{M}_f(R).$$

In other words,

$$(u_R)_*\colon K(\mathcal{MV}; w\mathcal{MV}) \xrightarrow{\simeq} K(R)$$

□

The proof used the approximation theorem. Finally, we define $\mathcal{S}$, the category of split modules. This subcategory of $\mathcal{MV}$ consists of the Mayer-Vietoris presentations

$$0 \longrightarrow 0 \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0.$$

Alternatively, one characterizes objects of this category as Mayer-Vietoris presentations in which the map κ is an isomorphism. This subcategory of $\mathcal{MV}$ inherits all the structures which are on $\mathcal{MV}$, but, of course, only the v-equivalences are relevant. Obviously it is a subcategory of $\mathcal{MV}^w$. We also used the approximation theorem to prove the following result.

Theorem 2.3. *The inclusion functor*

$$i : \mathcal{S} \longrightarrow \mathcal{MV}^w$$

induces a homotopy equivalence

$$vS_\bullet\mathcal{S} \longrightarrow vS_\bullet\mathcal{MV}^w.$$

In other words,

$$K(\mathcal{S}, v\mathcal{S}) \xrightarrow{\simeq} K(\mathcal{MV}^w, v\mathcal{MV}^w)$$

□

We may then interpret the diagram (2.1) as yielding another homotopy cartesian diagram.

$$\begin{tikzcd} K(\mathcal{S}, v\mathcal{S}) \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ K(A) \times K(B) \times K(C) \arrow[r] & K(R) \end{tikzcd}$$

Remark 2.4. The lower arrow is induced by tensoring modules up to R. The upper-left hand term may be analysed following the manner of Waldhausen. The result is not quite $\mathrm{Nil}(C; A', B') \times K(C) \times K(C)$, since rather interesting and delicate projective C-modules arise.

The category of Mayer-Vietoris presentations also inherits from the module categories a notion of mapping cylinder. This construction is necessary for the existence of the square of the fibration theorem, and it plays an important role in the new results.

For a category of right modules, the mapping cylinder is constructed in the following manner.

Definition 2.2 (Mapping cylinders for simplicial modules). Let $f\colon M \to N$ be a homomorphism of right modules. Form the pushout diagram

$$\begin{tikzcd} {[\partial\Delta^1]\mathbf{Z} \otimes M \cong M \oplus M} \arrow[r, "\mathrm{id}\oplus f"] \arrow[d, tail] & M \oplus N \arrow[d, tail, "i\oplus j"] \\ {[\Delta^1]\mathbf{Z} \otimes M} \arrow[r] & T(f) \end{tikzcd}$$

to obtain the *mapping cylinder object* $T(f)$ together with the *front inclusion* i and the *back inclusion* j. The universal property of the pushout provides the *canonical projection* $p\colon T(f) \to N$, such that the following diagram commutes.

$$
\begin{tikzcd}
M \arrow[r, tail, "i"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & N \arrow[l, tail, "j"'] \arrow[dl, equal] \\
& N &
\end{tikzcd}
$$

The mapping cylinder is a functor from a category of maps to another category of diagrams; it also happens to preserve cofibrations, when these are defined appropriately in the target category.

The category $\mathcal{MV}$ inherits the notion of mapping cylinder from the notions of mapping cylinder in $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$.

Idea. We also have the notion of mapping cone $C(f_-)$, factoring out the mapping cylinder by the front inclusion. Clearly, the mapping cone of a w-map f_- is an object of $\mathcal{MV}^w$. Thus, there is a natural way to reach $K_0(\mathcal{MV}^w, v\mathcal{MV}^w)$ from certain elements of $K_1(R)$. Is this the connecting homomorphism, and does the idea work more generally?

3 New results

The goal of this section is to develop an interpretation of the connecting homomorphism of the long exact sequence of homotopy groups associated to the homotopy cartesian square

$$
\begin{tikzcd}
vS_\bullet\mathcal{MV}^w \arrow[r, "f"] \arrow[d] & wS_\bullet\mathcal{MV}^w \arrow[d] \\
vS_\bullet\mathcal{MV} \arrow[r, "g"] & wS_\bullet\mathcal{MV}
\end{tikzcd}
\tag{3.1}
$$

To accomplish this we first record carefully a definition of the connecting homomorphism. Following an exercise from Spanier's book, if $g\colon (Y, y_0) \to (Y', y_0')$ is a base-point preserving map, write E_g for the homotopy fibre of g. This space is defined as the pullback via g of the path fibration $PY' \to Y'$, where PY' denotes the space of paths beginning at the base-point. There is then an exact sequence of spaces

$$\Omega Y \xrightarrow{\Omega g} \Omega Y' \xrightarrow{j} E_g \xrightarrow{p} Y \xrightarrow{g} Y',$$

where $j(\omega) = (y_0, \omega)$. One expression of the fact that the square is homotopy cartesian is to say that the homotopy fibres of the two horizontal rows are homotopy equivalent by a canonical map induced from the vertical arrows in the square diagram. Let us write E_g and E_h for the homotopy fibres of the top and bottom rows of the diagram of spaces obtained after geometric realization of diagram (2.1). In this situation there is a ladder diagram

$$\begin{tikzcd}
\Omega|vS_\bullet\mathcal{M}\mathcal{V}^w| \arrow[r, "\Omega f"] \arrow[d] & \Omega|wS_\bullet\mathcal{M}\mathcal{V}^w| \arrow[r, "j"] \arrow[d] & E_f \arrow[r, "p"] \arrow[d, "\simeq"] & |vS_\bullet\mathcal{M}\mathcal{V}^w| \arrow[r, "f"] \arrow[d] & |wS_\bullet\mathcal{M}\mathcal{V}^w| \arrow[d] \\
\Omega|vS_\bullet\mathcal{M}\mathcal{V}| \arrow[r, "\Omega g"] & \Omega|wS_\bullet\mathcal{M}\mathcal{V}| \arrow[r, "k"] & E_g \arrow[r, "q"] & |vS_\bullet\mathcal{M}\mathcal{V}| \arrow[r, "g"] & |wS_\bullet\mathcal{M}\mathcal{V}|
\end{tikzcd}$$

In these terms the connecting homomorphism $\partial_\#\colon \pi_r|wS_\bullet\mathcal{M}\mathcal{V}| \to \pi_{r-1}|vS_\bullet\mathcal{M}\mathcal{V}^w|$ is the following composition.

$$\pi_r|wS_\bullet\mathcal{M}\mathcal{V}| \xrightarrow{\cong} \pi_{r-1}\Omega|wS_\bullet\mathcal{M}\mathcal{V}| \xrightarrow{k_\#} \pi_{r-1}\,E_g \xrightarrow{\cong} \pi_{r-1}\,E_f \xrightarrow[\cong]{p_\#} \pi_{r-1}|vS_\bullet\mathcal{M}\mathcal{V}^w| \tag{3.2}$$

The main source of trouble for calculations is, of course, the link that is the inverse of the isomorphism induced by $E_f \xrightarrow{\simeq} E_g$. This is what we are trying to get around.

Starting from square (3.1) and working through Waldhausen's proof of the fibration theorem, one obtains a chain of equivalences of squares ending with the lefthand square in the next commuting diagram

$$\begin{tikzcd}
vS_\bullet\mathcal{M}\mathcal{V}^w \arrow[r, "f_1"] \arrow[d] & vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}^w, \mathcal{M}\mathcal{V}^w) \arrow[r] \arrow[d] & vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}^w) \arrow[d, equal] \\
vS_\bullet\mathcal{M}\mathcal{V} \arrow[r, "g_1"] & vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}, \mathcal{M}\mathcal{V}^w) \arrow[r, "h_1"] & vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}^w)
\end{tikzcd} \tag{3.3}$$

In this diagram the rows are fibrations-up-to-homotopy, and this implies that the lefthand square is homotopy-cartesian. Consequently, all the squares in the stages are homotopy-cartesian, including the one that interests us most. Chasing through the argument, we arrive at the following result describing the connecting homomorphism.

Proposition 3.1. *There is a diagram*

$$\begin{tikzcd}
\pi_{r-1}\Omega|wS_\bullet\mathcal{M}\mathcal{V}| \arrow[r, "k_\#"] \arrow[d, "\cong"'] & \pi_{r-1}\,E_g \arrow[r, "\cong"] & \pi_{r-1}\,E_f \arrow[r, "p_\#", "\cong"'] & \pi_{r-1}|vS_\bullet\mathcal{M}\mathcal{V}^w| \arrow[dl, "\cong"'] \\
\pi_{r-1}\Omega|vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}, \mathcal{M}\mathcal{V}^w)| \arrow[rr, "\Omega h_1"] & & \pi_{r-1}\Omega|vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}^w)|
\end{tikzcd} \tag{3.4}$$

that commutes up to sign, where h_1 arises from a functor, and the diagonal isomorphism is the canonical equivalence $|vS_\bullet\mathcal{M}\mathcal{V}^w| \to \Omega|vS_\bullet S_\bullet(\mathcal{M}\mathcal{V}^w)|$.

It appears that we have an interpretation of the connecting homomorphism in terms of a functor, a very nice way around the problem of inverting the homotopy equivalence $E_f \to E_g$.

Unfortunately, the one thing that can go wrong has gone wrong. The links chaining the square of (3.1) with the lefthand square of (3.3) aren't composable.

However, there is a way around this using a different sort of homotopy theory. We are concerned with elements of $\pi_r|wS_\bullet\mathcal{MV}|$. It is a result due to Kan that any element may represented by the realization of a simplicial map

$$f\colon K \longrightarrow \operatorname{diag} NwS_\bullet\mathcal{MV}.$$

Here K is a finite semisimplicial set arising from a subdivision of the simplicial r-sphere $\partial\Delta^{r+1}$, and the target is the diagonal simplicial set of the nerve of the simplicial category $wS_\bullet\mathcal{MV}$.

In several interesting and useful cases, the proof of the following proposition explains what to do. At the present level of technology, we need to start with a simplicial map

$$f\colon K \longrightarrow \operatorname{diag} NwS_m\mathcal{MV}.$$

We can create the following diagram.

$$\begin{tikzcd}
N\operatorname{simp}^{nd}(K) \arrow[r, "\simeq"] \arrow[d, dashed, "\phi_3"'] \arrow[dr, dashed, "\phi_2"'] \arrow[drr, "\phi_1"] & K \arrow[r, "f"] & NwS_m\mathcal{MV} \arrow[d] \\
vS_\bullet(S_m\mathcal{MV}, S_m\mathcal{MV}^w) \arrow[r] & N^w v\overline{w}S_m\mathcal{MV} \arrow[r, "i"'] & N^w vwS_m\mathcal{MV}
\end{tikzcd} \tag{3.5}$$

The top row consists of simplicial sets, and $N\operatorname{simp}^{nd}(K)$ is a barycentric subdivision of K. We view these all as simplicial categories, in which the only arrows are identities. At the lower right corner the term $N^w vwS_m\mathcal{MV}$ is the partial nerve of a bicategory of squares, hence, a simplical category, as is $N^w v\overline{w}S_m\mathcal{MV}$; $vS_\bullet(S_m\mathcal{MV}, S_m\mathcal{MV})$ is also a simplicial category, for each m.

In terms of the preceding diagram, given the simplicial map f as above, there exists a canonically defined simplicial map $\phi_1\colon N\operatorname{simp}^{nd}(K) \to N^w vwS_m\mathcal{MV}$ whose realization also represents the homotopy class of f.

Proposition 3.2. *Let $f\colon K \longrightarrow N_\bullet wS_m\mathcal{MV}$ be a simplicial map, and let $\phi_1\colon N\operatorname{simp}^{nd}(K) \to N^w vwS_m\mathcal{MV}$ be the canonical map derived derived from f. There are a simplicial map $\phi_2\colon N\operatorname{simp}^{nd}(K) \to N^w v\overline{w}S_m\mathcal{MV}$, and a simplicial homotopy H from $i\phi_2$ to ϕ_1, so that ϕ_2 is a lifting of ϕ_1 up to homotopy. There is also a simplicial map $\phi_3\colon N\operatorname{simp}^{nd}(K) \to vS_\bullet(S_m\mathcal{MV}, S_m\mathcal{MV}^w)$ lifting ϕ_2.*

Minimalist sketch. An extension of the mapping cylinder construction to an iterated mapping cylinder construction allows the construction of ϕ_2; the homotopy comes from the extension of the back projections. The lifting ϕ_3 is by an explicit construction of objects in the simplicial category $vS_\bullet(S_m\mathcal{MV}, S_m\mathcal{MV}^w)$. $\square$

Remark 3.3. How does the original idea translate into this framework? A w-equivalence $f_-\colon M_- \to M_-$ determines a map of a one simplex into $NwS_1\mathcal{MV}$. The subdivision of the one simplex produces two one-simplices, producing two objects of $vS_1(S_1\mathcal{MV}, S_1\mathcal{MV}^w)$, namely,

$$\begin{tikzcd} 0 \arrow[r, tail] & M \arrow[r, tail, "i"] \arrow[d, two heads] & T(f) \arrow[d, two heads] \\ & 0 \arrow[r, tail] & C(f) \arrow[d, two heads] \\ & & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r, tail] & M \arrow[r, tail, "j"] \arrow[d, two heads] & T(f) \arrow[d, two heads] \\ & 0 \arrow[r, tail] & C(M) \arrow[d, two heads] \\ & & 0 \end{tikzcd}$$

Under the functor h_1, these "map to" $C(f)$ and $C(M)$, respectively. The cone on M, $C(M)$, is v-equivalent to the trivial object, so it contributes nothing. (In fact, it is also true that for higher dimensional simplices, many simplices of the subdivision end up contributing nothing to the final result.)