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NEIGHBORHOODS OF STRATA IN MANIFOLD STRATIFIED SPACES

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Abstract. Strata in manifold stratified spaces are shown to have neighborhoods that are teardrops of manifold stratified approximate fibrations (under dimension and compactness assumptions). This is the best possible version of the tubular neighborhood theorem for strata in the topological setting. Applications are given to replacement of singularities, to the structure of neighborhoods of points in manifold stratified spaces, and to spaces of manifold stratified approximate fibrations.

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1. Introduction. The foundations of differential topology include the tubular neighborhood theorem: a smooth submanifold of a smooth manifold has a neighborhood that is the total space of a disc bundle over the submanifold. For locally flat topological submanifolds, the best result about neighborhoods (in high dimensions) is due to Edwards [**3**]: the submanifold has a mapping cylinder neighborhood given by a manifold approximate fibration (see also [**14**]).

For stratified spaces, the stratifications of Whitney are considered to be the correct theory in the smooth category. For Whitney stratified spaces, the tubular neighborhood theorem of Thom [**23**] and Mather [**16**], [**17**] says that each stratum has a neighborhood that is the total space of a bundle over the stratum, and the fiber of the bundle is the cone on the stratified link (see Goresky and MacPherson [**4**] for an exposition). As is the case for submanifolds, the structure on the neighborhoods is not part of the definition, and the proof of their existence is non-trivial.

In the topological category, Quinn [**19**] has introduced a natural stratification theory. The purpose of this paper is to establish the existence of a type of tubular neighborhood for strata in Quinn's stratified spaces, or *manifold stratified spaces*.

MAIN THEOREM. *Let X be a manifold stratified space with a stratum A satisfying:*
(1) *A has compact closure* $\mathrm{cl}(A)$ *in X,*
(2) *if Y and Z are distinct strata of X with $Z \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(Y)$, then* $\dim Y \geq 5$.
Then A has an MSAF teardrop neighborhood in X.

Teardrop neighborhoods of A in X are generalizations of mapping cylinder neighborhoods of A. Instead of being determined by a map to A, they are determined by a map to $A \times [0, \infty)$. Such maps are not hard to come by; the real significance is in

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the MSAF property. These initials stand for *manifold stratified approximate fibration*, an effective substitute for a fiber bundle.

Siebenmann [**22**] introduced a class of topologically stratified spaces earlier than Quinn, but Siebenmann's locally conelike spaces have proved to be too rigid to be considered the true topological analogue of the Whitney stratifications. Nevertheless, the Main Theorem above is new for Siebenmann's spaces.

Hughes, Taylor, Weinberger and Williams [**12**] have established the Main Theorem in the case of manifold stratified spaces with two strata. Many of the methods of [**12**] are used in the present paper.

For additional background information on the relationship among stratifications in various categories, see the survey paper by Hughes and Weinberger [**15**]. The Main Theorem was announced in [**6**] and that paper should be consulted for statements about applications. Also announced in [**6**] is a theory of neighborhoods of closed unions of strata. That theory, which uses the present results in a crucial way, has recently appeared in [**10**]. In that paper, *MSAF teardrop neighborhoods* are called *approximate tubular neighborhoods*.

I have benefited greatly from the interest of my collaborators on related projects: Andrew Ranicki, Larry Taylor, Shmuel Weinberger, and, especially, Bruce Williams.

2. Manifold stratified spaces. This section contains the basic definitions from the theory of stratifications as presented in [**6**], [**7**], [**8**], [**9**], [**12**], [**19**].

DEFINITION 2.1. A *stratification* of a space X consists of an index set $\mathcal{I}$ and a locally finite partition $\{X_i\}_{i\in\mathcal{I}}$ of locally closed subspaces of X (the X_i are pairwise disjoint and their union is X). For $i \in \mathcal{I}$, X_i is called the *i-stratum* and the closed set

$$X^i = \cup\{X_k \mid X_k \cap \mathrm{cl}(X_i) \neq \emptyset\}$$

is called the *i-skeleton*. We say that X is a *space with a stratification*.

For a space X with a stratification $\{X_i\}_{i\in\mathcal{I}}$, define a relation $\leq$ on the index set $\mathcal{I}$ by $i \leq j$ if and only if $X_i \subseteq \mathrm{cl}(X_j)$. The *Frontier Condition* is satisfied if for every $i, j \in \mathcal{I}$, $X_i \cap \mathrm{cl}(X_j) \neq \emptyset$ implies $X_i \subseteq \mathrm{cl}(X_j)$, in which case $\leq$ is a partial ordering of $\mathcal{I}$ and $X^i = \mathrm{cl}(X_i)$ for each $i \in \mathcal{I}$.

A map between spaces with stratifications is *stratum preserving* if it takes strata into strata.

If X is a space with a stratification, then a map $f : Z \times A \to X$ is *stratum preserving along A* if for each $z \in Z$, $f(\{z\} \times A)$ lies in a single stratum of X. In particular, a map $f : Z \times I \to X$ is a *stratum preserving homotopy* if f is stratum preserving along I. A homotopy $f : Z \times I \to X$ whose restriction to $Z \times [0, 1)$ is stratum preserving along $[0, 1)$ is said to be *nearly stratum preserving*.

DEFINITION 2.2. Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and $Y \subseteq X$.

(1) Y is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I$, $h_1(U) = Y$, and $h((U\setminus Y) \times [0, 1)) \subseteq X\setminus Y$.

(2) The *homotopy link* of Y in X is defined by

$$\mathrm{holink}(X, Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{ if and only if } t = 0\}.$$

(3) Y is *stratified forward tame* in X if there exist a neighborhood U of Y in X and a nearly stratum preserving homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t | Y = \text{inclusion} : Y \to X$ for each $t \in I$ and $h_1(U) = Y$.

(4) The *stratified homotopy link* of Y in X is defined by

$$\text{holink}_s(X, Y) = \{\omega \in \text{holink}(X, Y) \mid \text{ for some } i,\ \omega(t) \in X_i \text{ for all } t \in (0, 1]\}.$$

(5) Let $x_0 \in X_i \subseteq X$. The *local holink at* x_0 is

$$\text{holink}(X, x_0) = \{\omega \in \text{holink}_s(X, X_i) \mid \omega(0) = x_0\}.$$

All path spaces are given the compact-open topology. Evaluation at 0 defines maps $q : \text{holink}(X, Y) \to Y$ and $q : \text{holink}_s(X, Y) \to Y$, both called *holink evaluation*. There is a natural stratification of $\text{holink}_s(X, Y)$ into disjoint subspaces

$$\text{holink}_s(X, Y)_i = \{\omega \in \text{holink}_s(X, Y) \mid \omega(1) \in X_i\}.$$

The local holink at $x_0 \in X_i$ inherits a natural stratification from $\text{holink}_s(X, X_i)$.

DEFINITION 2.3. A space X with a stratification satisfying the Frontier Condition is a *manifold stratified space* if the following four conditions are satisfied:

(1) *Forward Tameness.* For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.

(2) *Normal Fibrations.* For each $k > i$, the holink evaluation

$$q : \text{holink}(X_i \cup X_k, X_i) \to X_i$$

is a fibration.

(3) *Compactly Dominated Local Holinks.* For each i and each $x_0 \in X_i$, there exist a compact subset C of the local holink $\text{holink}(X, x_0)$ and a stratum preserving homotopy

$$h : \text{holink}(X, x_0) \times I \to \text{holink}(X, x_0)$$

such that $h_0 = id$ and $h_1(\text{holink}(X, x_0)) \subseteq C$.

(4) *Manifold strata property.* X is a locally compact, separable metric space, each stratum X_i is a topological manifold (without boundary) and X has only finitely many nonempty strata.

If X is only required to satisfy conditions (1) and (2), then X is a *homotopically stratified space*.

REMARK. The definition of manifold stratified space given above agrees with the one given in [**6**] except that the local holinks condition is apparently weaker than the compactly dominated holinks property stated there. The current formulation should be considered the correct one and agrees with [**9**]. I plan to clarify the relationship between these conditions and the reverse tameness condition of Quinn in a future paper. For more information, see [**19**, Prop. 2.15, Lem. 4.6] and [**11**, Chap. 8,9].

DEFINITION 2.4. Let X and Y be spaces with stratifications $\{X_i\}_{i \in \mathcal{I}}$ and $\{Y_j\}_{j \in \mathcal{J}}$, respectively, and let $p : X \to Y$ be a map.

(1) p is a *stratified fibration* provided that given any space Z and any commuting diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & Y \end{array}$$

with F a stratum preserving homotopy, there exists a *stratified solution*; i.e., a stratum preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\tilde{F} = F$. The diagram above is a *stratified homotopy lifting problem*.

(2) p is a *stratified approximate fibration* provided that given any stratified homotopy lifting problem, there exists a *stratified controlled solution*; i.e., a map $\tilde{F} : Z \times I \times [0, 1) \to X$ that is stratum preserving along $I \times [0, 1)$ such that $\tilde{F}(z, 0, t) = f(z)$ for each $(z, t) \in Z \times [0, 1)$ and the function $\bar{F} : Z \times I \times I \to Y$ defined by $\bar{F}|Z \times I \times [0, 1) = p\tilde{F}$ and $\bar{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous.

(3) p is a *manifold stratified approximate fibration* (MSAF) if X and Y are manifold stratified spaces and p is a proper stratified approximate fibration.

(4) If α is an open cover of Y, then p is a *stratified α-fibration* provided that given any stratified homotopy lifting problem, there exists a *stratified α-solution*; i.e., a stratum preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\tilde{F}$ is α-close to F.

(5) p is a *manifold approximate fibration* (MAF) if p is a MSAF and X and Y have only one stratum each; i.e., X and Y are manifolds.

REMARKS. (1) In [**8**] and [**9**] the map $\bar{F}$ in Definition 2.4(2) was incorrectly assumed to be stratum preserving along $I \times I$. The results in [**8**] and [**9**] are correct when this modification is made. At any rate, this distinction is not important in this paper because the results here concern spaces Y with only a single stratum.

(2) In the case that Y is unstratified (that is, consists of a single stratum) and is an ANR and $p : X \to Y$ is a stratified α-fibration for every open cover α of Y, then p is a stratified approximate fibration. See [**13**, Lem. 12.11]. The converse is essentially true as well, but we do not need it. See [**13**, Lem. 12.10].

The following result illustrates properties of manifold stratified spaces, which will be used repeatedly.

THEOREM 2.5 ([**7**], [**19**]). *Let X be a manifold stratified space and let $Y \subseteq X$ be a closed union of some of the strata of X. Then:*

(1) *Y is stratified forward tame in X, and*

(2) *the evaluation $q : \mathrm{holink}_s(X, Y) \to Y$ is a stratified fibration.*

3. Teardrop neighborhoods. This section contains the basic teardrop construction as well as a reduction of the proof of the Main Theorem to a special case.

Given spaces X, Y and a map $p : X \to Y \times \mathbb{R}$, the *teardrop* of p is the space denoted by $X \cup_p Y$ whose underlying set is the disjoint union $X \amalg Y$ with the minimal topology such that

(1) $X \subset X \cup_p Y$ is an open embedding, and

(2) the function $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ defined by

$$c(x) = \begin{cases} p(x), & \text{if } x \in X \\ (x, +\infty), & \text{if } x \in Y \end{cases}$$

is continuous.

This is a generalization of the construction of the open mapping cylinder of a map $g : X \to Y$. Namely, $\overset{\circ}{\mathrm{cyl}}(g)$ is the teardrop $(X \times \mathbb{R}) \cup_{g \times id} Y$. However, not all teardrops are open mapping cylinders because not all maps to $Y \times \mathbb{R}$ can be split as a product. See [**12**] for more about the teardrop construction.

If X is a space with a stratification and $A \subseteq X$, we say A has an MSAF *teardrop neighborhood in* X if there is an open neighborhood U of A and an MSAF $p : U\backslash A \to A \times \mathbb{R}$ such that the natural function $(U\backslash A) \cup_p A \to U$ is a homeomorphism. This is equivalent to saying that p is an MSAF and the natural extension $\tilde{p} : U \to A \times (-\infty, +\infty]$ is continuous. In this case, $\tilde{p}$ is also an MSAF when $A \times (-\infty, +\infty]$ is given the natural stratification (see [**9**, Prop. 7.1], [**12**]).

The following main result of [**9**] shows that the teardrop construction yields manifold stratified spaces.

THEOREM 3.1 [**9**]. *If X and Y are manifold stratified spaces and $p : X \to Y \times \mathbb{R}$ is a manifold stratified approximate fibration, then the teardrop $X \cup_p Y$ with the natural stratification is a manifold stratified space.*

The special case of the Main Theorem will now be stated.

THEOREM 3.2. *The Main Theorem holds whenever* cl(A)$\backslash A$ *consists of at most one point.*

The proof of Theorem 3.2 will be given in § 7. The Main Theorem of § 1 follows from Theorem 3.2 and the following result as is shown in Corollary 3.4 below.

PROPOSITION 3.3. *If X is a manifold stratified space and Z is a compact union of strata of X, then the quotient space $Y = X/Z$ with the natural stratification is a manifold stratified space.*

Proof. Y has a single point stratum corresponding to $\{Z\}$; the other strata are homeomorphic to strata of X. The forward tameness condition follows from the fact that Z is stratified forward tame in X by Theorem 2.5 above. The compactly dominated local holinks condition follows from Proposition 5.6 below.

COROLLARY 3.4. *Theorem* 3.2 *implies the Main Theorem.*

Proof. Let X and A be given as in the Main Theorem, let $B = \mathrm{cl}(A)$ and let $Z = B\backslash A$. If $Z = \emptyset$, then the result follows immediately from Theorem 3.2. Otherwise Z is compact and we form the quotient space $Y = X/Z$, which is a manifold stratified space by Proposition 3.3. Moreover, Y has a stratum corresponding to A. Theorem 3.2 implies that A has an MSAF teardrop neighborhood in Y. Since $Y\backslash\{Z\}$ is stratum preserving homeomorphic to $X\backslash Z$, the result follows.

REMARK. There is also a version of the Main Theorem in which it is not assumed that the stratum A has compact closure in X. Let X be a manifold stratified space with finitely many strata $\{X_i\}_{i\in\mathcal{I}}$ and let A be a stratum of X. Consider the one-point compactification $X' = X \cup \{x_\infty\}$ to be a space with a stratification whose strata are $\{X_i\}_{i\in\mathcal{I}} \cup \{x_\infty\}$. Assume that

(1) X' is a manifold stratified space, and

(2) if Y is any stratum of X such that $\mathrm{cl}(A) \cap \mathrm{cl}(Y) \neq \emptyset$, then $\dim(Y) \geq 5$.

Then it follows that A has an MSAF teardrop neighborhood in X. This is because the Main Theorem implies that A has an MSAF teardrop neighborhood in X'. Note that condition (1) is essentially a tameness condition at infinity for the non-compact strata of X. For example, it says that the non-compact minimal strata of X are manifolds with a tame end (see [**11**]).

4. Stratified sucking. This section establishes sucking phenomena in a stratified setting. This tool is due to Chapman [**2**] and was further developed in [**5**]. We will need the following result of Quinn [**19**].

THEOREM 4.1 (Stratified Isotopy Extension [**19**]). *Suppose X is a manifold stratified space, X^i is a skeleton and U is any neighborhood of X^i. Suppose that if there exist indices $j < k$ such that $j \leq i$, then $\dim(X_k) \geq 5$. If $h : X^i \times I \to X^i \times I$ is a stratum preserving isotopy, then there exists an extension $\tilde{h} : X \times I \to X \times I$ of h to a stratum-preserving isotopy such that $\tilde{h}$ is supported on U. Moreover, if $C \subseteq V \subseteq X$ with C closed and V open and $h_t|(V \cap X^i) =$ inclusion for each $t \in I$, then $\tilde{h}$ may be chosen so that $\tilde{h}_t|C =$ inclusion for each $t \in I$.*

REMARKS. As mentioned above, Quinn [**19**] has verified the Stratified Isotopy Extension Theorem. The relative version is not stated in [**19**], but it follows from the proof. See the relative version mentioned in the proof of Theorem 2.3 in [**18**] (which is the basis for Quinn's proof in [**19**]).

For manifold stratified spaces with at most two strata, there is an independent proof of the Stratified Isotopy Extension Theorem in [**12**], which also includes a parameterized version. We plan to eventually provide a parameterized version for manifold stratified spaces with any number of strata (cf. [**6**]). It was originally envisioned that the proof would be logically independent of [**19**]. However, that is not clear now. It is important to realize that the present paper relies heavily on Quinn's Stratified Isotopy Extension Theorem.

THEOREM 4.2 (*Stratified Sucking*). *Suppose X is a manifold stratified space with no strata of dimension less than 5 and Y is a manifold without boundary. For every open cover α of Y, there exists an open cover β of Y such that if $p : X \to Y$ is a proper stratified β-fibration, then p is properly α-homotopic to a manifold stratified approximate fibration.*

REMARK. Chapman [**2**] has verified Theorem 4.2 in the special case that X is a manifold and we rely on his proof below.

Proof of Theorem 4.2. Once the appropriate engulfing results are established [**2**, § 3], the proof follows an outline that is now quite familiar: wrapping up [**2**, § 4], handle lemmas [**2**, § 5] and, finally, the proof follows by making improvements over handles and taking a limit of β_n-fibrations [**2**, § 6]. Observe that the limit argument of [**2**, § 6] is still valid because we are assuming that Y is unstratified (i.e., Y consists of a single stratum), so there is no problem with limits of solutions collapsing onto lower strata. We also need to use the remark following Definition 2.4 above to conclude that the limit is a stratified approximate fibration.

In order for Chapman's machine to work in the current context we need to make sure that the necessary engulfing isotopies exist. As in [**2**, § 3], we are considering maps $p| : X \backslash X_i \to Y$ where Y has an $\mathbb{R}$ factor. Certain isotopies in the $\mathbb{R}$ direction need to be lifted to X. The key engulfing lemma is stated below.

To conclude this section, we state and indicate the proof of the key engulfing lemma that is needed to start Chapman's machine and obtain Theorem 4.2. For notation, Z is a compact polyhedron and Y is a space that contains $Z \times \mathbb{R}$ as an open subset. Projection onto Z is denoted by p_1.

LEMMA 4.3. *Suppose X is a manifold stratified space with no strata of dimension less than 5. For every $\epsilon > 0$ there exists $\delta > 0$ so that if $p : X \to Y$ is a proper stratified δ-fibration over $Z \times [-3, 3]$, then there is a stratum preserving homeomorphism $h : X \to X$ such that*

(1) $p^{-1}(Z \times (-\infty, 1]) \subseteq hp^{-1}(Z \times (-\infty, 0))$,

(2) *there is a stratum preserving isotopy $h_s : \mathrm{id}_X \simeq h$, $0 \leq s \leq 1$ that is a $(p_1 p)^{-1}(\epsilon)$-homotopy supported on $p^{-1}(Z \times [-2, 2])$.*

Proof. Chapman's proof of the corresponding unstratified result [**2**, Lem. 3.4] is to work locally in Z, obtaining a finite number of isotopies, which are then composed. In the stratified case, one also works locally in Z, which ensures that the final isotopy is small when measured in Z. The new wrinkle is to work inductively up through the strata of X. Each time the isotopy is defined on a stratum, use Theorem 4.1 to extend it to a neighborhood of the corresponding skeleton (the relative version is needed here because we are working locally in Z). At the next step the isotopy on X can be constructed to agree with the lifted isotopy on the previous skeleton (cf. the "Concluding Remarks" of [**2**, § 3]). (In reading the proof of Chapman's Lemma 3.3, take note that the last sentence of the first paragraph in the proof should read "... lies in the complement of $f^{-1}(\Gamma(v))$.")

5. Compactly dominated local holinks. In this section let X be a locally compact, homotopically stratified metric space with only finitely many strata. Further suppose that the strata are ANRs and that X has compactly dominated local holinks. Let A be a stratum of X such that $B = \mathrm{cl}(A)$ is compact. Recall from [**7**] that

$$\mathrm{P}_{\mathrm{nsp}}(X, B) = \{\omega \in X^I \mid \omega(0) \in B \text{ and } \omega \text{ is nearly stratum preserving in the sense that } \omega((0,1]) \text{ lies in a single stratum of X}\}.$$

The stratification of X induces a natural stratification of $\mathrm{P}_{\mathrm{nsp}}(X, B)$ in which the stratum of a path ω is determined by the stratum of X that contains its terminal point $\omega(1)$. Let

$$\mathrm{holink_s}^{+}(X, B) = \mathrm{holink_s}(X, A) \cup \{\omega \in (B\backslash A)^I \mid \omega \text{ is a constant path}\}.$$

We interpret $\mathrm{holink_s}^{+}(X, B)$ as $\mathrm{holink_s}(X, A) \cup (B\backslash A)$, where each point in $B\backslash A$ is identified with the constant path at that point. Note that $\mathrm{holink_s}^{+}(X, B) \subseteq \mathrm{P}_{\mathrm{nsp}}(X, B)$ and as such inherits a natural stratification.

Let d be a metric for X and let $\delta : B \to [0, +\infty)$ be a map such that $\delta^{-1}(0) = B\backslash A$. Let

$$\mathrm{holink_s}^{\delta}(X, A) = \{\omega \in \mathrm{holink_s}(X, A) \mid d(\omega(0), \omega(t)) < \delta(\omega(0)) \text{ for all } t \in I\},$$
$$\mathrm{holink_s}^{\delta}(X, B) = \mathrm{holink_s}^{\delta}(X, A) \cup (B\backslash A) \subseteq \mathrm{holink_s}^{+}(X, B).$$

LEMMA 5.1. *The inclusion* $\mathrm{holink_s}^{\delta}(X, B) \to \mathrm{holink_s}^{+}(X, B)$ *is a stratum preserving fiber homotopy equivalence (both are spaces over B via the holink evaluation).*

Proof. The technique of proof comes from Quinn [**19**, Lem. 2.4(i)]. The idea is to shrink paths along themselves towards their initial points. A partition of unity is used to piece this local shrinking together to provide a homotopy inverse for the inclusion.

LEMMA 5.2. *The holink evaluation* $\mathrm{holink_s}^{\delta}(X, A) \to A$ *is a stratified fibration.*

Proof. This follows from the fact that holink evaluation $\mathrm{holink_s}(X, A) \to A$ is a stratified fibration [**7**, Cor. 6.2]: lifting problems for $\mathrm{holink_s}^{\delta}(X, A) \to A$ have stratified solutions in $\mathrm{holink_s}(X, A)$; those solutions can be shrunk into $\mathrm{holink_s}^{\delta}(X, A)$ by another partition of unity construction. In particular, there is a stratum preserving and fiber preserving deformation

$$\beta : \mathrm{holink_s}^{+}(X, B) \times I \to \mathrm{holink_s}^{+}(X, B)$$

such that

(1) $\beta_0 = \mathrm{id}$,
(2) β is rel $B\backslash A$,
(3) $(\beta_t\omega)(s) \in \omega(I)$ for all $s, t \in I$ and $\omega \in \mathrm{holink_s}^{+}(X, B)$,
(4) $\beta_1(\mathrm{holink_s}^{+}(X, B)) \subseteq \mathrm{holink_s}^{\delta}(X, B)$.

LEMMA 5.3. *If* $h : \mathrm{holink_s}^{\delta}(X, A) \times I \to \mathrm{holink_s}^{\delta}(X, A)$ *is any fiber preserving homotopy, then h extends continuously to* $\tilde{h} : \mathrm{holink_s}^{\delta}(X, B) \times I \to \mathrm{holink_s}^{\delta}(X, B)$ *via the identity; that is,* $\tilde{h}(\omega_b, t) = \omega_b$ *for all* $t \in I$ *where* ω_b *denotes the constant path at* $b \in B\backslash A$.

Proof. It suffices to show that the adjoint

$$\hat{h} : \mathrm{holink_s}^{\delta}(X, B) \times I \times I \to X; \quad \hat{h}(\omega, s, t) = \tilde{h}(\omega, s)(t),$$

is continuous at points in $(B\backslash A) \times I \times I$. Thus, let $(\omega_b, s_0, t_0) \in (B\backslash A) \times I \times I$ and let $(\omega_n, s_n, t_n) \in \mathrm{holink_s}^{\delta}(X, A) \times I \times I$, $n = 1, 2, 3, \ldots$ be a sequence converging to (ω_b, s_0, t_0). Since $\hat{h}(\omega_b, s_0, t_0) = b$ and $\hat{h}(\omega_n, s_n, t_n) = \tilde{h}(\omega_n, s_n)(t_n)$, we need to show that $\tilde{h}(\omega_n, s_n)(t_n) \to b$ as $n \to \infty$. But $\omega_n \to \omega_b$ implies that $\omega_n(0) \to \omega_b(0) = b$. Also note $d(\tilde{h}(\omega_n, s_n)(0), \tilde{h}(\omega_n, s_n)(t_n)) < \delta(\tilde{h}(\omega_n, s_n)(0))$. Since h is fiber preserving, $\tilde{h}(\omega_n, s_n)(0) = \omega_n(0)$. Thus, $d(\omega_n(0), \tilde{h}(\omega_n, s_n)(t_n)) < \delta(\omega_n(0))$. Since $\delta(\omega_n(0)) \to 0$, it follows that $d(b, \tilde{h}(\omega_n, s_n)(t_n)) \to 0$ as required.

PROPOSITION 5.4. *There exists a compact set* $C \subseteq \mathrm{holink_s}^{+}(X, B)$ *together with a stratum preserving and fiber preserving* (*over B*) *homotopy*

$$d : \mathrm{holink_s}^{+}(X, B) \times I \to \mathrm{holink_s}^{+}(X, B)$$

such that

(1) $d_0 = \mathrm{id}$,
(2) $d_1(\mathrm{holink_s}^{+}(X, B)) \subseteq C$,
(3) $d_t|(B\backslash A) = \mathrm{inclusion}$ *for each* $t \in I$ (*in particular,* $B\backslash A \subseteq C$).

Proof. Since the inclusion $\mathrm{holink_s}^{\delta}(X, B) \to \mathrm{holink_s}^{+}(X, B)$ is a stratum preserving fiber homotopy equivalence (Lemma 5.1), it suffices to define the homotopy on $\mathrm{holink_s}^{\delta}(X, B)$. Moreover, by Lemma 5.3, d only needs to be defined on $\mathrm{holink_s}^{\delta}(X, A)$. To this end use the facts that $\mathrm{holink_s}^{\delta}(X, A) \to A$ is a stratified fibration (Lemma 5.2) and that A is an ANR to conclude that $\mathrm{holink_s}^{\delta}(X, A) \to A$ has local stratum preserving fiber homotopy trivializations. Combine this observation with the fact that the fibers of $\mathrm{holink_s}^{\delta}(X, A) \to A$ are compactly dominated (being stratum preserving homotopy equivalent to the local holinks) to construct locally finite open countable covers $\{U_k\}$ and $\{V_k\}$ of A with $\mathrm{cl}(V_k) \subseteq U_k$ for $k = 1, 2, 3, \ldots$, stratum preserving and fiber preserving homotopies

$$d^k : \mathrm{holink_s}^\delta(X, A) \times I \to \mathrm{holink_s}^\delta(X, A),$$

and compact subsets $C_k \subseteq \mathrm{holink_s}^\delta(X, \mathrm{cl}(V_k))$ such that

(1) $d_0^k = \mathrm{id}$,

(2) $d_1^k(\mathrm{holink_s}^\delta(X, \mathrm{cl}(V_k))) \subseteq C_k$,

(3) $d_t^k(\mathrm{holink_s}^\delta(X, A\backslash U_k)) =$ inclusion for all $t \in I$

(cf. [**12**, Lem. 6.6]). Define $D_t = \lim_{k\to\infty} d_t^k \circ d_t^{k-1} \circ \cdots \circ d_t^1$ (noting that this composition is locally finite) to get a stratum preserving and fiber preserving homotopy

$$D : \mathrm{holink_s}^\delta(X, A) \times I \to \mathrm{holink_s}^\delta(X, A)$$

and a subset $C' \subseteq \mathrm{holink_s}^\delta(X, A)$ such that

(1) $D_0 = \mathrm{id}$,

(2) $D_1(\mathrm{holink_s}^\delta(X, A)) \subseteq C'$,

(3) for each $x \in A$ there exists a compact neighborhood N_x of x in A such that $q^{-1}(N_x) \cap C'$ is compact where $q : \mathrm{holink_s}^\delta(X, A) \to A$ is holink evaluation. (In fact, one can take $N_x = \mathrm{cl}(V_k)$ where $x \in V_k$.)

Now it can be seen that $C = C' \cup (B\backslash A)$ is compact and that the extension d of D given by Lemma 5.3 fulfills the requirements.

LEMMA 5.5. *Let $Z \subseteq X$ be a compact and suppose Z is a union of strata of X and A is a maximal stratum of Z (i.e., A is disjoint from the closure of any other stratum of Z). For every neighborhood U of Z in X there exist a neighborhood V of A in X with $V \subseteq U$, a compact subset $K \subseteq U\backslash A$ and a stratum preserving homotopy $g : (V\backslash A) \times I \to U$ such that $g_0 =$* inclusion *and $g_1(V\backslash A) \subseteq K$.*

Proof. Choose a map $\delta : Z \to [0, +\infty)$ such that $\delta^{-1}(0) = Z\backslash A$ and the δ-neighborhood $N_\delta(A)$ of A in X satisfies $N_\delta(A) \subseteq U$. By the proof of Proposition 5.4 there exist a compact subset $C \subseteq \mathrm{holink_s}^\delta(X, Z)$ and a stratum preserving and fiber preserving homotopy $d : \mathrm{holink_s}^\delta(X, Z) \times I \to \mathrm{holink_s}^\delta(X, Z)$ such that $d_0 = \mathrm{id}$, $d_1(\mathrm{holink_s}^\delta(X, Z)) \subseteq C$ and $d_t|(Z\backslash A) =$ inclusion for all $t \in I$. Let $K = \{\omega(1) \mid \omega \in C\}$. Then K is compact and $K \subseteq (N_\delta(A)\backslash A) \cup (Z\backslash A)$. Since A is stratified forward tame in X (Theorem 2.5), there exist a neighborhood V of A in X with $V \subseteq N_\delta(A)$ and a nearly stratum preserving homotopy $h : V \times I \to N_\delta(A)$ such that $h_0 =$ inclusion and $h_1(V) \subseteq A$. By making V smaller we may assume that the track of each point in $V\backslash A$ is an element of $\mathrm{holink_s}^\delta(X, A)$. That is, there is an induced map $\hat{h} : V\backslash A \to \mathrm{holink_s}^\delta(X, A)$. If this is combined with the deformation d of $\mathrm{holink_s}^\delta(X, A)$ into C followed by evaluation of holinks at 1, there is an induced stratum preserving homotopy $g : (V\backslash A) \times I \to U$ such that $g_0 =$ inclusion and $g_1(V\backslash A) \subseteq K$.

PROPOSITION 5.6. *If $Z \subseteq X$ is compact, Z is a union of strata of X and the quotient space X/Z is given the natural stratification, then X/Z has compactly dominated local holinks.*

Proof. X/Z has a stratum consisting of the single point corresponding to $\{Z\}$. The other strata are homeomorphic to strata of X. The compactly dominated local holinks condition only has to be checked at the point $\{Z\}$. We can use [**9**, Lem. 5.3] and transfer the problem to a statement about Z in X: show that given a neighborhood U of Z in X there exist a neighborhood V of Z in X with $V \subseteq U$, a compact subset $K \subseteq U\backslash Z$

and a stratum preserving homotopy $g : (V\backslash Z) \times I \to U$ such that $g_0 =$ inclusion and $g_1(V\backslash Z) \subseteq K$. We proceed by induction on the number n of strata of Z. It is vacuously true for $n = 0$, so assume $n > 0$ and the statement is true for fewer than n strata. Let Y be a maximal stratum of Z. Let W be a compact neighborhood of Z in X with $W \subseteq U$ (recall we are assuming X is locally compact). By the inductive hypothesis there exist a neighborhood V_1 of $Z\backslash Y$ in X with $V_1 \subseteq \operatorname{int} W$, a compact subset $K_1 \subseteq W\backslash(Z\backslash Y)$ and a stratum preserving homotopy $g^1 : (V_1\backslash(Z\backslash Y)) \times I \to W$ such that $g^1_0 =$ inclusion and $g^1_1(V_1\backslash(Z\backslash Y)) \subseteq K_1$. Let $\rho^1 : X \to I$ be a map such that $V_1' = (\rho^1)^{-1}(1) \subseteq V_1$ is a compact neighborhood of $Z\backslash Y$ and $\rho^1(X\backslash V_1) = 0$. Define $\tilde{g}^1 : (X\backslash(Z\backslash Y)) \times I \to X$ by

$$\tilde{g}^1(x, t) = \begin{cases} g^1(x, \rho^1(x) \cdot t) & \text{if } x \in V_1\backslash(Z\backslash Y) \\ x & \text{if } x \notin V_1. \end{cases}$$

Note that $\tilde{g}^1_0 =$ inclusion and $\tilde{g}^1_1(V_1'\backslash(Z\backslash Y)) \subseteq K_1$. By Lemma 5.5 there exist a neighborhood V_2 of Y in X with $V_2 \subseteq \operatorname{int} W$ and a compact subset $K_2 \subseteq W\backslash Y$ for which there is a stratum preserving homotopy $g^2 : (V_2\backslash Y) \times I \to W$ such that $g^2_0 =$ inclusion and $g^2_1(V_2\backslash Y) \subseteq K_2$. Let $\rho^2 : X\backslash \mathrm{Bd}(Y) \to I$ be a map (where Bd(Y) $= \operatorname{cl} Y\backslash Y$) such that $V_2' = (\rho^2)^{-1}(1) \subseteq V_2$ is a closed neighborhood of Y in $X\backslash\mathrm{Bd}(Y)$ and $\rho^2((X\backslash\mathrm{Bd}(Y)\backslash V_2) = 0$. Define $\tilde{g}^2 : (X\backslash Z) \times I \to X$ by

$$\tilde{g}^2(x, t) = \begin{cases} g^2(x, \rho^2(x) \cdot t) & \text{if } x \in V_2\backslash Y \\ x & \text{if } x \notin V_2 \cup Z. \end{cases}$$

Note that $\tilde{g}^2_0 =$ inclusion and $\tilde{g}^2_1(V_2'\backslash Y) \subseteq K_2$. Define $g : (X\backslash Z) \times I \to X$ by $g(x, t) = \tilde{g}^2_t \circ \tilde{g}^1_t(x)$. Of course, g is stratum preserving, $g_0 =$ inclusion, $V = V_1' \cup V_2'$ is a neighborhood of Z in X and $g_t(V\backslash Z) \subseteq W$ for all $t \in I$. It remains to show that $g_1(V\backslash Z)$ is contained in a compact subset of $W\backslash Z$. To this end first note that

$$\begin{aligned} g_1(V\backslash Z) &= \tilde{g}^2_1\tilde{g}^1_1((V_1' \cup V_2')\backslash Z) \\ &= \tilde{g}^2_1\tilde{g}^1_1(V_1'\backslash Z) \cup \tilde{g}^2_1\tilde{g}^1_1(V_2'\backslash Z) \\ &\subseteq \tilde{g}^2_1(K_1\backslash Z) \cup \tilde{g}^2_1\tilde{g}^1_1(V_2'\backslash Z). \end{aligned}$$

Since

$$\begin{aligned} V_2'\backslash Z &= ((V_2' \cap V_1')\backslash Z) \cup ((V_2'\backslash V_1')\backslash Z) \\ &\subseteq (V_1'\backslash Z) \cup ((V_2'\backslash \operatorname{int} V_1')\backslash Z), \end{aligned}$$

it follows that

$$g_1(V\backslash Z) \subseteq \tilde{g}^2_1(K_1\backslash Z) \cup \tilde{g}^2_1\tilde{g}^1_1((V_2'\backslash \operatorname{int} V_1')\backslash Z).$$

Now let $K_3 = \tilde{g}^1_1(V_2'\backslash \operatorname{int} V_1')$ and note that K_3 is compact and $K_3 \cap (Z\backslash Y) = \emptyset$. Thus

$$g_1(V\backslash Z) \subseteq \tilde{g}^2_1(K_1\backslash Z) \cup \tilde{g}^2_1(K_3\backslash Z)$$

(it is helpful to recall that the $\tilde{g}^i$ are stratum preserving) and the proof is completed by the following

CLAIM. *If C is a compact subset of W and $C \cap (Z\backslash Y) = \emptyset$, then $\tilde{g}^2_1(C\backslash Z)$ is contained in a compact subset of $W\backslash Z$.*

Proof of Claim. Note that

$$\begin{aligned} C\backslash Z &= ((V_2' \cap C)\backslash Y) \cup ((W\backslash \operatorname{int} V_2') \cap C) \\ &\subseteq (V_2'\backslash Y) \cup ((W\backslash \operatorname{int} V_2')) \cap C). \end{aligned}$$

Of course, $\tilde{g}_1^2(V_2'\backslash Y) \subseteq K_2$. Moreover, $(W\backslash \operatorname{int} V_2') \cap C$ is compact and misses Z (putting it in the domain of $\tilde{g}_1^2$) from which it follows that $\tilde{g}_1^2((W\backslash \operatorname{int} V_2') \cap C)$ is compact.

6. Homotopy near a stratum. We are working towards a proof of Theorem 3.2 which will be completed in § 7. Recall that the main result of this paper is that a stratum A in a manifold stratified space X has an MSAF teardrop neighborhood (under compactness and dimension restrictions). Also recall that Theorem 3.2 is the special case that $\mathrm{cl}(A)\backslash A$ is a single point (or empty). This section establishes the preliminary homotopy structure on neighborhoods of such A without insisting that the strata of X be manifolds. This homotopy structure will be combined with sucking in the presence of manifold strata in § 7 in order to get MSAF teardrop neighborhoods.

The three propositions in this section require that the closure of A be stratified forward tame in X. The first two propositions homotopically relate neighborhoods of A to mapping cylinders of maps to A (in fact, the mapping cylinders are mapping cylinders of certain holink evaluations to A). The third proposition adds the compactly dominated holinks property and the normal fibrations property in order to get a weak lifting property of a deleted neighborhood of A over $A \times \mathbb{R}$. The compactly dominated local holinks property is used to get a better homotopical relation to the mapping cylinder of the holink evaluation. The normal fibrations property is used to show that the mapping cylinder is the mapping cylinder of a stratified fibration, and the lifting property follows.

NOTATION. If $c : A \to \mathbb{R}$ is a map, then we use the following notation:

$$A \times [c, +\infty) = \{(a, t) \in A \times \mathbb{R} \mid c(a) \leq t < +\infty\}.$$

Similar notation defines $A \times [c, +\infty]$, $A \times (-\infty, c]$, etc. Let X be a locally compact separable metric space with a stratification containing A as a stratum. Assume the Frontier Condition, that $B = \mathrm{cl}(A)$ is compact and that $B\backslash A = \{b_0\}$ is a single point so that there is a natural identification of B with the one point compactification $A \cup \{+\infty\} = A \cup \{b_0\}$. Let $\mathrm{holink_s}^+(X, B)$ be the subspace of $\mathrm{P_{nsp}}(X, B)$ as defined at the beginning of § 5. There are three holink evaluation maps $(\omega \mapsto \omega(0))$ that we will use:

$$\begin{aligned} q &: \mathrm{holink_s}(X, B) \to B \\ q^+ &: \mathrm{holink_s}^+(X, B) \to B \\ q_A &: \mathrm{holink_s}(X, A) \to A. \end{aligned}$$

Note that we have the following relations among the various holink spaces:

$$\begin{aligned} \mathrm{holink_s}(X, B) &= \mathrm{holink_s}(X, A) \cup q^{-1}(b_0) \\ \mathrm{holink_s}^+(X, B) &= \mathrm{holink_s}(X, A) \cup \{b_0\} \\ \mathrm{holink_s}(X, A) &= \mathrm{holink_s}(X, B) \cap \mathrm{holink_s}^+(X, B) \end{aligned}$$

(and the two unions are disjoint unions). Of course, the holink evaluations above agree on their common domain; that is,

$$q_A = q|\mathrm{holink_s}(X, A) = q^+|\mathrm{holink_s}(X, A).$$

For the evaluation map $q : \mathrm{holink_s}(X, B) \to B$, identify

$$\overset{\circ}{\mathrm{cyl}}(q) = [\mathrm{holink_s}(X, B) \times \mathbb{R}] \cup_{q \times \mathrm{id}_{\mathbb{R}}} B$$

and similarly for $\overset{\circ}{\mathrm{cyl}}(q^+)$ and $\overset{\circ}{\mathrm{cyl}}(q_A)$. There are three teardrop collapses on these open mapping cylinders:

$$Q : \overset{\circ}{\mathrm{cyl}}(q) \to B \times (-\infty, +\infty]$$

$$Q^+ : \overset{\circ}{\mathrm{cyl}}(q^+) \to B \times (-\infty, +\infty]$$

$$Q_A : \overset{\circ}{\mathrm{cyl}}(q_A) \to A \times (-\infty, +\infty]$$

which agree on their common domain $\overset{\circ}{\mathrm{cyl}}(q_A)$. The mapping cylinder $\overset{\circ}{\mathrm{cyl}}(q)$ has a natural stratification whose strata are either of the form $\mathrm{holink_s}(X, B)_i \times \mathbb{R}$ or X_i where X_i is a stratum of B. The stratifications of $\overset{\circ}{\mathrm{cyl}}(q^+)$ and $\overset{\circ}{\mathrm{cyl}}(q_A)$ are similar. Let d be a metric for X. Give $A \times \mathbb{R}$ the metric

$$d'((x_1, t_1), (x_2, t_2)) = \max\{d(x_1, x_2), |t_1 - t_2|\}.$$

In any metric space, $N(x, \epsilon)$ denotes the open ϵ-neighborhood of x.

REMARK. The results to follow are also valid in the simpler case $B \backslash A = \emptyset$, but we concentrate on the harder case.

PROPOSITION 6.1. *Suppose B is stratified forward tame in X. Then there exist a compact neighborhood Y of B in X and maps*

$$f : Y \to \overset{\circ}{\mathrm{cyl}}(q), \qquad g : \overset{\circ}{\mathrm{cyl}}(q) \to Y$$

together with homotopies

$$F : igf \simeq i : Y \to X, \qquad G : fg \simeq \mathrm{id} : \overset{\circ}{\mathrm{cyl}}(q) \to \overset{\circ}{\mathrm{cyl}}(q)$$

with $i : Y \to X$ the inclusion such that f, g, F, G are stratum preserving and rel B.

Proof[1]. Since B is stratified forward tame in X, there exist a compact neighborhood Y of B and a nearly stratum preserving homotopy $H : Y \times I \to X$ rel B with $H_0 =$ inclusion and $H_1(Y) = B$. Let $\hat{H} : Y \backslash B \to \mathrm{holink_s}(X, B)$ be the adjoint of

[1]The proof of this proposition is based on [**12**, Prop. 6.5] which in turn is based on [**11**, Prop. 9.13]. However, the current proof is simpler than either of the other two. This is due to the use of the refined shrinking argument given here. In fact, the proof in [**12**] should have incorporated such an argument in order to make the map g continuous. In other words, the current proof should be viewed as a replacement of the argument in [**12**, Prop. 6.5].

$\overset{\circ}{H}$, $\hat{H}(x)(t) = H(x, 1-t)$. Define $p : X \to (0, +\infty]$ by $p(x) = 1/d(x, B)$ and $f : Y \to \overset{\circ}{\mathrm{cyl}}(q)$ by

$$f(x) = \begin{cases} (\hat{H}(x), p(x)) \in \mathrm{holink_s}(X, B) \times (0, +\infty), & \text{if } x \in Y \setminus B, \\ x, & \text{if } x \in B. \end{cases}$$

Note that $Qf : Y \to B \times (-\infty, +\infty]$ is given by $x \mapsto (H(x, 1), p(x))$ which is clearly continuous. It follows from [**12**, 3.4] that f is continuous.

As preparation for the definition of g, choose a sequence $M_1 \leq M_2 \leq M_3 \leq \cdots$ of positive numbers converging to infinity such that if $n \in \{1, 2, 3, \ldots\}$, $x \in Y$ and $p(x) \geq n$, then $\mathrm{diam} H(\{x\} \times I) \leq 1/M_n$. Indeed, using the fact that $x \mapsto \mathrm{diam} H(\{x\} \times I)$ defines a continuous function $Y \to [0, \infty)$, one may let

$$1/M_n = \max\{\mathrm{diam} H(\{x\} \times I) \mid x \in Y \cap p^{-1}([n, +\infty])\}.$$

The required properties follow from the facts that $Y \cap p^{-1}([n, +\infty])\}$ is compact for each n, that

$$B = \bigcap_{n=1}^{\infty} p^{-1}([n, +\infty]),$$

and the homotopy H is rel B.

Now use the sequence just constructed to specify a certain subspace of $\overset{\circ}{\mathrm{cyl}}(q)$. If a level of the mapping cylinder is close to B, then we want only those holink elements in that level which are of small diameter (the smallness determined by the closeness of the level to B). Precisely,

$$\overset{\circ}{\mathrm{cyl}}(q)_\infty = \{(\omega, t) \in \overset{\circ}{\mathrm{cyl}}(q) \setminus B \mid \text{ for } n = 1, 2, 3, \ldots, \text{ and } t \geq n, \\ \mathrm{Im}(\omega) \subseteq Y \text{ and } \mathrm{diam}\, \omega \leq 1/M_n\} \cup B \subseteq \overset{\circ}{\mathrm{cyl}}(q).$$

We will show that there is a stratum preserving deformation R of $\overset{\circ}{\mathrm{cyl}}(q)$ into $\overset{\circ}{\mathrm{cyl}}(q)_\infty$ rel B. The idea is an extension of the idea behind Lemma 5.1: paths are to be shrunk along themselves towards their initial points, but now the amount of shrinking must increase near B in $\overset{\circ}{\mathrm{cyl}}(q)$. We first need a map to measure the amount of shrinking.

CLAIM. *There exists a map* $\rho : \mathrm{holink_s}(X, B) \times (-\infty, +\infty] \to I$ *such that*

(1) $\rho^{-1}(0) = \mathrm{holink_s}(X, B) \times \{+\infty\}$, *and*

(2) *if* $n \in \{1, 2, 3, \ldots\}$, $t \geq n$ *and* $\omega \in \mathrm{holink_s}(X, B)$, *then* $\mathrm{diam}\omega([0, \rho(\omega, t)]) \leq 1/M_n$.

Proof of Claim. This is an elementary partition of unity argument. Let $(\omega, t) \in \mathrm{holink_s}(X, B) \times \mathbb{R}$, let n_t be the largest positive integer such that $t > n_t$ (or let $n_t = 1$ if $t \leq 1$), and choose $c_{(\omega,t)} \in (0, 1]$ such that $\mathrm{diam}\omega([0, c_{(\omega,t)}]) < 1/M_{n_t+1}$. Moreover, if $t > 0$, require $c_{(\omega,t)} < 1/t$. Let $U(\omega, t)$ be a neighborhood of (ω, t) in $\mathrm{holink_s}(X, B) \times \mathbb{R}$ such that if $(\omega', t') \in U(\omega, t)$, then $\mathrm{diam}\omega'([0, c_{(\omega,t)}]) < 1/M_{n_t+1}$ and $t - 1 < t' < t + 1$.

There exists a locally finite refinement $\mathcal{U} = \{U_\alpha\}$ of

$$\{U(\omega, t) \mid (\omega, t) \in \mathrm{holink_s}(X, B) \times \mathbb{R}\}.$$

For each α, choose $(\omega_\alpha, t_\alpha) \in \mathrm{holink_s}(X, B) \times \mathbb{R}$ such that $U_\alpha \subseteq U(\omega_\alpha, t_\alpha)$. There exists a partition of unity $\{\sigma_\alpha : \mathrm{holink_s}(X, B) \times \mathbb{R} \to I\}$ subordinate to $\mathcal{U}$. Define

ρ by setting $\rho(\omega, +\infty) = 0$ and $\rho(\omega, t) = \sum_\alpha \sigma_\alpha(\omega, t) \cdot c_{(\omega_\alpha, t_\alpha)}$. The continuity of ρ follows from the condition $c_{(\omega,t)} < 1/t$. To verify the second property above, suppose $t \geq n$ and $\omega \in \text{holink}_s(X, B)$. Then $n \leq n_t + 1$ (from which it follows that $1/M_n \geq 1/M_{n_t+1}$.) Let $C = \max\{c_{(\omega_\alpha,t_\alpha)} \mid \sigma_\alpha(\omega, t) \neq 0\}$. Then $\rho(\omega, t) \leq \sum_\alpha \sigma_\alpha(\omega, t)C = C$ and $\text{diam}\omega([0, \rho(\omega, t)]) \leq \text{diam}\omega([0, C]) \leq 1/M_{n_t+1} \leq 1/M_n$. This completes the proof of the claim.

The map ρ induces a map $\tilde{\rho} : \text{holink}_s(X, B) \times (-\infty, +\infty] \times I \to X^I$ defined by $\tilde{\rho}(\omega, t, s)(u) = \omega((1-s)u + su\rho(\omega, t))$. Note that $\tilde{\rho}(\omega, t, 0) = \omega$ and $\tilde{\rho}(\omega, t, 1)(u) = \omega(u\rho(\omega, t))$. Think of $\tilde{\rho}$ as a shrinking homotopy. Use it to define a deformation $R : \overset{\circ}{\text{cyl}}(q) \times I \to \overset{\circ}{\text{cyl}}(q)$ by

$$\begin{cases} R((\omega, t), s) = (\tilde{\rho}(\omega, t, s), t), & \text{if } (\omega, t) \in \overset{\circ}{\text{cyl}}(q) \setminus B \\ R(x, s) = x, & \text{if } x \in B. \end{cases}$$

Note that:
(1) R_0 is the identity,
(2) $R_1(\overset{\circ}{\text{cyl}}(q)) \subseteq \overset{\circ}{\text{cyl}}(q)_\infty$,
(3) $R_s(\overset{\circ}{\text{cyl}}(q)_\infty) \subseteq \overset{\circ}{\text{cyl}}(q)_\infty$ for all $s \in I$,
(4) R is stratum preserving, fiber preserving over $(-\infty, +\infty]$ and rel B.

The deformation R shows that the inclusion $\overset{\circ}{\text{cyl}}(q)_\infty \to \overset{\circ}{\text{cyl}}(q)$ is a homotopy equivalence.

Define $g : \overset{\circ}{\text{cyl}}(q) \to Y$ by

$$\begin{cases} g(\omega, t) = \tilde{\rho}(\omega, t, 1)(1) = \omega(\rho(\omega, t)), & \text{if } (\omega, t) \in \overset{\circ}{\text{cyl}}(q) \setminus B \\ g(x) = x, & \text{if } x \in B. \end{cases}$$

To see that g is continuous at $x \in B$, suppose $(\omega_n, t_n) \in \overset{\circ}{\text{cyl}}(q) \setminus B$ with $\omega_n(0) \to x$ and $t_n \to \infty$. Since $\text{diam}\omega_n([0, \rho(\omega_n, t_n)]) \to 0$ and $\omega_n(0) \to x$, it follows that $\omega_n(\rho(\omega_n, t_n)) \to x$. In other words, $g = e \circ R_1$ where $e : \overset{\circ}{\text{cyl}}(q)_\infty \to Y$ is defined by

$$\begin{cases} e(\omega, t) = \omega(1), & \text{if } (\omega, t) \in \overset{\circ}{\text{cyl}}(q) \setminus B \\ e(x) = x, & \text{if } x \in B. \end{cases}$$

The point is that e would not be continuous at points of B if it were defined on all of $\overset{\circ}{\text{cyl}}(q)$ instead of just the subspace $\overset{\circ}{\text{cyl}}(q)_\infty$.

Define $F : Y \times I \to X$ by

$$F(x, t) = H(x, (1-t)(1 - \rho(\hat{H}(x), p(x)))).$$

Clearly, $F : igf \simeq i$.

We will define $G : \overset{\circ}{\text{cyl}}(q) \times I \to \overset{\circ}{\text{cyl}}(q)$ in two stages corresponding to $I = [0, 1/2] \cup [1/2, 1]$. We first need an auxiliary map $\gamma : \text{holink}_s(X, B) \times \mathbb{R} \times [0, 1/2] \to X^I$ defined by

$$\gamma(\omega, t, s)(u) = \begin{cases} \hat{H}(\omega(\rho(\omega, t)(1 - (1-u)2s)))\left(\frac{u}{1-(1-u)2s}\right), & \text{if } (s, u) \neq (1/2, 0) \\ \omega(0), & \text{if } (s, u) = (1/2, 0). \end{cases}$$

Here is a way to think about γ. Fix $(\omega, t) \in \text{holink}_s(X, B) \times \mathbb{R}$. Then $\gamma(\omega, t, \cdot)(\cdot) : [0, 1/2] \times I \to X$ can be described as a composition of three maps. The first maps the rectangle $[0, 1/2] \times I$ to the square $I \times I$ via $(s, u) \mapsto (2s, u)$. The second maps the square $I \times I$ to the triangle $T = \{(s, u) \mid 0 \le s \le 1, 0 \le u \le 1 - s\}$ via $(s, u) \mapsto ((1 - u)s, u)$. The third maps the triangle T into X via

$$(s, u) \mapsto \begin{cases} \hat{H}(\omega(\rho(\omega, t)(1 - s)))(u/1 - s), & \text{if } s \neq 1 \\ \omega(0), & \text{if } s = 1. \end{cases}$$

Note that:

(1) $\gamma(\omega, t, 0)(u) = \hat{H}(\omega(\rho(\omega, t))(u)$,
(2) $\gamma(\omega, t, 1/2)(u) = \hat{H}(\omega(\rho(\omega, t)u)(1) = \omega(\rho(\omega, t)u)$,
(3) $\gamma(\omega, t, s)(1) = \hat{H}(\omega(\rho(\omega, t))(1) = \omega(\rho(\omega, t))$,
(4) $\gamma(\omega, t, s)(0) = \hat{H}(\omega(\rho(\omega, t)(1 - 2s))(0) \in B$,
(5) $\gamma(\omega, t, 1/2)(0) = \hat{H}(\omega(0))(0) \in B$.

Now define $G : \overset{\circ}{\text{cyl}}(q) \times [0, 1/2] \to \overset{\circ}{\text{cyl}}(q)$ by

$$\begin{cases} G((\omega, t), s) = (\gamma(\omega, t, s), (1 - 2s)p(\omega(\rho(\omega, t))) + 2st), \\ G(x, s) = x, \end{cases}$$

$$\text{if } \begin{cases} (\omega, t, s) \in \text{holink}_s(X, B) \times \mathbb{R} \times [0, 1/2], \\ (x, s) \in B \times [0, 1/2]. \end{cases}$$

Note that $G_0 = fg$ and $G((\omega, t), 1/2) = (\gamma(\omega, t, 1/2), t)$ where $\gamma(\omega, t, 1/2)(u) = \omega(\rho(\omega, t)u)$.

In order to finish the definition of G, define another auxiliary map

$$\beta : \text{holink}_s(X, B) \times \mathbb{R} \times [1/2, 1] \to X^I$$

by

$$\beta(\omega, t, 1/2)(u) = \omega(\rho(\omega, t)u(2 - 2s) + (2s - 1)u).$$

Note that $\beta(\omega, t, 1/2) = \gamma(\omega, t, 1/2)$ and $\beta(\omega, t, 1) = \omega$.

Now define $G : \overset{\circ}{\text{cyl}}(q) \times [1/2, 1] \to \overset{\circ}{\text{cyl}}(q)$ by

$$\begin{cases} G((\omega, t), s) = (\beta(\omega, t, s), t), & \text{if } (\omega, t, s) \in \text{holink}_s(X, B) \times \mathbb{R} \times [1/2, 1] \\ G(x, s) = x, & \text{if } (x, s) \in B \times [1/2, 1]. \end{cases}$$

Note that the two definitions of $G_{1/2}$ agree so that we have defined a stratum preserving homotopy $G : fg \simeq \text{id rel } B$.

The next proposition is a refinement of the previous one. The focus changes from the mapping cylinder $\overset{\circ}{\text{cyl}}(q)$ to the mapping cylinder $\overset{\circ}{\text{cyl}}(q^+)$ in order to have more control near $\{b_0\}$.

PROPOSITION 6.2. *Suppose B is stratified forward tame in X. Then there exist a compact neighborhood Y of B in X, a neighborhood $\tilde{Y}$ of A in X such that $B \subseteq \tilde{Y} \subseteq Y$ and maps*

$$\tilde{f} : Y \to \overset{\circ}{\text{cyl}}(q^+), \qquad \tilde{g} : \overset{\circ}{\text{cyl}}(q^+) \to Y$$

together with homotopies

$$\tilde{F} : i\tilde{g}\tilde{f} \simeq i : Y \to X, \qquad \tilde{G} : \tilde{f}\tilde{g} \simeq \text{id} : \overset{\circ}{\text{cyl}}(q^+) \to \overset{\circ}{\text{cyl}}(q^+)$$

with $i : Y \to X$ *the inclusion such that*

(1) $\tilde{f}|\tilde{Y}, \tilde{g}, \tilde{F}|\tilde{Y} \times I$ *are stratum preserving and rel B,*

(2) $\tilde{G}|[\overset{\circ}{\mathrm{cyl}}(q^+)\backslash(\{b_0\} \times \mathbb{R})] \times I$ *is stratum preserving,*

$$\tilde{G}(\{b_0\} \times (-\infty, +\infty] \times I) = \{b_0\} \times (-\infty, +\infty]$$

and $\tilde{G}$ *is rel B.*

Proof. Let $f : Y \to \overset{\circ}{\mathrm{cyl}}(q)$ be as in Proposition 6.1. Let

$$\tilde{Y} = Y \backslash f^{-1}(q^{-1}(b_0) \times \mathbb{R}).$$

Clearly, $\tilde{Y}$ is a neighborhood of A (but not of B) in X and $B \subseteq \tilde{Y} \subseteq Y$.

In order to define $\tilde{f}$ we first define auxiliary maps. Let $H : Y \times I \to X$ be the deformation from the proof of Proposition 6.1 together with the adjoint $\hat{H} : Y\backslash B \to \mathrm{holink}_s(X, B)$. Choose a map $\delta : B \to I$ such that $\delta^{-1}(0) = \{b_0\}$. Since $f^{-1}(q^{-1}(b_0) \times \mathbb{R}) = H_1^{-1}(b_0)\backslash\{b_0\}$, δ may be chosen so that the δ-neighborhood about A in Y is contained in $\tilde{Y}$; that is,

$$\bigcup_{a \in A} \{y \in Y \mid d(y, a) < \delta(a)\} \subseteq \tilde{Y}.$$

Define

$$\alpha : \mathrm{holink}_s(X, B) \to \mathrm{holink}_s{}^+(X, B)$$

by setting $\alpha(\omega)(t) = \omega(t \cdot \delta(\omega(0))$ for every $\omega \in \mathrm{holink}_s(X, B)$ and $t \in I$. Since the function spaces involve the compact space I mapping to the metric space X, the topology is that of uniform convergence; hence, it is easy to see that α is continuous. (Note that the image of α need not lie in $\mathrm{holink}_s{}^\delta(X, B)$ as defined in § 5.) Even though α need not be stratum preserving (because $\alpha^{-1}(b_0) = q^{-1}(b_0)$ and $q^{-1}(b_0)$ might meet several strata), it is stratum preserving on the complement of $q^{-1}(b_0)$. Moreover, α is fiber preserving over B; that is, $q^+\alpha = q$. In particular, there is an induced map

$$\hat{\alpha} : \overset{\circ}{\mathrm{cyl}}(q) \to \overset{\circ}{\mathrm{cyl}}(q^+)$$

defined by

$$(\omega, t) \mapsto (\alpha(\omega), t) \text{ for } (\omega, t) \in \mathrm{holink}_s(X, B) \text{ and } b \mapsto b \text{ for } b \in B.$$

(The continuity of $\hat{\alpha}$ follows from the continuity criteria [**12**] because $Q^+\hat{\alpha} = Q$.) Note that

$$\hat{\alpha}(q^{-1}(b_0) \times (-\infty, +\infty]) = \{b_0\} \times (-\infty, +\infty].$$

Now define $\tilde{f}$ to be the composition

$$\tilde{f} : Y \xrightarrow{f} \overset{\circ}{\mathrm{cyl}}(q) \xrightarrow{\bar{\alpha}} \overset{\circ}{\mathrm{cyl}}(q^+).$$

Note that $\tilde{f}$ is stratum preserving on $\tilde{Y}$ and $\tilde{f}(Y\backslash\tilde{Y}) = \{b_0\}$.

We also note, for use in the proof of Proposition 6.3 below that $Q^+\tilde{f} = Qf$ (because $Q^+\hat{\alpha} = Q$).

In order to define the remaining maps, we need to make some modifications in the proof of Proposition 6.1. In particular, let $\rho : \mathrm{holink_s}(X, B) \times (-\infty, +\infty] \to I$ and $\{M_n\}_{n=1}^{\infty}$ be as in the proof of 6.1. By another elementary partition of unity argument, there exists a map

$$\sigma : \mathrm{holink_s}^{+}(X, B) \to I$$

such that:

(1) $\sigma^{-1}(0) = \{b_0\}$, and

(2) $\mathrm{diam}\omega([0, \sigma(\omega)]) \leq \delta(\omega(0))$ for all $\omega \in \mathrm{holink_s}^{+}(X, B)$.

Define $\hat{\rho} : \mathrm{holink_s}^{+}(X, B) \times (-\infty, +\infty] \to I$ by

$$\hat{\rho}(\omega, t) = \begin{cases} \sigma(\omega) \cdot \rho(\omega, t), & \text{if } (\omega, t) \in \mathrm{holink_s}(X, A) \times (\infty, +\infty] \\ 0, & \text{if } (\omega, t) \in \{b_0\} \times (-\infty, +\infty]. \end{cases}$$

Note that:

(1) $\hat{\rho}^{-1}(0) = \mathrm{holink_s}^{+}(X, B) \times \{+\infty\} \cup \{b_0\} \times (-\infty, +\infty]$, and

(2) if $n \in \{1, 2, 3, \ldots\}$, $t \geq n$ and $\omega \in \mathrm{holink_s}^{+}(X, B)$, then

$$\mathrm{diam}\omega([0, \hat{\rho}(\omega, t)]) \leq \min\{1/M_n, \delta(\omega(0))\}.$$

The map $\hat{\rho}$ induces a map $\rho^* : \mathrm{holink_s}^{+}(X, B) \times (-\infty, +\infty] \times I \to X^I$ defined by $\rho^*(\omega, t, s)(u) = \omega((1 - s)u + su\hat{\rho}(\omega, t))$. Note that $\rho^*(\omega, t, 0) = \omega$ and $\rho^*(\omega, t, 1)(u) = \omega(u\hat{\rho}(\omega, t))$. Use ρ^* to define a deformation $\hat{R} : \mathring{\mathrm{cyl}}(q^+) \times I \to \mathring{\mathrm{cyl}}(q^+)$ by

$$\begin{cases} \hat{R}((\omega, t), s) = (\rho^*(\omega, t, s), t), & \text{if } (\omega, t) \in \mathring{\mathrm{cyl}}(q^+) \backslash B \\ \hat{R}(x, s) = x, & \text{if } x \in B. \end{cases}$$

Analogous to $\mathring{\mathrm{cyl}}(q)_\infty$, we need a subspace of $\mathring{\mathrm{cyl}}(q^+)$ that not only controls diameters of holink elements in mapping cylinder levels close to B, but also controls diameters of all holink elements (regardless of mapping cylinder level) whose initial points are close to b_0. Define

$$\mathring{\mathrm{cyl}}(q^+)_\infty = \{(\omega, t) \in \mathring{\mathrm{cyl}}(q)_\infty \cap \mathring{\mathrm{cyl}}(q^+) \backslash B \mid \mathrm{diam}\omega \leq \delta(\omega(0))\} \cup \{b_0\} \times \mathbb{R} \cup B \subseteq \mathring{\mathrm{cyl}}(q^+).$$

Note that:

(1) $\hat{R}_0$ is the identity,

(2) $\hat{R}_1(\mathring{\mathrm{cyl}}(q^+)) \subseteq \mathring{\mathrm{cyl}}(q^+)_\infty$,

(3) $\hat{R}_s(\mathring{\mathrm{cyl}}(q^+)_\infty) \subseteq \mathring{\mathrm{cyl}}(q^+)_\infty$ for all $s \in I$,

(4) $\hat{R}$ is stratum preserving, fiber preserving over $(-\infty, +\infty]$ and rel $(\{b_0\} \times \mathbb{R}) \cup B$.

The deformation $\hat{R}$ shows that the inclusion $\mathring{\mathrm{cyl}}(q^+)_\infty \to \mathring{\mathrm{cyl}}(q^+)$ is a homotopy equivalence.

Define $\hat{e} : \mathring{\mathrm{cyl}}(q^+)_\infty \to Y$ by

$$\begin{cases} \hat{e}(\omega, t) = \omega(1), & \text{if } (\omega, t) \in \mathring{\mathrm{cyl}}(q^+)_\infty \backslash B \\ \hat{e}(x) = x, & \text{if } x \in B. \end{cases}$$

The point is that $\hat{e}$ would not be continuous at points of B if it were defined on all of $\mathring{\text{cyl}}(q^+)$ instead of just the subspace $\mathring{\text{cyl}}(q^+)_\infty$.

Now $\tilde{g} : \mathring{\text{cyl}}(q^+) \to Y$ can be defined by $\tilde{g} = \hat{e} \circ \hat{R}_1$.

The definitions of the homotopies $\tilde{F}$ and $\tilde{G}$ are similar enough to the definitions of F and G in 6.1 that the details are omitted.

The next proposition establishes a type of fibration property for a neighborhood of A. It is the main homotopy information used in the next section.

PROPOSITION 6.3. *Suppose, in addition to the standing assumptions of this section, that X is a homotopically stratified space with only finitely many strata, that the strata are ANRs, and that X satisfies the compactly dominated local holinks property. For every sequence $\{\epsilon_i\}_{i=1}^{\infty}$ of positive numbers there exist a neighborhood N of A in $X \setminus \{b_0\}$ and a proper map $p : N \to A \times (-\infty, +\infty]$ such that*

(1) $p^{-1}(A \times \{+\infty\}) = A$ *and* $p| : A \to A \times \{+\infty\}$ *is the identity,*
(2) $p^{-1}(A \times (0, +\infty))$ *is open in* X,
(3) p *has the following lifting property:*

given any space Z and any commuting diagram of maps

$$\begin{array}{ccc} Z & \xrightarrow{h} & p^{-1}(A \times (0, +\infty)) \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p|} \\ Z \times I & \xrightarrow{H} & A \times (0, +\infty) \end{array}$$

there exists a stratum preserving homotopy $\tilde{H} : Z \times I \to N$ such that $\tilde{H}(z, 0) = h(z)$ for each $z \in Z$ and $p\tilde{H}$ is $\mathcal{E}$-close to H where $\mathcal{E}$ is the collection of open subsets of $A \times \mathbb{R}$ given by

$$\mathcal{E} = \{N(x, \epsilon_i) \times (i - 1, i + 2) \mid x \in A \text{ and } i = 1, 2, 3, \ldots\}.$$

Proof. It follows from [7, Theorem 6.3] that B is stratified forward tame in X so that the previous propositions apply. Let Y, $\tilde{Y}, \tilde{f}, \tilde{g}, \tilde{F}, \tilde{G}$ be as in Proposition 6.2. Let $C \subseteq \text{holink}_s{}^+(X, B)$ and

$$d : \text{holink}_s{}^+(X, B) \times I \to \text{holink}_s{}^+(X, B)$$

be given by Proposition 5.4 (we are assuming in the hypothesis all the standing assumptions on X in § 5). Reverse the parameter by setting $D_s = d_{1-s}$. Thus, $D_1 = \text{id}$ and $D_0(\text{holink}_s{}^+(X, B)) \subseteq C$. Define $\hat{D} : \mathring{\text{cyl}}(q^+) \times I \to \mathring{\text{cyl}}(q^+)$ by

$$\hat{D}_s = \begin{cases} D_s \times \text{id}_{\mathbb{R}} & \text{on holink}_s{}^+(X, B) \times \mathbb{R} \\ \text{id} & \text{on } B. \end{cases}$$

Define $g' : \mathring{\text{cyl}}(q^+) \to Y$ by $g' = \tilde{g}\hat{D}_0$. Define $F' : Y \times I \to X$ by

$$F'_s = \begin{cases} i\tilde{g}\hat{D}_{2s}\tilde{f}, & \text{if } 0 \le s \le 1/2 \\ \tilde{F}_{2s-1}, & \text{if } 1/2 \le s \le 1, \end{cases}$$

where $i : Y \to X$ is the inclusion. Note that $F' : ig'\tilde{f} \simeq i$. Define $G' : \overset{\circ}{\mathrm{cyl}}(q^+) \times I \to \overset{\circ}{\mathrm{cyl}}(q^+)$ by

$$G'_s = \begin{cases} \tilde{G}_{2s}\hat{D}_0, & \text{if } 0 \le s \le 1/2 \\ \hat{D}_{2s-1}, & \text{if } 1/2 \le s \le 1. \end{cases}$$

Note that $G' : \tilde{f}g' \simeq \mathrm{id}$.

Choose a proper map $n_0 : A \to [0, +\infty)$ such that

I. $F'((Q^+\tilde{f})^{-1}(A \times [n_0, +\infty)) \times I) \subseteq \tilde{Y}$, and

II. $Q^+G'((Q^+)^{-1}(A \times [n_0, +\infty)) \times I) \subseteq A \times [0, +\infty)$.

Why does such an n_0 exist? Since $F'_t| : A \to X$ is the inclusion for all $t \in I$ and $\tilde{Y}$ is a neighborhood of A in X, there exists an open neighborhood U of A in $\tilde{Y}$ such that $F'(U \times I) \subseteq \tilde{Y}$. Since $\tilde{Y}$ is a neighborhood of A in Y, we may assume that U is open in Y and in X. Since Y is compact, both $K = Y \setminus U$ and $Qf(K)$ are compact. Moreover, $Qf(K) \cap (A \times \{+\infty\}) = \emptyset$. Thus, there is a proper map $n_0 : A \to [0, +\infty)$ such that $Qf(K) \cap (A \times [n_0, +\infty)) = \emptyset$, from which it follows that $(Qf)^{-1}(A \times [n_0, +\infty)) \subseteq U$. Since $Q^+\tilde{f} = Qf$ (as was pointed out in the proof of Proposition 6.2 above), it follows that

$$(Q^+\tilde{f})^{-1}(A \times [n_0, +\infty)) \subseteq U.$$

Property I now follows. For Property II, since $G'_t| : B \to \overset{\circ}{\mathrm{cyl}}(q^+)$ is the inclusion for all $t \in I$ and $(Q^+)^{-1}(A \times [0, +\infty)) = Q_A^{-1}(A \times [0, +\infty))$ is a neighborhood of A in $\overset{\circ}{\mathrm{cyl}}(q^+)$ and in $\overset{\circ}{\mathrm{cyl}}(q_A)$, there exists a neighborhood V of A in $\overset{\circ}{\mathrm{cyl}}(q_A)$ such that $G'(V \times I) \subseteq (Q^+)^{-1}(A \times [0, +\infty))$. Now we just need to make sure that $Q_A^{-1}(A \times [n_0, +\infty)) \subseteq V$. This can be done if $Q_A(V)$ is a neighborhood of $A \times \{+\infty\}$ in $A \times (-\infty, +\infty]$. It is indeed such a neighborhood, as follows from [**12**, Lemma 3.6].

For $i \ge 1$, choose proper maps $n_i : A \to [i, +\infty)$ inductively such that $n_i \ge n_{i-1}$ and the following five properties hold:

(1) $Q^+\tilde{f}F'((Q^+\tilde{f})^{-1}(A \times [n_0, n_{i-1}]) \times I) \subseteq A \times [0, n_i]$,

(2) $Q^+G'((Q^+)^{-1}(A \times [n_0, n_{i-1}]) \times I) \subseteq A \times [0, n_i]$,

(3) $Q^+\tilde{f}F'((Q^+\tilde{f})^{-1}(A \times [n_i, +\infty)) \times I) \subseteq A \times [n_{i-1}, +\infty)$,

(4) $Q^+G'((Q^+)^{-1}(A \times [n_i, +\infty)) \times I) \subseteq A \times [n_{i-1}, +\infty)$.

(5) The tracks of points over $A \times [n_i, +\infty)$ under the homotopies $\mathrm{proj}_A Q^+\tilde{f}F'$ and $\mathrm{proj}_A Q^+G'$ have diameter less than ϵ_i in A.

Assuming $i \ge 1$ and n_{i-1} has been defined, we will show, for each of the five properties, that a map n_i can be defined satisfying that property. Then a proper map bigger than each of those will satisfy all of the properties.

Property (1): Let $W = (Q^+\tilde{f})^{-1}(A \times [n_0, n_{i-1}])$. We use the notation

$$B \times [n_0, n_{i-1}] = A \times [n_0, n_{i-1}] \cup \{(b_0, +\infty)\} \subseteq B \times (-\infty, +\infty].$$

Since $A \times [n_0, n_{i-1}] \subseteq B \times [n_0, n_{i-1}]$, the map $Q^+\tilde{f}$ is $Qf \circ \mathrm{inclusion} : \tilde{Y} \to B \times (-\infty, +\infty]$ (as pointed out above), and $(Qf)^{-1}(B \times [n_0, n_{i-1}]) \subseteq Y$, it follows that W is contained in a compact subset K of Y such that $K \cap A = \emptyset$. Then $F'(K \times I)$ is also compact and $F'(K \times I) \cap A = \emptyset$ (however $F'(K \times I)$ need not be contained in Y). Moreover, $Q^+\tilde{f}(F'(K \times I) \cap \tilde{Y}) \subseteq Qf(F'(K \times I) \cap Y)$. Since $Qf(F'(K \times I) \cap Y)$ is compact and misses $A \times \{+\infty\}$, there exists a map $n_i : A \to [i, +\infty)$ such that $Qf(F'(K \times I)) \cap Y) \subseteq A \times (-\infty, n_i)$. It follows that $Q^+\tilde{f}F'(W \times I) \subseteq A \times [0, n_i]$ as required.

Property (2): Let $W = (Q^+)^{-1}(A \times [n_0, n_{i-1}])$. Recall from the beginning of the proof that C is a certain compact subset of $\text{holink}_s{}^+(X, B)$ containing b_0. We use the notation

$$C \times [n_0, n_{i-1}] = \{(z, t) \in C \times \mathbb{R} \mid Q^+(z, t) \in A \times [n_0, n_{i-1}]\} \cup \{b_0\} \subseteq \overset{\circ}{\text{cyl}}(q^+).$$

Note that C is a compact subset of $\overset{\circ}{\text{cyl}}(q^+)$, that

$$\hat{D}_0(W) = (D_0 \times id_{\mathbb{R}})(W) \subseteq C \times [n_0, n_{i-1}],$$

and that $\hat{D}_0(W)$ misses B. In particular, $G'(W \times [0, 1/2]) \subseteq \tilde{G}(C \times [n_0, n_{i-1}] \times I)$ which is compact and misses A. Thus, $Q^+G'(W \times [0, 1/2])$ is contained in a compact subset of $B \times (-\infty, +\infty]$ which misses $B \times \{+\infty\}$. On the other hand, $G'(W \times [1/2, 1]) = \hat{D}(W \times I) \subseteq W$ (because $\hat{D}$ is fiber preserving over $\mathbb{R}$). Thus, $Q^+G'(W \times [1/2, 1]) \subseteq A \times [n_0, n_{i-1}]$. It follows that a map n_i exists with the property that $Q^+G'(W \times I) \subseteq A \times (-\infty, n_i]$. Property II of n_0 supplies the remaining detail.

Property (3): The proof of this follows the proof of Property I of n_0 given above, with n_i and $(Q^+\tilde{f})^{-1}(A \times [n_{i-1}, +\infty))$ playing the roles of n_0 and $\tilde{Y}$, respectively.

Property (4): The proof of this follows the proof of Property II of n_0 given above, with n_{i-1} and n_i playing the roles of 0 and n_0, respectively.

Property (5): This uses the fact that F' and G' are rel B.

Let $N = \tilde{Y}$ and let $\gamma : A \times \mathbb{R} \to A \times \mathbb{R}$ be a homeomorphism that is fiber preserving over A, supported on $A \times [-1, +\infty)$ and takes the graph of n_i onto the horizontal line $A \times \{i\}$ for $i = 1, 2, 3, \ldots$. Then $p = \gamma Q^+\tilde{f}|N$ satisfies the conclusions of the proposition. In particular, $p^{-1}(A \times (0, +\infty)) = (Q^+\tilde{f})^{-1}(A \times (n_0, +\infty))$ is open in X because $(Q^+\tilde{f})^{-1}(A \times (n_0, +\infty)) = (Qf)^{-1}(A \times (n_0, +\infty)) \cap \tilde{Y} = (Qf)^{-1}(A \times (n_0, +\infty))$ which is open in U, hence, is open in X.

The remainder of the proof consists of providing more details on the lifting property. To this end suppose we are given a lifting problem

$$\begin{array}{ccc} Z & \xrightarrow{\ h\ } & p^{-1}(A \times (0, +\infty)) \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p|} \\ Z \times I & \xrightarrow{\ H\ } & A \times (0, +\infty) \end{array}$$

We will define a stratified $\mathcal{E}$-solution by constructing and piecing together two homotopies. The first homotopy $\hat{H}$ will be an $\mathcal{E}$-lift but it will not have h as the initial level. The second homotopy will correct this. To begin note that $p|$ fits into the following commuting diagram

$$\begin{array}{ccccc} p^{-1}(A \times (0, +\infty)) & \xrightarrow{\ \tilde{f}\ } & \overset{\circ}{\text{cyl}}(q^+)\backslash(B \cup (\{b_0\} \times \mathbb{R})) & \xrightarrow{\ =\ } & \text{holink}_s(X, A) \times \mathbb{R} \\ {\scriptstyle p|}\downarrow & & & & \downarrow{\scriptstyle Q^+| = q_A \times \text{id}_{\mathbb{R}}} \\ A \times (0, +\infty) & \xrightarrow{\ \gamma^{-1}\ } & A \times (0, +\infty) & \xrightarrow{\text{inclusion}} & A \times \mathbb{R} \end{array}$$

We now use the assumption that X is a homotopically stratified metric space with finitely many strata to conclude by [**7**, Corollary 6.2] that $q_A : \text{holink}_s(X, A) \to A$ is a stratified fibration. It follows that there exists a stratum preserving homotopy

$H^* : Z \times I \to \text{holink}_s(X, A) \times (0, +\infty)$ such that $H_0^* = \tilde{f}h$ and $\gamma Q^+ H^* = H$. Let $\hat{H} = g' H^* : Z \times I \to N$.

We will now show that $p\hat{H}$ and H are $\mathcal{E}$-close. Let $(z, t) \in Z \times I$ and let $i \geq 0$ be such that $H(z, t) \in A \times (i, i+1]$. Then $\gamma^{-1} H(z, t) \in A \times (n_i, n_{i+1}]$ and so $H^*(z, t) \in (Q^+)^{-1}(A \times (n_i, n_{i+1}])$. From (2) and (4) it follows that $Q^+ G_0' H^*(z, t) \in A \times [n_{i-1}, n_{i+2}]$. Since $G_0' = \tilde{f} g'$ we have

$$p\hat{H}(z, t) = \gamma Q^+ \tilde{f} g' H^*(z, t) \in A \times [i-1, i+2].$$

By (5) it follows that $p\hat{H}(z, t) \in N(x, \epsilon_i) \times [i-1, i+2]$ where $x = \text{proj}_A H(z, t)$. Since $H(z, t) \in N(x, \epsilon_i) \times (i, i+1]$, it follows that $p\hat{H}$ and H are $\mathcal{E}$-close.

Now we have to make up for the fact that $\hat{H}_0$ need not equal h. Since $\hat{H}_0 = g' \tilde{f} h$ and $F' : \tilde{i} g' \tilde{f} \simeq \tilde{i}$ we can, as a first approximation, define $\tilde{H} : Z \times [-1, 1] \to N$ by

$$\tilde{H}(z, t) = \begin{cases} F'(h(z), -t) & \text{if } -1 \leq t \leq 0 \\ \hat{H}(z, t) & \text{if } 0 \leq t \leq 1. \end{cases}$$

We will now see that the tracks $pF'(h(z) \times I)$ are $\mathcal{E}$-small. From this it follows from a standard argument that $\tilde{H}$ can be reparameterized by traveling along $[-1, 0]$ quite rapidly. The resulting homotopy will be our desired solution. So let $z \in Z$ and let $i \geq 0$ be such that $\gamma Q^+ \tilde{f} h(z) \in A \times (i, i+1]$. Then $Q^+ \tilde{f} h(z) \in A \times (n_i, n_{i+1}]$. From (1) and (3) it follows that $Q^+ \tilde{f} F'(h(z) \times I) \subseteq A \times [n_{i-1}, n_{i+2}]$. Thus, $pF'(h(z) \times I) \subseteq A \times [i-1, i+2]$. From (5) it follows that $\text{proj}_A Q^+ \tilde{f} F'(h(z) \times I)$ has diameter less than ϵ_i. Thus, $pF'(h(z) \times I) \subseteq N(ph(z), \epsilon_i) \times [i-1, i+2]$ as desired.

7. Completion of the proof of the main theorem. We begin by fully stating the theorem that will be proved in this section.

THEOREM 7.1. *Let X be a manifold stratified space with a stratum A satisfying:*
(1) *A has compact closure* $\text{cl}(A) = B$ *in X,*
(2) $\dim A \geq 5$*,*
(3) *$B \backslash A$ consists of a single point b_0.*
Then A has an MSAF teardrop neighborhood in X.

If, in the statement of Theorem 7.1, condition (3) is replaced by "$B \backslash A = \emptyset$," and condition (2) is replaced by "if X_i is a stratum of X such that $\text{cl}(X_i) \cap A \neq \emptyset$, then $\dim X_i \geq 5$," then the resulting statement is also true and its proof is simpler than the proof of Theorem 7.1. Therefore, we make no further mention of its proof. These two results (Theorem 7.1 and its simplification) together make up Theorem 3.2. Thus, the proof of Theorem 3.2 is completed by this section. In turn, recall that the Main Theorem of this paper follows from Theorem 3.2, as was established by Corollary 3.4.

A word of explanation might be useful. If A were compact, then the epsilonics in this section (and, hence, in the rest of the paper) would be considerably easier. For non-compact A, our assumption that $\text{cl}(A)$ is a compact union of strata in a manifold stratified space implies that A is a manifold (with $\dim A \geq 5$) having finitely many tame ends in the sense of Siebenmann. Even though the ends of A might not be collarable, it is true that they have periodic structure (namely, they are the infinite cyclic cover of a MAF over the circle). It is this periodic structure on the ends of A, which is one of the main results in [**11**], that allows us to deal with the non-compactness of A. I do not know if the Main Theorem of this paper would be true without this assumption.

We will assume the hypothesis and notation of Theorem 7.1 for the remainder of this section. Since B is a manifold stratified space with two strata A and $\{b_0\}$, and $\dim(A) \geq 5$, it follows from **[11]** or **[12]** that b_0 has a neighborhood in B with MAF teardrop structure. That is, there is a map $\pi : B \to [0, +\infty]$ such that $\pi^{-1}(+\infty) = b_0$ and π is a manifold approximate fibration over $(0, +\infty)$. We also use π to denote its restriction $\pi : A \to [0, +\infty)$. The hypotheses of Proposition 6.3 are satisfied.

If $b > 0$ and $\epsilon = \{\epsilon_i\}_{i=1}^{\infty}$ is a sequence of positive numbers, define a collection of open subsets of $A \times \mathbb{R}$ covering $A \times [1, +\infty)$ by

$$\mathcal{U}_{b,\epsilon} = \{N(x, \epsilon_i) \times (t - b, t + b) \mid (x, t) \in A \times [i - 1, i + 1] \text{ and } i = 1, 2, 3, \ldots\}.$$

Before giving the proof of Theorem 7.1, we need the following embellishment of the Stratified Sucking Theorem 4.2. This is the place where the manifold condition on the strata is used.

LEMMA 7.2. *For every $b > 0$ and sequence $\epsilon = \{\epsilon_i\}_{i=1}^{\infty}$ of positive numbers, there exist $c > 0$ and a sequence $\delta = \{\delta_i\}_{i=1}^{\infty}$ of positive numbers such that if $M \subseteq X, j : M \to A \times \mathbb{R}$ is a proper stratified $\mathcal{U}_{c,\delta}$-fibration over $A \times (1/2, +\infty)$, $j^{-1}(A \times (0, +\infty))$ is an open subspace of X, and the strata of M are of dimension greater than or equal to 5, then j is properly $\mathcal{U}_{b,\epsilon}$-homotopic rel $j^{-1}(A \times (-\infty, 0])$ to a map $j' : M \to A \times \mathbb{R}$ with j' a stratified approximate fibration over $A \times (1, +\infty)$.*

Proof. There are two aspects of this lemma that make it different from Theorem 4.2. The first is the local-and-relative aspect (approximate fibration properties *over* $A \times (1/2, +\infty)$ and the homotopy $j \simeq j'$ is *rel* $j^{-1}(A \times (-\infty, 0])$). The second is the use of special types of open covers ($\mathcal{U}_{c,\delta}$ and $\mathcal{U}_{b,\epsilon}$) rather than arbitrary open covers.

To deal with the first aspect, we note that the local-and-relative version of Theorem 4.2 goes like this:

> Suppose X and Y are as in Theorem 4.2, $C \subseteq \operatorname{int} D \subseteq D \subseteq U \subseteq Y$, with U open in Y and C and D closed in Y. For every open cover α of Y there exists an open cover β of Y such that if $p : X \to Y$ is a proper map that is a stratified β-fibration over D, then p is properly α-homotopic rel $p^{-1}(Y \backslash U)$ to a map that is an MSAF over C.

The proof of this follows from the proof of Theorem 4.2 because the argument is a local handle-by-handle one.

For the second aspect, the question is, "if we are given an open cover of the form $\alpha = \mathcal{U}_{b,\epsilon}$, why does the proof of Theorem 4.2 yield an open cover of the form $\beta = \mathcal{U}_{c,\delta}$?" The answer is that the size of the members of β are determined by the size of the handles of $A \times \mathbb{R}$, hence by the size of handles in A. If A is compact, there is no problem. However, if A is non-compact, then we have to get a handlebody decomposition of A in which handles do not decrease in size (without a positive lower limit) as one moves towards infinity in A. There are two (closely related) ways to see why this should be the case. First, $\pi^{-1}((0, +\infty))$ is the infinite cyclic cover of a closed manifold $\hat{A}$ with a manifold approximate fibration $\hat{A} \to S^1$ **[11]**. One can pull back handles in $\hat{A}$ to get handles with a periodic structure in A. Alternatively, use the Approximate Isotopy Covering Property of manifold approximate fibrations **[5]**, **[11]**, **[13]** as follows. For $i = 1, 2, 3, \ldots$ let $\tau_i : [0, +\infty) \to [0, +\infty)$ be the homeomorphism that is translation by i on $[1/2, +\infty)$ and is linear elsewhere,

$$\tau_i(t) = \begin{cases} (1 + 2i)t, & 0 \leq t \leq 1/2 \\ t + i, & t \geq 1/2. \end{cases}$$

Let $\tilde{\tau}_i : A \to A$ be a homeomorphism that approximately covers τ_i; that is, $\pi\tilde{\tau}_i$ is close to $\tau_i\pi$ (they should be within a distance of 1/4). The homeomorphisms $\tilde{\tau}_i$ give the desired handles.

Proof of Theorem 7.1. Choose a sequence $\epsilon = \{\epsilon_i\}_{i=1}^{\infty}$ of positive numbers such that $\lim_{i\to\infty} \epsilon_i = 0$. For the number $b = 1$ and the sequence ϵ, Lemma 7.2 provides another number $c > 0$ and sequence $\delta = \{\delta_i\}_{i=1}^{\infty}$ of positive numbers. We can assume that $\delta_1 \geq \delta_2 \geq \delta_3 \geq \dots$ and that $c < 1$. For the sequence δ, Proposition 6.3 provides a neighborhood N of A in $X\backslash\{b_0\}$ and a proper map $p : N \to A \times (-\infty, +\infty]$ such that

(1) $p^{-1}(A \times \{+\infty\}) = A$ and $p| : A \to A \times \{+\infty\}$ is the identity,
(2) $p^{-1}(A \times (0, +\infty))$ is open in X,
(3) $p| : N\backslash A \to A \times \mathbb{R}$ is a stratified $\mathcal{E}$-fibration over $A \times (0, +\infty)$ where

$$\mathcal{E} = \{N(x, \delta_i) \times (i-1, i+2) \mid x \in A \text{ and } i = 1, 2, 3, \dots\}.$$

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be the homeomorphism defined by

$$\gamma(t) = \begin{cases} t, & t \leq 0 \\ ct/2, & t \geq 0. \end{cases}$$

We will now verify that the map $\tilde{p} = (\mathrm{id}_A \times \gamma) \circ p| : N\backslash A \to A \times \mathbb{R}$ is a stratified $\mathcal{U}_{c,\delta}$-fibration over $A \times [1/2, +\infty)$. To this end suppose we are given a lifting problem

$$\begin{array}{ccccc} Z & \xrightarrow{h} & N\backslash A & = & N\backslash A \\ {\scriptstyle\times 0}\downarrow & & & & \downarrow{\scriptstyle\tilde{p}} \\ Z \times I & \xrightarrow{H} & A \times [1/2, +\infty) & \xrightarrow{\text{inclusion}} & A \times \mathbb{R}. \end{array}$$

If $H^* = (\mathrm{id}_A \times \gamma^{-1}) \circ H$, then there is a commuting diagram

$$\begin{array}{ccccc} Z & \xrightarrow{h} & N\backslash A & = & N\backslash A \\ {\scriptstyle\times 0}\downarrow & & & & \downarrow{\scriptstyle p|} \\ Z \times I & \xrightarrow{H^*} & A \times [1/c, +\infty) & \xrightarrow{\text{inclusion}} & A \times \mathbb{R}. \end{array}$$

which is itself a lifting problem. Because $p|$ is a stratified $\mathcal{E}$-fibration over $A \times (0, +\infty)$, there is a stratum preserving homotopy $\tilde{H} : Z \times I \to N\backslash A$ such that $\tilde{H}(z, 0) = h(z)$ for all $z \in Z$ and $p\tilde{H}$ is $\mathcal{E}$-close to H^*. We need to show that $\tilde{p}\tilde{H} = (\mathrm{id}_A \times \gamma) \circ p\tilde{H}$ is $\mathcal{E}$-close to $H = (\mathrm{id}_A \times \gamma) \circ H^*$. Therefore, let $(z_0, t_0) \in Z \times I$ be given. Since $p\tilde{H}$ is $\mathcal{E}$-close to H^*, there exists $E \in \mathcal{E}$ such that

$$p\tilde{H}(z_0, t_0), H^*(z_0, t_0) \in E.$$

Say $E = N(x_0, \delta_j) \times (j-1, j+2)$ for some $x_0 \in A$ and $j \in \{1, 2, 3, \dots\}$. Then

$$\tilde{p}\tilde{H}(z_0, t_0), H(z_0, t_0) \in (\mathrm{id}_A \times \gamma)(E) = N(x_0, \delta_j) \times (c(j-1)/2, c(j+2)/2).$$

We need an element of $\mathcal{U}_{c,\delta}$ which contains $N(x_0, \delta_j) \times (c(j-1)/2, c(j+2)/2)$. Recall

$$\mathcal{U}_{c,\delta} = \{N(x, \delta_i) \times (t-c, t+c) \mid (x, t) \in A \times [i-1, i+1] \text{ and } i = 1, 2, 3, \dots\}.$$

The interval $(c(j-1)/2, c(j+2)/2)$ is of length $3c/2$. Let t_1 denote the midpoint of that interval. If we can show that $i-1 \leq t_1 \leq i+1$ for some $i \in \{1, 2, 3, \ldots\}$ such that $i \leq j$, then $N(x_0, \delta_j) \subseteq N(x_0, \delta_i)$ and $(c(j-1)/2, c(j+2)/2) \subseteq (t_1 - c, t_1 + c)$. Hence, $N(x_0, \delta_i) \times (t_1 - c, t_1 + c)$ would be the desired member of $\mathcal{U}_{c,\delta}$. Thus, we are reduced to proving: there exists $i \in \{1, 2, 3, \ldots\}$ such that $i - 1 \leq t_1 \leq i + 1$ and $i \leq j$. Noting that $t_1 = (2cj + c)/4$ and that $[t_1]$ (the greatest integer less than or equal to t_1) is positive, we can let $i = [t_1] + 1$. It follows easily from the facts $0 < c < 1$ and $j \geq 1$ that $[t_1] + 1 \leq j$. This completes the proof that $\tilde{p} : N \backslash A \to A \times \mathbb{R}$ is a stratified $\mathcal{U}_{c,\delta}$-fibration over $A \times [1/2, +\infty)$.

Define an isotopy $\gamma_s : \mathbb{R} \to \mathbb{R}$, $s \in I$ by

$$\gamma_s(t) = \begin{cases} t & (t \leq 0) \\ (1-s)t + sct/2 & (t \geq 0). \end{cases}$$

This induces a proper homotopy $(\mathrm{id}_A \times \gamma_s)p|(N\backslash A) : p| \simeq \tilde{p}$, $S \in I$. Moreover, $\tilde{p}$ extends via the identity $A \to A \times \{+\infty\}$ to a map $\hat{p} : N \to A \times (-\infty, +\infty]$ for which there is a proper homotopy $p|N \simeq \hat{p}$ rel A. By our choices of c and δ, Lemma 7.2 implies that $\tilde{p}$ is properly $\mathcal{U}_{1,\epsilon}$-homotopic rel $\tilde{p}^{-1}(A \times (-\infty, 0])$ to a map $p' : N \backslash A \to A \times \mathbb{R}$ which is a stratified approximate fibration over $A \times [1, +\infty)$. Since $\lim_{i\to\infty} \epsilon_i = 0$, it follows that the homotopy $\tilde{p} \simeq p'$ extends to a homotopy $\hat{p} \simeq \hat{p}'$ rel $A \cup \tilde{p}^{-1}(A \times (-\infty, 0])$ where $\hat{p}'$ is the extension of p' to N via the identity $A \to A \times \{+\infty\}$. It follows that $\hat{p}'$ gives the desired MSAF teardrop structure of A in X.

8. Applications. Several types of applications are presented in this section. These should be viewed only as examples of the possibilities. For a fuller list of the types of problems for which teardrop technology is suited, see [**6**] and [**12**]. Weinberger's book [**24**] describes the usefulness of teardrop neighborhoods for solving classification problems.

8.1 REPLACEMENT OF SINGULARITIES. We study the problem of replacing a minimal stratum of a manifold stratified space by another manifold (or manifold stratified space) without changing the complement. This is related to the problem of replacing fixed sets of group actions on manifolds addressed by Cappell and Weinberger [**1**] and it is expected that this technique will have applications to topological locally linear actions.

THEOREM 8.1.1. *Suppose X is a manifold stratified space with a compact minimal stratum B and all strata of $X \backslash B$ are of dimension ≥ 5. Let Y be a manifold stratified space such that there exists a MSAF $p : B \times \mathbb{R} \to Y \times \mathbb{R}$. Then there exists a manifold stratified space Z containing Y as a closed union of strata such that $X \backslash B$ and $Z \backslash Y$ are stratum preserving homeomorphic.*

Proof. The pair (Z, Y) is constructed as follows. Use the Main Theorem to find an open neighborhood U of B in X for which there is a MSAF $f : U \backslash B \to B \times \mathbb{R}$ making U a teardrop neighborhood, $X = (X \backslash B) \cup (U \cup_f B)$. Then $pf : U \backslash B \to Y \times \mathbb{R}$ is an MSAF and $Z = (X \backslash B) \cup (U \cup_{pf} Y)$ is a manifold stratified space by Theorem 3.1.

The pair (Z, Y) is called a *blow down* of (X, B). Simple examples occur when there exists a manifold approximate fibration $f : B \to Y$. However, in that case, Z is the obvious quotient space of X induced by f. In particular, we can always blow B down to a point.

More interesting is the case when B does *not* fiber over Y in any nice way. For example, suppose Y is a closed manifold with $\dim Y = \dim B \geq 5$ and that B and Y are h-cobordant, but not homeomorphic. Then (as is well known) $B \times \mathbb{R}$ and $Y \times \mathbb{R}$ are homeomorphic and we can blow (X, B) down to (Z, Y).

8.2 NEIGHBORHOODS OF POINTS. Points in Whitney stratified spaces have conical neighborhoods that are of the form of an euclidean space cross the cone on a compact space (see [**4**]). Siebenmann [**22**] used this property as his definition for locally conelike topologically stratified spaces. Quinn's manifold stratified spaces [**19**] have conical neighborhoods up to homotopy. The next result describes up to homeomorphism what neighborhoods of points look like in manifold stratified spaces. We offer two different views of the neighborhoods.

THEOREM 8.2.1. *Let X be a manifold stratified space containing a point x_0 in a stratum A with* $\dim A = n$.

(1) *If A satisfies the compactness and dimension hypothesis in the Main Theorem, then there exist a manifold stratified space M with an MSAF $p : M \to \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and a stratum preserving open embedding $h : M \cup_p \mathbb{R}^n \to X$ that carries $\mathbb{R}^n$ onto a neighborhood of x_0 in A and* 0 *onto x_0.*

(2) *If $n \geq 5$, then there exist a manifold stratified space N with an MSAF $r : N \to \mathbb{R}$ and an open embedding $g : N \cup_r \{+\infty\} \to X$ such that $g(+\infty) = x_0$.*

Proof. (1) By the Main Theorem A has an MSAF teardrop neighborhood in X; say it is given by an MSAF $\tilde{p} : \tilde{M} \to A \times \mathbb{R}$. If $\mathbb{R}^n$ is an euclidean neighborhood of x_0 in A with 0 identified to x_0, then $p = \tilde{p}| : M = \tilde{p}^{-1}(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ is the desired MSAF.

(2) If $\mathcal{S} = \{X_i\}$ is the given stratification of X, then consider the new stratification obtained by introducing x_0 as a stratum: $\mathcal{S}' = \{X_i \mid X_i \neq A\} \cup \{A \backslash x_0\} \cup \{x_0\}$. If X with the stratification $\mathcal{S}'$ is a manifold stratified space, then the result follows from part (1) applied to the stratum x_0. In order to establish the required properties for $\mathcal{S}'$, first note that the forward tameness condition at x_0 follows from [**9**, Lemma 5.2]. The only other nontrivial property that requires verification is the compact domination of the local holink at x_0 in $\mathcal{S}'$.

To this end we establish some notation. Note that we may assume that A is a minimal stratum of X because the lower strata do not affect the result. As usual $q : \mathrm{holink}_s(X, A) \to A$ is holink evaluation. Let $F_0 = \mathrm{holink}(A, x_0)$ (which is homotopy equivalent to S^{n-1}) and let $F_1 = q^{-1}(x_0)$. Let $F_2 = \{\omega \in X^I \mid \omega$ is nearly stratum preserving and $\omega(t) = x_0$ if and only if $t = 0\}$. Of course, F_2 is just the local holink at x_0 in $\mathcal{S}'$ and is what we need to show is compactly dominated (in a stratum preserving way). Moreover, F_1 is the local holink at x_0 in the original stratification, so it is compactly dominated.

Let U be an open neighborhood of A in X for which there exists a nearly stratum preserving deformation $d : U \times I \to X$ of U to A in X rel A (Theorem 2.5(1)). Let $F_1' = \{\omega \in F_1 \mid \mathrm{Im}(\omega) \subseteq U\}$ and $F_2' = \{\omega \in F_2 \mid \mathrm{Im}(\omega) \subseteq U\}$. The usual shrinking arguments (cf. [**7**], [**19**]) show that the inclusions $F_1' \to F_1$ and $F_2' \to F_2$ are stratum preserving homotopy equivalences. In particular, F_1' is compactly dominated and it suffices to show that F_2' is compactly dominated. Let $c(F_1')$ denote the cone on F_1', $([0, 1] \times F_1')/\{(0, \sigma) \sim (0, \sigma')\}$. The vertex is denoted v and the cone is given the teardrop topology (cf. [**12**]). We will show that $F_0 \times c(F_1')$ dominates F_2' (in a stratum preserving way to be explained below). In order to define a map $f : F_2' \to F_0 \times c(F_1')$, first let $\alpha : F_2' \to [0, 1]$ be a map such that $\alpha^{-1}(0) = F_0$. Now define f by

$$f(\omega) = \begin{cases} (d_1 \circ \omega, [\alpha(\omega), \omega]) & \text{if } \omega \in F_1' \\ (d_1 \circ \omega, v) & \text{if } \omega \in F_0. \end{cases}$$

Using properties of the teardrop topology, it is easy to verify that f is continuous.

In order to define a map $g : F_0 \times c(F_1') \to F_2'$, recall that $q : \mathrm{P}_{\mathrm{nsp}}(U, A) \to A$ is a stratified fibration [**7**, Thm. 6.1] where $\mathrm{P}_{\mathrm{nsp}}(U, A)$ denotes the space of nearly stratum preserving paths in U with initial point in A (see § 5). Consider the following stratified homotopy lifting problem:

$$\begin{array}{ccc} F_0 \times F_2' & \longrightarrow & \mathrm{P}_{\mathrm{nsp}}(U, A) \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ F_0 \times I \times F_2' & \longrightarrow & A \end{array}$$

where the top horizontal map is $(\omega, \sigma) \mapsto \sigma$ and the bottom horizontal map is $(\omega, s, \sigma) \mapsto \omega(s)$. Let $G : F_0 \times I \times F_2' \to \mathrm{P}_{\mathrm{nsp}}(U, A)$ be a stratum preserving solution. In particular, $G(\omega, 0, \sigma) = \sigma$ and $G(\omega, s, \sigma)(0) = \omega(s)$. Define $G' : F_0 \times I \times F_2' \to \mathrm{P}_{\mathrm{nsp}}(U, A)$ by $G'(\omega, s, \sigma)(t) = G(\omega, t, \sigma)(st)$. It is easy to see that G' induces a function $g' : F_0 \times c(F_2') \to \mathrm{P}_{\mathrm{nsp}}(U, A)$ that would be continuous if the cone were given the quotient topology. However, it need not be continuous in the teardrop topology, but we now modify it so that it is. Using a partition of unity argument (cf. [**12**, Lemma 4.3]) construct a map $\varphi : F_0 \times I \times F_2' \times [0, 1] \to [0, 1]$ such that $\varphi(\omega, t, \sigma, s) = 0$ if only if $s = 0$, and such that

$$\operatorname{diam}\{G(\omega, t, \sigma)(s') \mid 0 \leq s' \leq \varphi(\omega, t, \sigma, s)\} \leq s$$

for each (ω, t, σ, s). In particular, $G(\omega, t, \sigma)(st\varphi(\omega, t, \sigma, s))$ is s-close to $\omega(t)$.

Now define $\tilde{G} : F_0 \times I \times F_2' \to \mathrm{P}_{\mathrm{nsp}}(U, A)$ by

$$\tilde{G}(\omega, s, \sigma)(t) = G(\omega, t, \sigma)(st\varphi(\omega, t, \sigma, s)).$$

The function $g : F_0 \times c(F_2') \to \mathrm{P}_{\mathrm{nsp}}(U, A)$, $(\omega, [s, \sigma]) \mapsto \tilde{G}(\omega, s, \sigma)$ induced by $\tilde{G}$ is now continuous.

Define a homotopy $H : F_2' \times I \to F_2'$ by

$$H(\omega, u)(t) = \tilde{G}(d_1 \circ \omega, ut, \omega)((1 - u)t + u\alpha(\omega)t).$$

Thus, $H_0 = \mathrm{id}$ and $H_1 = gf$. It only remains to discuss the strata of $F_0 \times c(F_2')$. They are of the form $F_0 \times \{v\}$ or $F_0 \times (0, 1] \times Z$ where Z is a stratum of F_2'. One observes that the compact domination of $F_0 \times c(F_2')$ respects this stratification, as does the homotopy H.

One amusing consequence of the result above is that an inductive definition (on the number of strata) of manifold stratified spaces can be given, except for low dimensional uncertainties. One pleasing aspect of this definition is that it illustrates a striking resemblance to Siebenmann's definition [**22**].

More explicitly, a *strong manifold stratified space with one stratum* is a manifold. Suppose $k > 1$ and that strong manifold stratified spaces with fewer than k strata have been defined. A space with a stratification $\{X_i\}$ containing k strata and satisfying the

Frontier Condition and the Manifold Strata Property (2.3(4)) is a *strong manifold stratified space* provided for each $x \in X$ with $x \in X_i$, $\dim X_i = n$, there exist a strong manifold stratified space L_x with fewer than k strata with an MSAF $p : L_x \to \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and a stratum preserving open embedding $h : L_x \cup_p \mathbb{R}^n \to X$ that carries $\mathbb{R}^n$ onto a neighborhood of x in X_i and 0 onto x.

Corollary 8.2.2. *Every strong manifold stratified space is a manifold stratified space. Conversely, every manifold stratified space with at most one stratum of dimension less than* 5 *is a strong manifold stratified space.*

Proof. The first statement follows from Theorem 3.1. The converse follows from Theorem 8.2.1(1).

For a discussion of some of the low dimensional problems with manifold stratified spaces, see Quinn [**20**], [**21**].

8.3 Spaces of manifold stratified approximate fibrations. The following result is parameterized version of the Stratified Sucking Theorem 4.2. For notation, Δ^k is the standard k-simplex. A map $p : X \times \Delta^k \to Y \times \Delta^k$ is *fiber preserving* if it commutes with the projections to Δ^k.

Theorem 8.3.1 (Parametric Stratified Sucking). *Suppose X is a manifold stratified space with no strata of dimension less than* 5, *and Y is a manifold without boundary. For every open cover α of $Y \times \Delta^k$, there exists an open cover β of Y such that if $p : X \times \Delta^k \to Y \times \Delta^k$ is a fiber preserving and for each $t \in \Delta^k$, $p_t : X \to Y$ is a proper stratified β-fibration, then p is fiber preserving properly α-homotopic to a map $\tilde{p} : X \times \Delta^k \to Y \times \Delta^k$ such that for each $t \in \Delta^k$, $\tilde{p}_t : X \to Y$ is a stratified manifold approximate fibration. Moreover, if $p_t : X \to Y$ is given to be a stratified manifold approximate fibration for each $t \in \partial\Delta^k$, then the homotopy $p \simeq \tilde{p}$ can be required to be rel $X \times \partial\Delta^k$.*

Proof. One observes that the engulfing constructions in the proof of Theorem 4.2 imply a Δ^k-parameter version of those constructions. The source for this is [**5**, §§2–5] where Chapman's unparameterized machine is made to work with parameters.

Corollary 8.3.2. *Suppose X is a manifold stratified space with no strata of dimension less than* 5 *and Y is a closed manifold. Then the space* MSAF(X, Y) *of manifold stratified approximate fibrations from X to Y is locally k-connected for each $k \geq 0$.*

Proof. As in the unstratified case [**5**] this follows directly from Theorem 8.3.1. One also needs to consult [**13**, § 13] to see how to eliminate the assumption in [**5**] that Y has a handlebody.

8.4 A loose end. According to Hughes and Ranicki [**11**, Prop. 17.20] every ANR band is simple homotopy equivalent to one whose infinite cyclic cover is proper homotopic to an approximate fibration. However, the proof relied on a stratified sucking result promised by [**6**]. The missing result follows from Theorem 4.2 as the final proposition shows.

Proposition 8.4.1. *Suppose $(M, \partial M)$ is a manifold with boundary considered as a manifold stratified space with two strata:* int(M) *and ∂M. Suppose further that* $\dim(M) \geq 6$ *and $p : M \to \mathbb{R}$ is a proper stratified bounded fibration (that is, a proper stratified b-fibration for some $b > 0$). Then p is boundedly homotopic to a manifold stratified approximate fibration.*

Proof. This is a standard application of Theorem 4.2 (see [**11**, Cor. 16.10]). The idea is to follow p by the map $\mathbb{R} \to \mathbb{R}$, $x \mapsto x/L$, for some large $L > 0$. The composition $M \to \mathbb{R}$ is a stratified ϵ-fibration for a small $\epsilon > 0$. Constants $\epsilon > 0$ can be used instead of the open covers in Theorem 4.2 because of the homogeneous metric on $\mathbb{R}$.

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