

**NIL PHENOMENA IN TOPOLOGY (14–15 APRIL 2007)**  
**VANDERBILT UNIVERSITY (NASHVILLE, TN U.S.A.)**  
**PROBLEM SESSION**

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**IO.** (Ivonne Ortiz, Miami of Ohio)

Does  $NK_n(\mathbb{Z}[G], \alpha) = 0$  if and only if  $NK_n(\mathbb{Z}[G]) = 0$ ?

**DJP.** (Daniel Juan-Pineda, Morelia)

Is  $NK_2(\mathbb{Z}[G]) \neq 0$  for some non-trivial finite group  $G$ ? It is known that  $NK_1(\mathbb{Z}[G]) = 0$  if and only if  $G$  has  $p$ -rank 1. It also is known that  $NK_1(\mathbb{Z}[G], \alpha) = 0$  if  $C_p \times C_p \not\subset G$ .

**CH.** (Christian Haesemeyer, Illinois)

It seems to be known that  $NK_2(\mathbb{Z}[G])$  is non-zero for cyclic groups  $G$  and conjectured that it is non-vanishing for all finite groups. Likewise, it is known that  $NK_1(\mathbb{F}_p[C_{p^n}])$  does not vanish, and one might guess that  $NK_1(\mathbb{F}_p[G]) \neq 0$  for any finite group whose order is divisible by  $p$ .

Note that this is suggestive, since  $\mathbb{Z}$  has dimension 1 and  $\mathbb{F}_p$  has dimension 0. Maybe the following might be true:

Let  $R$  be a commutative ring of dimension  $d$  and  $G$  a finite group such that  $R[G]$  is not regular (is that equivalent to “some residue characteristic of  $R$  divides the order of  $G$ ”?). Then  $NK_{d+1}(R[G]) \neq 0$ , or at least,  $RG$  is not  $K_{d+1}$ -regular.

Compare to the following conjecture of Vorst:

If  $R$  is a finite type algebra over some field, of dimension  $d$ , then  $R$  is not  $K_{d+1}$ -regular. (We recently proved this in the characteristic zero case.)

**IH1.** (Ian Hambleton, McMaster)

For a finite group  $G$ , it is known that  $NK_n(\mathbb{Z}[G]) \cong NK_n(\widehat{\mathbb{Z}}_{|G|}[G])$ . Suppose  $G$  is a finite  $p$ -group. How does one study the following relative group?

$$NK_n(\widehat{\mathbb{Z}}_p[G]) \longrightarrow \mathbb{F}_p[G]$$

This may have implications in TC-theory.

*References:* [Dundas] [Hesselholt]

**IH2.** (Ian Hambleton, McMaster)

Suppose  $\mathbb{Z}[G]$  is contained in a maximal order  $\Gamma$ . Consider the cartesian square

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathbb{Z}[G]/\Gamma & \longrightarrow & \Gamma/n\Gamma. \end{array}$$

Is there a Mayer-Vietoris sequence in  $NK_n$  for all  $n \geq 1$ ?

*Reference:* [Suslin-Voevodsky]

**RS.** (Ross Staffeldt, New Mexico State)

Consider the *Nil*-terms associated to fundamental group diagrams

$$G_1 \xleftarrow{i_1} G_0 \xrightarrow{i_2} G_2$$

where  $i_1$  and  $i_2$  are injective.

**AR.** (Andrew Ranicki, Edinburgh)

$K$ - or  $L$ -theory of  $G_1 *_{G_0} G_2$  in the non-injective case? The methods of non-commutative localization produce long exact sequences in the injective case, and probably lead to spectral sequences in the general case. Can someone identify a condition that leads to collapsing spectral sequences? Can someone treat, by any ad-hoc methods whatsoever, an example that provides some insight?

**BW1.** (Bruce Williams, Notre Dame)

Burghlelea, Lashof, and Rothenberg have shown, for all  $i$ , that

$$\pi_i(C(M \times S^1)) \cong \pi_i(C(M \times I)) \oplus \pi_{i-1}(C \times I) \oplus \pi_i(NC(M))$$

where  $NC(M)$  is defined as the homotopy fiber of a map

$$CEmb(M, M \times S^1) \longrightarrow CEmb(M, M \times R^1).$$

Give a direct geometric definition of  $NC(M)$ .

*Reference:* [BLR75]

**BW2.** (Bruce Williams, Notre Dame)

Let  $\mathcal{C}(M) := \text{hocolim}_i C(M \times I^i)$  be the stable concordance space. Then (BW1) implies

$$\mathcal{C}(M \times S^1) \simeq \mathcal{C}(M) \times \Omega^{-1}\mathcal{C}(M) \times NC(M).$$

The algebraic decomposition

$$A(M \times S^1) \simeq A(M) \times \Omega^{-1}A(M) \times N_-A(M) \times N_+A(M)$$

of Klein, Waldhausen, et.al. formally implies that  $NC(M)$  decomposes as a product of two equivalent pieces

$$N_{\pm}C(M) \simeq \Omega N_{\pm}A(M).$$

Give a geometric interpretation of this decomposition.

*References:* [HKV<sup>+</sup>01] [HKV<sup>+</sup>02]

**BW3.** (Bruce Williams, Notre Dame)

Do we get a decomposition of  $NC(M)$  outside the concordance stable range? Does it help that  $M = N \times I \times I^j$  for some  $j$ ?

**BW4.** (Bruce Williams, Notre Dame)

Algebraic constructions and computations of Hesselholt-Madsen and Grunewald-Klein-Macko yield non-trivial elements in  $\pi_0 N_{\pm}A(M)$ . This formally implies the existence of non-trivial elements in  $\pi_0 N_{\pm}\mathcal{C}(M)$  and  $\pi_0 NTOP(M)$ . Give geometric descriptions of these elements.

*References:* [Hesselholt-Madsen] [GKM] [HP00] [Jahren-Rognes-Waldhausen]

**JG1.** (Joachim Grunewald, Bonn)

Vogel generalizes the class of regular rings and asks the question whether the  $K$ -theoretic  $Nil$ -groups still vanish for this class of rings. A positive answer to this conjecture would extend to class of groups with trivial Whitehead group.

*Reference:* [Bih]

**JG2.** (Joachim Grunewald, Bonn)

Recall that  $NA_{\pm}(point) \neq 0$ . Is there a notion of “regular” for a space  $X$  such that  $NA_{\pm}(X) = 0$ ?

*Reference:* [GKM]

**FC-JG.** (Frank Connolly and Joachim Grunewald, Notre Dame and Bonn)

Let  $G$  be a finite group. Is  $NK_n(\mathbb{Z}[G], \alpha)$  finitely generated over the Verschiebung algebra  $\mathcal{V}$ ? Connolly and da Silva have shown that  $NK_0(\mathbb{Z}[G])$  is a finitely generated  $\mathcal{V}$ -module. Guin-Walery and Loday have shown that  $NK_2(\mathbb{Z}[C_p])$  is a cyclic module over  $\mathcal{V}/p\mathcal{V}$ .

*References:* [CdS95] [GWL81]

**FC1.** (Frank Connolly, Notre Dame)

Suppose  $\Gamma$  is a virtual PD group and its Farrell cohomology,  $\widehat{H}^*(\Gamma; \mathbb{Z})$ , is periodic. Assume  $\Gamma$  has no elements of order 2. Is  $\Gamma = \pi_1(M)$  for some closed manifold  $M$  such that  $\widetilde{M}$  has the homotopy type of a sphere?

**FC2.** (Frank Connolly, Notre Dame)

$hdim(X)$  = the minimum dimension of a CW complex of homotopy type  $X$ . For a crystallographic group  $\Gamma$  of rank  $n$ , it is known that  $hdim B_{vc}(\Gamma) = n + 1$  if  $n \geq 2$ . Show  $hdim(\Gamma) = n$  if  $\Gamma$  is virtually nilpotent of (nilpotent) rank  $n$  but is not virtually abelian.

*Reference:* [CFH]

**FC3.** (Frank Connolly, Notre Dame)

$UNil_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$  is an infinite direct sum of cyclic modules over  $\mathcal{V} = \mathbb{Z}[V_1, V_2, \dots]$ . Is  $UNil_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$  finitely generated over a slightly larger algebra of operators?

Joachim Grunewald remarks that the addition of the Frobenius operators is not sufficient for finite generation in the case of  $NA(point)$ .

*Reference:* [GKM]

**SP.** (Stratos Prassidis, Canisius)

How does the Bass-Heller-Swan type splitting for  $Wh$ -theory, given by Hughes-Prassidis, relate to the H utteman-Klein-Vogell-Waldhausen-Williams splitting for  $A$ -theory?

*References:* [HP00] [HKV+01]

**JD1.** (Jim Davis, Indiana)

What is a computation of  $L_*(\mathbb{Z}[x, y])$ ?

**JD2.** (Jim Davis, Indiana)

Formulate the Farrell-Jones isomorphism conjecture in  $L$ -theory with respect to the family  $CFIN = CYCLICS \cup FINITES$ , using twisted transfers to eliminate the types of virtually cyclic groups which surject onto  $D_\infty$ .

**QK.** (Qayum Khan, Vanderbilt)

What is a computation of  $L_*(\mathbb{Z}[C_4 *_{C_2} C_4])$ ? Observe that the  $\mathbb{Z}[C_2]$ -bimodule structure on  $\mathbb{Z}[C_4 - C_2]$  is standard but the involution is non-standard.

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