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Products and adjunctions of manifold stratified spaces ☆

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Abstract

Basic topological constructions of manifold stratified spaces and stratified approximate fibrations are studied. These include products of manifold stratified spaces, products and compositions of stratified approximate fibrations and Euclidean stabilization of stratified approximate fibrations. The main result shows that the adjunction of two manifold stratified spaces via a manifold stratified approximate fibration is a manifold stratified space.

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1. Introduction

The purpose of this paper is to establish some basic properties of constructions involving manifold stratified spaces. These are the spaces introduced by Quinn [14] as the natural setting for the study of topological stratified phenomena. Manifold stratified spaces are the topological analogues of the smoothly stratified sets of Thom [16] and Mather [13] (see Goresky and MacPherson [4] for an exposition). Earlier Siebenmann [15] had introduced a category of topologically stratified spaces less inclusive than Quinn's, but the locally conelike condition of Siebenmann has proved to be too rigid (see [12]).

The theory of manifold stratified spaces is closely related to the theory of stratified approximate fibrations. In this paper we study constructions of manifold stratified spaces and stratified approximate fibrations. The main result is that the adjunction of two manifold stratified spaces via a stratified approximate fibration is also a manifold stratified space (Theorem 6.2). Other constructions studied here include products of manifold stratified

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spaces, products and compositions of stratified approximate fibrations and Euclidean stabilization of stratified approximate fibrations.

Most of the results established in this paper are crucial to [9] which contains a proof of the Approximate Tubular Neighborhood Theorem for manifold stratified spaces.

2. Manifold stratified spaces

This section contains the basic definitions from the theory of topological stratifications as presented in [5–7,14]. Quinn [14] is the original source for most of these ideas.

Definition 2.1. A *stratification* of a space X consists of an index set $\mathcal{I}$ and a locally finite partition $\{X_i\}_{i\in\mathcal{I}}$ of locally closed subspaces of X (the X_i are pairwise disjoint and their union is X). For $i\in\mathcal{I}$, X_i is called the *i-stratum* and the closed set

$$X^i=\bigcup\bigl\{X_k \mid X_k\cap \mathrm{cl}(X_i)\neq\emptyset\bigr\}$$

is called the *i-skeleton*. We say X is a *space with a stratification*.

For a space X with a stratification $\{X_i\}_{i\in\mathcal{I}}$, define a relation $\leqslant$ on the index set $\mathcal{I}$ by $i\leqslant j$ if and only if $X_i\subseteq \mathrm{cl}(X_j)$. The *Frontier Condition* is satisfied if for every $i,j\in\mathcal{I}$, $X_i\cap\mathrm{cl}(X_j)\neq\emptyset$ implies $X_i\subseteq\mathrm{cl}(X_j)$, in which case $\leqslant$ is a partial ordering of $\mathcal{I}$ and $X^i=\mathrm{cl}(X_i)$ for each $i\in\mathcal{I}$.

If X is a space with a stratification, then a map $f:Z\times A\to X$ is *stratum preserving along A* if for each $z\in Z$, $f(\{z\}\times A)$ lies in a single stratum of X. In particular, a map $f:Z\times I\to X$ is a *stratum preserving homotopy* if f is stratum preserving along I. A homotopy $f:Z\times I\to X$ whose restriction to $Z\times[0,1)$ is stratum preserving along $[0,1)$ is said to be *nearly stratum preserving*.

Definition 2.2. Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and $Y\subseteq X$.

(1) Y is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h:U\times I\to X$ such that $h_0=\text{inclusion}:U\to X$, $h_t|Y=\text{inclusion}:Y\to X$ for each $t\in I$, $h_1(U)=Y$, and $h((U\setminus Y)\times[0,1))\subseteq X\setminus Y$.

(2) The *homotopy link* of Y in X is defined by

$$\mathrm{holink}(X,Y)=\bigl\{\omega\in X^I \mid \omega(t)\in Y \text{ if and only if } t=0\bigr\}.$$

(3) Let $x_0\in X_i\subseteq X$. The *local holink at x_0* is

$$\mathrm{holink}(X,x_0)=\bigl\{\omega\in\mathrm{holink}(X,X_i)\ \mid\ \omega(0)=x_0 \text{ and } \omega(t)\in X_j \text{ for some } j,\ \text{for all } t\in(0,1]\bigr\}.$$

All path spaces are given the compact-open topology. Evaluation at 0 defines a map $q:\mathrm{holink}(X,Y)\to Y$ called *holink evaluation*. There is a natural stratification of $\mathrm{holink}(X,x_0)$ into disjoint subspaces

$$\mathrm{holink}(X,x_0)_j=\bigl\{\omega\in\mathrm{holink}(X,x_0)\mid\omega(1)\in X_j\bigr\}.$$

Definition 2.3. A space X with a stratification satisfying the Frontier Condition is a *manifold stratified space* [1] if the following four conditions are satisfied:

(1) *Forward Tameness*. For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.

(2) *Normal Fibrations*. For each $k > i$, the holink evaluation

$$q : \mathrm{holink}(X_i \cup X_k, X_i) \to X_i$$

is a fibration.

(3) *Compactly dominated local holinks*. For each $x_0 \in X$, there exist a compact subset C of the local holink $\mathrm{holink}(X, x_0)$ and a stratum preserving homotopy

$$h : \mathrm{holink}(X, x_0) \times I \to \mathrm{holink}(X, x_0)$$

such that $h_0 = \mathrm{id}$ and $h_1(\mathrm{holink}(X, x_0)) \subseteq C$.

(4) *Manifold strata property*. X is a locally compact, separable metric space, each stratum X_i is a topological manifold (without boundary) and X has only finitely many nonempty strata.

If X is only required to satisfy conditions (1) and (2), then X is a *homotopically stratified space*.

Definition 2.4. A subset A of a space X with a stratification is a *pure subset* if A is closed and is a union of strata of X.

3. Stratified approximate fibrations

In this section we give the formal definitions of various types of stratified approximate fibrations. These maps are the analogues of the approximate fibrations of Coram and Duvall [3], but the formulation given here is modeled on [11]. In Remarks 3.3 below we correct some aspects of the definitions given previously.

Definition 3.1. Let X and Y be spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively, and let $p : X \to Y$ be a map.

(1) p is a *stratified fibration* provided that given any space Z and any commuting diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & Y \end{tikzcd}$$

with F a stratum preserving homotopy, there exists a *stratified solution*; i.e., a stratum preserving homotopy $\widetilde{F} : Z \times I \to X$ such that $\widetilde{F}(z, 0) = f(z)$ for each $z \in Z$ and $p\widetilde{F} = F$. The diagram above is a *stratified homotopy lifting problem*.

[1] It has been suggested that *stratifold* would be a better name.

(2) p is a *strong stratified approximate fibration* provided that given any stratified homotopy lifting problem, there exists a *strong stratified controlled solution*; i.e., a map $\widetilde{F}: Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\widetilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\overline{F}: Z \times I \times I \to Y$ defined by $\overline{F}|Z \times I \times [0,1) = p\widetilde{F}$ and $\overline{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous and stratum preserving along $I \times I$.

(3) p is a *weak stratified approximate fibration* provided that given any stratified homotopy lifting problem, there exists a *weak stratified controlled solution*; i.e., a map $\widetilde{F}: Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\widetilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\overline{F}: Z \times I \times I \to Y$ defined by $\overline{F}|Z \times I \times [0,1) = p\widetilde{F}$ and $\overline{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous.

(4) p is a *manifold stratified approximate fibration* (MSAF) if X and Y are manifold stratified spaces and p is a proper weak stratified approximate fibration.

(5) If α is an open cover of Y, then p is a *stratified α-fibration* provided that given any stratified homotopy lifting problem, there exists a *stratified α-solution*; i.e., a stratum preserving homotopy $\widetilde{F}: Z \times I \to X$ such that $\widetilde{F}(z,0) = f(z)$ for each $z \in Z$ and $p\widetilde{F}$ is α-close to F.

(6) p is a *manifold approximate fibration* (MAF) if p is an MSAF and X and Y have only one stratum each (i.e., they are manifolds).

(7) If $Y \times \mathbb{R}$ is given the natural stratification (see Section 4), then a map $p: X \to Y \times \mathbb{R}$ is a *stratified rectangularly controlled fibration* (SRCF) if p is a stratified $(\alpha \times \beta)$-fibration for all open covers α of Y and β of $\mathbb{R}$. Here $\alpha \times \beta$ denotes the open cover $\{U \times V \mid U \in \alpha, V \in \beta\}$ of $Y \times \mathbb{R}$.

Example 3.2. Here is an example of a weak stratified approximate fibration $p: X \to Y$ which is not a strong stratified approximate fibration (in fact, p is an MSAF). Let $X = \mathbb{R}^2$, a manifold stratified space with a single stratum. Let W be a Warsaw circle in $\mathbb{R}^2$ with singular arc A (the arc of non-manifold points of W). Let $Y = X/A$ and let $p: X \to Y$ be the quotient map. Of course, p is cell-like and $Y \cong \mathbb{R}^2$. The image $p(W)$ of W is a circle Y_0 in Y. Stratify Y with two strata: Y_0 and $Y \setminus Y_0$. Then Y is a manifold stratified space and it follows easily that p is an MSAF (in fact, p is a MAF when Y is left unstratified). However, p is not a strong stratified approximate fibration because there are paths in Y_0 which cannot be approximately lifted to X without leaving $p^{-1}(Y_0) = W$.

Remarks and Corrections 3.3. The difference between a *strong* and a *weak* stratified approximate fibration is in whether or not the map $\overline{F}: Z \times I \times I \to Y$ is required to be stratum preserving along $I \times I$. In my previous papers [5–7] the two notions were confused. It is hoped that the following remarks will clarify the issue.

(1) The results in [5] hold when "stratified approximate fibrations" are interpreted as "strong stratified approximate fibrations". This change in terminology should be made in Definition 5.4, Remark 5.5, Lemma 7.5, Corollary 7.6 and Proposition 8.6. I suspect that Proposition 8.6 also holds for weak stratified approximate fibrations.

(2) The results in [6] hold when “stratified approximate fibrations” are interpreted as “weak stratified approximate fibrations”. Thus, Definition 4.1(2) should be corrected and the terminology changed in Propositions 4.2, 5.5, 5.7, 5.8, 5.10, 5.11 and Theorem 7.3. I do not know if in Theorem 7.3 one may conclude that p is also a strong stratified approximate fibrations, but I suspect so. As an example, note that if $p : X \to Y$ is the weak stratified approximate fibration in Example 3.2 above, then $\mathrm{cyl}(p)$ is a homotopically stratified space, but p is not a strong stratified approximate fibration.

(3) The results in [7] hold when “stratified approximate fibrations” are interpreted as “weak stratified approximate fibrations”. Thus the definition in Section 2 should be corrected and the terminology should be changed in the Main Theorem, Proposition 3.2 and Theorems 4.3, 5.4, 6.1 and 7.1. I do not know if the Main Theorem also holds for proper SRCFs.

(4) If X and Y are manifold stratified spaces, Y has only a single stratum and $p : X \to Y$ is an MSAF, then p is a strong stratified approximate fibration.

(5) If $p : X \to Y$ is a proper stratified α-fibration for every open cover α of Y and X and Y are manifold stratified spaces, I do not know if p must be an MSAF. The answer in the special case where $p : X \to Y \times \mathbb{R}$ is also unknown to me.

(6) SRCFs are included to illustrate some of the puzzles which remain in the stratified world. For example, I do not know if every proper SRCF $p : X \to Y \times \mathbb{R}$ where X and Y are manifold stratified spaces is an MSAF, but I suspect that it need not be so. Moreover, I do not know if every proper SRCF $p : X \to Y \times \mathbb{R}$ where X and Y are manifold stratified spaces is a stratified γ-fibration for every open cover γ of $Y \times \mathbb{R}$. The problem is in getting a local to global result.

The following result on mappings cylinders is the main result from [6] and will be used several times in this paper.

Theorem 3.4 [6, Theorem 5.11]. *Let $p : X \to Y$ be a proper map between locally compact homotopically stratified metric spaces with only finitely many strata and suppose the strata of Y are path connected. Then p is a weak stratified approximate fibration if and only if $\mathrm{cyl}(p)$ with the natural stratification is a homotopically stratified space.*

Corollary 3.5. *If $p : X \to Y$ is a manifold stratified approximate fibration between manifold stratified spaces with only finitely many strata, then $\mathrm{cyl}(p)$ with the natural stratification is a manifold stratified space.*

Proof. All of the conditions except the compactly dominated local holinks condition follow from Theorem 3.4. The remaining condition follows from [7, Theorem 6.1]. □

Theorem 6.2 below on adjunctions of manifold stratified spaces generalizes Corollary 3.5, but its proof uses Corollary 3.5. See Cappell and Shaneson [1] for other conditions which insure that a mapping cylinder is a manifold stratified space.

4. Products, compositions and Euclidean stabilization

In this section we establish facts about products of stratified spaces, products and compositions of stratified approximate fibrations and Euclidean stabilization.

Products of spaces with stratifications

Here is the notation used in the first result. Let X and Y be spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively. The natural stratification on $X \times Y$ is given by

$$\big\{X_i \times Y_j \mid (i, j) \in \mathcal{I} \times \mathcal{J}\big\}.$$

It is elementary to verify that this is indeed a locally finite partition by locally closed subsets. Moreover, if the stratifications of X and Y satisfy the Frontier Condition, then so does the stratification of $X \times Y$. In fact, the induced partial order on $\mathcal{I} \times \mathcal{J}$ is described by

$$(i, j) \leqslant (k, l) \quad \text{if and only if } i \leqslant k \text{ and } j \leqslant l.$$

Proposition 4.1.

(1) *If the stratifications of X and Y satisfy the forward tameness condition, then so does the natural stratification of $X \times Y$.*
(2) *If the stratifications of X and Y satisfy the normal fibrations condition, then so does the natural stratification of $X \times Y$.*
(3) *If the stratifications of X and Y satisfy the compactly dominated local holinks condition, then so does the natural stratification of $X \times Y$.*
(4) *If X and Y are manifold stratified spaces, then so is $X \times Y$* (*with the natural stratification*).

Proof. The proof of (1) is straightforward and is left to the reader. For (2), suppose $(i, j) < (k, l)$ and show that

$$\operatorname{holink}\big((X_i \times Y_j) \cup (X_k \times Y_l), X_i \times Y_j\big) \to X_i \times Y_j$$

is a fibration. There are three cases which give $(i, j) < (k, l)$:

(i) $i < k$ and $j < l$,
(ii) $i = k$ and $j < l$,
(iii) $i < k$ and $j = l$.

In each of these cases, there is, respectively, a natural identification of

$$\operatorname{holink}\big((X_i \times Y_j) \cup (X_k \times Y_l), X_i \times Y_j\big) \to X_i \times Y_j$$

with the natural map:

(i) $\operatorname{holink}(X_i \cup X_k, X_i) \times \operatorname{holink}(Y_j \cup Y_l, Y_j) \to X_i \times Y_j$,
(ii) $X_i \times \operatorname{holink}(Y_j \cup Y_l, Y_j) \to X_i \times Y_j$,
(iii) $\operatorname{holink}(X_i \cup X_k, X_i) \times Y_j \to X_i \times Y_j$.

Each of these is a fibration by hypothesis.

For (3) let $(x_0, y_0) \in X_i \times Y_j$ be given. The result follows immediately from the following lemma.

Lemma 4.2. *There exists a stratum preserving homotopy equivalence between the local holink* $\mathrm{holink}(X \times Y, (x_0, y_0))$ *and the join* $\mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0)$.

Proof of Lemma 4.2. The join $\mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0)$ is the quotient space of

$$\mathrm{holink}(X, x_0) \times I \times \mathrm{holink}(Y, y_0)$$

obtained by making the identifications $(\omega, 0, \sigma) \sim (\omega, 0, \sigma')$ and $(\omega, 1, \sigma) \sim (\omega', 1, \sigma)$ for all $\omega, \omega' \in \mathrm{holink}(X, x_0)$ and $\sigma, \sigma' \in \mathrm{holink}(Y, y_0)$. The natural strata of $\mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0)$ are indexed by $\mathcal{A} = \{(k, l) \in \mathcal{I} \times \mathcal{J} \mid (i, j) < (k, l)\}$ and are given by

$$\begin{aligned}&\big(\mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0)\big)_{(k,l)}\\ &\quad = \begin{cases} \mathrm{holink}(X, x_0)_k \times (0, 1) \times \mathrm{holink}(Y, y_0)_l, & i < k,\ j < l,\\ \mathrm{holink}(X, x_0)_k \times \{0\} \times \mathrm{holink}(Y, y_0), & i < k,\ j = l,\\ \mathrm{holink}(X, x_0) \times \{1\} \times \mathrm{holink}(Y, y_0)_l, & i = k,\ j < l,\end{cases}\end{aligned}$$

where the sets on the right hand side are identified with their images in

$$\mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0).$$

The strata of $\mathrm{holink}(X \times Y, (x_0, y_0))$ are also indexed by $\mathcal{A}$ and are given by

$$\mathrm{holink}\big(X \times Y, (x_0, y_0)\big)_{(k,l)} = \big\{\omega \in \mathrm{holink}\big(X \times Y, (x_0, y_0)\big) \mid \omega(1) \in X_k \times Y_l\big\}.$$

In order to begin the definition of a homotopy equivalence φ, first let

$$\rho : \mathrm{holink}\big(X \times Y, (x_0, y_0)\big) \to I$$

be a map such that $\rho^{-1}(0) = \{\omega \mid \omega(1) \in X \times Y_j\}$ and $\rho^{-1}(1) = \{\omega \mid \omega(1) \in X_i \times Y\}$. Define

$$\varphi : \mathrm{holink}\big(X \times Y, (x_0, y_0)\big) \to X^I * Y^I$$

by

$$\varphi(\omega) = \big[p_X \circ \omega, \rho(\omega), p_Y \circ \omega\big]$$

where $p_X : X \times Y \to X$ and $p_y : X \times Y \to Y$ are the projections. The following three facts imply that φ is well-defined and has image in $\mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0)$:

(1) If $0 < \rho(\omega) < 1$, then $p_x \circ \omega \in \mathrm{holink}(X, x_0)$ and $p_Y \circ \omega \in \mathrm{holink}(Y, y_0)$.
(2) If $\rho(\omega) = 0$, then $p_X \circ \omega \in \mathrm{holink}(X, x_0)$.
(3) If $\rho(\omega) = 1$, then $p_X \circ \omega \in \mathrm{holink}(Y, y_0)$.

We only verify the first since the others are similar. So suppose $0 < \rho(\omega) < 1$. Then $\omega(1) \in X_k \times Y_l$ for some $i < k$ and $j < l$. Thus, $\omega((0, 1]) \subseteq X_k \times Y_l$ and $\omega(0) = (x_0, y_0)$. The result follows. It is also easy to verify that φ takes strata to strata and preserves indices.

In order to define a stratum preserving homotopy inverse for φ, adopt the following notation: given any path $\omega : I \to Z$ in a space Z and any $t \in I$, let $t \cdot \omega$ be the path $(t \cdot \omega)(s) = \omega(ts)$. Define $\psi : \mathrm{holink}(X, x_0) * \mathrm{holink}(Y, y_0) \to \mathrm{holink}(X \times Y, (x_0, y_0))$ by

$\psi([\sigma, t, \tau]) = (1-t)\cdot\sigma \times t\cdot\tau$. It is not too difficult to verify that $\varphi \circ \psi$ and $\psi \circ \varphi$ are stratum preserving homotopic to the respective identities. $\square$

Finally, to complete the proof of Proposition 4.1, note that (4) follows immediately from (1), (2) and (3). $\square$

Products of stratified approximate fibrations

The next result concerns products of stratified approximate fibrations.

Proposition 4.3. *If $f_i : X_i \to Y_i$, $i = 1, 2$, are weak* (*respectively, strong*) *stratified approximate fibrations between spaces with stratifications, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also a weak* (*respectively, strong*) *stratified approximate fibration* (*where the product spaces are given the natural stratifications*).

Proof. A stratified lifting problem

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & X_1 \times X_2 \arrow[d, "f_1 \times f_2"] \\ Z \times I \arrow[r, "F"] & Y_1 \times Y_2 \end{tikzcd}$$

induces stratified lifting problems

$$\begin{tikzcd} Z \arrow[r, "p_i f"] \arrow[d, "\times 0"'] & X_i \arrow[d, "f_i"] \\ Z \times I \arrow[r, "p_i F"] & Y_i \end{tikzcd}$$

where p_i denotes projection to the ith factor, $i = 1, 2$. Weak (respectively, strong) stratified controlled solutions $\widetilde{F}_i : Z \times I \times [0,1) \to X_i$ of these two problems induce a weak (respectively, strong) stratified controlled solution $\widetilde{F} : Z \times I \times [0,1) \to X_1 \times X_2$ of the original problem defined by $\widetilde{F}(z,s,t) = (\widetilde{F}_1(z,s,t), \widetilde{F}_2(z,s,t))$, as is easily verified. $\square$

The proof of the following result is very similar to the proof of Proposition 4.3 so we omit it.

Proposition 4.4. *If $f_i : X_i \to Y_i$, $i = 1, 2$, are stratified α_i-fibrations between spaces with stratifications where α_i is an open cover of Y_i, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a stratified* $(\alpha_1 \times \alpha_2)$*-fibration.*

Compositions of stratified approximate fibrations

The following fact concerns compositions of stratified approximate fibrations.

Proposition 4.5. *If $f : X \to Y$ and $g : Y \to Z$ are weak* (*respectively, strong*) *stratified approximate fibrations between metric spaces with stratifications, then $gf : X \to Z$ is also a weak* (*respectively, strong*) *stratified approximate fibration.*

Proof. A stratified lifting problem

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "\times 0"'] & X \arrow[d, "gf"] \\ W \times I \arrow[r, "H"] & Z \end{tikzcd}$$

induces a stratified lifting problem

$$\begin{tikzcd} W \arrow[r, "fh"] \arrow[d, "\times 0"'] & Y \arrow[d, "g"] \\ W \times I \arrow[r, "H"] & Z \end{tikzcd}$$

with a stratified controlled solution $\widetilde{H}_1 : W \times I \times [0,1) \to Y$. Let $p_W : W \times [0,1) \to W$ be projection. Then there is an induced stratified lifting problem

$$\begin{tikzcd} W \times [0,1) \arrow[r, "hp_W"] \arrow[d, "\times 0"'] & X \arrow[d, "f"] \\ W \times I \times [0,1) \arrow[r, "\widetilde{H}_1"] & Y \end{tikzcd}$$

with a stratified controlled solution $\widetilde{H}_2 : W \times I \times [0,1) \times [0,1) \to X$. One might think that the map $\widetilde{H} : W \times I \times [0,1) \to X$ defined by $\widetilde{H}(w,s,t) = \widetilde{H}_2(w,s,t,t)$ is a stratified controlled solution of the original problem. However, the map $gf\widetilde{H}$ need not be continuously extendible to all of $W \times I \times I$ via H. The construction can be corrected as follows. First, since X and Z are metric, it can be assumed that W is also a metric space (cf. [5, Remark 5.5]). Then a partition of unity argument allows one to find a map $\varphi : W \times I \times [0,1) \to [0,1)$ so that $\widetilde{H}$ defined by $\widetilde{H}(w,s,t) = \widetilde{H}_2(w,s,t,\varphi(w,s,t))$ is a stratified controlled solution (cf. [5, Lemma 8.1]). $\square$

The next example provides a simple example of a stratified fibration and is needed in [9].

Proposition 4.6 (Corner Collapse). *Let Y be a space with a stratification. Define $r : Y \times [0,+\infty) \times [0,+\infty) \to Y \times [0,+\infty)$ by $r(x,s,t) = (x, s+t)$. Then r is a stratified fibration and maps strata into strata.*

Proof. The stratifications of $Y \times [0,+\infty) \times [0,+\infty)$ and $Y \times [0,+\infty)$ are the natural ones. For example, the strata of $Y \times [0,+\infty) \times [0,+\infty)$ are of the form $Y_i \times \{0\} \times \{0\}$, $Y_i\{0\} \times (0,+\infty)$, $Y_i \times (0,+\infty) \times \{0\}$ or $Y_i \times (0,+\infty) \times (0,+\infty)$ where Y_i is a stratum of Y. It is clear that r maps strata into strata. To see that r is a stratified fibration, suppose there is a stratified lifting problem:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & Y \times [0,+\infty) \times [0,+\infty) \arrow[d, "r"] \\ Z \times I \arrow[r, "F"] & Y \times [0,+\infty) \end{tikzcd}$$

Write $F(z,t) = (F_1(z,t), F_2(z,t)) \in Y \times [0,+\infty)$ and $f(z) = (f_1(z), f_2(z), f_3(z)) \in Y \times [0,+\infty) \times [0,+\infty)$ for each $(z,t) \in Z \times I$. Define $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3) : Z \times I \to Y \times [0,+\infty) \times [0,+\infty)$ for $(z,t) \in Z \times I$ by

$$\begin{aligned}
\widetilde{F}_1(z,t) &= F_1(z,t) \in Y, \\
\widetilde{F}_2(z,t) &= \frac{f_2(z)}{f_2(z)+f_3(z)} \cdot F_2(z,t) \in [0,+\infty) \quad \text{if } f_2(z)+f_3(z) \neq 0, \\
\widetilde{F}_3(z,t) &= \frac{f_3(z)}{f_2(z)+f_3(z)} \cdot F_2(z,t) \in [0,+\infty) \quad \text{if } f_2(z)+f_3(z) \neq 0, \\
\widetilde{F}_2(z,t) &= \widetilde{F}_3(z,t) = 0 \in [0,+\infty) \qquad \text{if } f_2(z)+f_3(z) = 0.
\end{aligned}$$

It is easy to verify that $\widetilde{F}$ is a stratified solution of the given problem. □

Euclidean stabilization of stratified approximate fibrations

The next result concerns stabilizing a map $p : X \to Y$ by crossing with the identity on $\mathbb{R}$. It characterizes when the map $p \times \mathrm{id}_{\mathbb{R}}$ is a strong stratified approximate fibration, a weak stratified approximate fibration, or an SRCF.

Proposition 4.7. *Let X and Y be metric spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively, and let $p : X \to Y$ be a map.*

(A) *The following are equivalent*:
  (1) *p is a strong stratified approximate fibration,*
  (2) *$p \times \mathrm{id}_{\mathbb{R}} : X \times \mathbb{R} \to Y \times \mathbb{R}$ is a strong stratified approximate fibration.*

(B) *The following are equivalent*:
  (1) *p is a weak stratified approximate fibration,*
  (2) *$p \times \mathrm{id}_{\mathbb{R}}$ is a weak stratified approximate fibration,*
  (3) *$p \times \mathrm{id}_{\mathbb{R}}$ is a stratified α-fibration for every open cover α of $Y \times \mathbb{R}$.*

(C) *The following are equivalent*:
  (1) *p is a stratified α-fibration for every open cover α of Y,*
  (2) *$p \times \mathrm{id}_{\mathbb{R}}$ is a stratified rectangularly controlled fibration.*

Proof. (A) (1) implies (2) follows from Proposition 4.3 above.

(A) (2) implies (1). Let

$$\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
{\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\
Z \times I & \xrightarrow{F} & Y
\end{array}$$

be a stratified homotopy lifting problem. There is an induced stratified homotopy lifting problem

$$\begin{array}{ccc}
Z & \xrightarrow{i_X f} & X \times \mathbb{R} \\
{\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p\times \mathrm{id}_{\mathbb{R}}} \\
Z \times I & \xrightarrow{i_Y F} & Y \times \mathbb{R}
\end{array}$$

where $i_X(x) = (x, 0)$ and $i_Y(y) = (y, 0)$ for all $x \in X$ and $y \in Y$. If $\widetilde{F} : Z \times I \times [0, 1) \to X \times \mathbb{R}$ is a strong stratified controlled solution of the induced problem, then it is easy to check that $p_X \widetilde{F} : Z \times I \times [0, 1) \to X$ is a strong stratified controlled solution to the original problem where p_X denotes projection to X.

(B) (1) implies (2) follows from Proposition 4.3 above.

(B) (2) implies (3) follows from a straightforward modification of the proof of the unstratified case in [11, 12.10].

(B) (3) implies (1). Let

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & Y \end{tikzcd}$$

be a stratified homotopy lifting problem. There is an induced stratified homotopy lifting problem

$$\begin{tikzcd} Z \times \mathbb{R} \arrow[r, "f \times \mathrm{id}_{\mathbb{R}}"] \arrow[d, "\times 0"'] & X \times \mathbb{R} \arrow[d, "p \times \mathrm{id}_{\mathbb{R}}"] \\ Z \times I \times \mathbb{R} \arrow[r, "F \times \mathrm{id}_{\mathbb{R}}"] & Y \times \mathbb{R}. \end{tikzcd}$$

Give $Y \times \mathbb{R}$ the product metric and let α be an open cover of $Y \times \mathbb{R}$ such that if $U \in \alpha$, $n \in \mathbb{Z}_+$ and $U \cap Y \times [n, +\infty) \neq \emptyset$, then $\operatorname{diam} U < 1/n$. Let $F' : Z \times I \times \mathbb{R} \to X \times \mathbb{R}$ be a stratified α-solution of the induced problem. Define $\widetilde{F} : Z \times I \times [0, +\infty) \to X$ by $\widetilde{F}(z, s, t) = p_X F'(z, s, t)$ where p_X denotes the projection to X. Note that $\widetilde{F}$ is stratum preserving along $I \times [0, +\infty)$ and that $\widetilde{F}(z, s, t) = f(z)$ for every $(z, t) \in Z \times [0, +\infty)$. It remains to show that the function $\overline{F} : Z \times I \times [0, +\infty] \to Y$ defined by extending $p\widetilde{F}$ via $F \times \mathrm{id}_{\{+\infty\}}$ is continuous at points of $Z \times I \times \{+\infty\}$. For this it suffices to consider a sequence $\{(z_i, s_i, t_i)\}_{i=1}^{\infty}$ in $Z \times I \times [0, +\infty)$ converging to $(z_0, s_0, +\infty) \in Z \times I \times \{+\infty\}$ and show that $\{p\widetilde{F}(z_i, s_i, t_i)\}_{i=1}^{\infty}$ converges to $F(z_0, s_0)$. To this end let $\varepsilon > 0$ be given. There exists an integer N such that it $i \geqslant N$, then $t_i \in [n, +\infty)$ for some $n \in \mathbb{Z}_+$ with $n \geqslant 1/\varepsilon$. Let $i \geqslant N$. Then $(p \times \mathrm{id}_{\mathbb{R}})F'(z_i, s_i, t_i)$ is α-close to $(F(z_i, s_i), t_i)$. Hence there exists $U \in \alpha$ such that $(p \times \mathrm{id}_{\mathbb{R}})F'(z_i, s_i, t_i), (F(z_i, s_i), t_i) \in U$. In particular, $U \cap [n, +\infty) \neq \emptyset$, so $\operatorname{diam} U < 1/n \leqslant \varepsilon$. Thus, $p_Y(p \times \mathrm{id}_{\mathbb{R}})F'(z_i, s_i, t_i)$ and $F(z_i, s_i)$ are ε-close. But $p_Y(p \times \mathrm{id}_{\mathbb{R}})F'(z_i, s_i, t_i) = pp_X F'(z_i, s_i, t_i) = p\widetilde{F}(z_i, s_i, t_i)$. Since $F(z_i, s_i) \to F(z_0, s_0)$, we may assume N is so large that $F(z_i, s_i)$ is ε-close to $F(z_0, s_0)$ for $i \geqslant N$. Thus, $p\widetilde{F}(z_i, s_i, t_i)$ is 2ε-close to $F(z_0, s_0)$ for $i \geqslant N$.

(C) (1) implies (2) follows from Proposition 4.4 above.

(C) (2) implies (1). Let

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & Y \end{tikzcd}$$

be a stratified homotopy lifting problem. There is an induced stratified homotopy lifting problem

$$\begin{tikzcd} Z \arrow[r, "i_X f"] \arrow[d, "\times 0"'] & X \times \mathbb{R} \arrow[d, "p\times \mathrm{id}_{\mathbb{R}}"] \\ Z \times I \arrow[r, "i_Y F"] & Y \times \mathbb{R}. \end{tikzcd}$$

Let β be the open cover of $\mathbb{R}$ consisting of all open intervals of length one. If $\widetilde{F} : Z \times I \to X \times \mathbb{R}$ is a stratified $(\alpha \times \beta)$-solution of the induced problem, then it is easy to check that $p_X \widetilde{F} : Z \times I \to X$ is a stratified α-solution to the original problem. $\square$

Remark 4.8. It is just a matter of definitions that any of the conditions in (A) imply those in (B), which in turn imply those of (C). Example 3.2 shows that (B) need not imply (A). I do not know an example which shows (C) need not imply (B), but I conjecture that such an example exists. See also Remark 3.3(6).

5. Lemmas on compactly dominated local holinks

This section contains a couple of technical results which will be used in the next section on adjunctions. The hardest aspect in the proof of the adjunction theorem is establishing the compactly dominated holinks condition and that is the purpose of the results of this section.

Lemma 5.1. *Let X be a homotopically stratified, locally compact metric space with only finitely many strata and suppose the strata are ANRs. Let x_0 be a point in a stratum X_i of X. The following are equivalent*:

(1) *X has a compactly dominated local holink at x_0.*
(2) *For every neighborhood U of x_0 there exist a neighborhood V of x_0 with $V \subseteq U$, a compact subset $K \subseteq U \setminus X_i$, and a stratum preserving deformation $g : (V \setminus X_i) \times I \to U$ such that $g_1(V \setminus X_i) \subseteq K$.*
(3) *For every neighborhood U of x_0 there exists a stratum preserving deformation $h : X \setminus (U \cap X_i) \times I \to X$ such that*
   (i) *$h_t|(X \setminus U) = \text{inclusion} : X \setminus U \to X$ for every $t \in I$,*
   (ii) *$h_t(U \setminus X_i) \subseteq U$ for every $t \in I$,*
   (iii) *there exists a neighborhood W of x_0 such that $W \subseteq U$ and $\mathrm{cl}\, h_1(W \setminus X_i) \cap X_i = \emptyset$.*

Proof. The proof that (1) and (2) are equivalent is in [7, 5.3] (although the condition $K \subseteq U \setminus X_i$ was mistakenly left out of the statement in [7]).

(2) implies (3). Let U be a neighborhood of x_0 and let V, K and g be as in (2). Let $\rho : X \to I$ be a map such that $x_0 \in \mathrm{int}(\rho^{-1}(1)) \subseteq \rho^{-1}(1) \subseteq V$ and $X \setminus V \subseteq \rho^{-1}(0)$. Define $h : X \setminus (U \cap X_i) \times I \to X$ by

$$h(x,t)=\begin{cases} x & \text{if } x\in X\setminus U,\\ g\big(x,\rho(x)\cdot t\big) & \text{if } x\in V\setminus X_i.\end{cases}$$

The conditions (i) and (ii) of (3) clearly hold. To verify condition (iii), let $W=\rho^{-1}(1)$. Then $h_1(W\setminus X_i)=g_1(W\setminus X_i)\subseteq K$, so $\mathrm{cl}(h_1(W\setminus X_i)\subseteq K\subseteq U\setminus X_i$.

(3) implies (2). Let U be a neighborhood of x_0. By local compactness, we may assume that U is compact. Let h and W be given by (3). Let $V=W$ and $g=h|(W\setminus X_i)\times I$. Then $g_1(V\setminus X_i)=h_1(W\setminus X_i)\subseteq \mathrm{cl}(h_1(W\setminus X_i))\subseteq U\setminus X_i$. Since U is compact, $K=\mathrm{cl}(h_1(W\setminus X_i))$ is also compact. □

Lemma 5.2. *Let X be a manifold stratified space with only finitely many strata, let $A\subseteq X$ be a compact pure subset of X, and let C be a closed subset of A. For every open neighborhood U of C in X there exist a neighborhood V of C in X with $V\subseteq U$, a compact subset $K\subseteq U\setminus A$ and a stratum preserving deformation $h:(V\setminus A)\times I\to U\setminus A$ such that $h_1(V\setminus A)\subseteq K$.*

Proof. There is a proof given in [8, Section 5] for the special case $C=A$. If that proof is examined, then one finds that the following is established (where d is a metric for X and $N_\delta(S)$ denotes the δ-neighborhood of a subset $S\subseteq X$):

> For every sufficiently small $\delta>0$ there is a compact subset $L\subseteq X\setminus A$ and a stratum preserving deformation
>
> $$g:\big(N_\delta(A)\setminus A\big)\times I\to X$$
>
> such that g is a δ-homotopy (i.e., $d(x,g(x,t))<\delta$ for each (x,t)) and the image of g is in L.

Given U choose $\delta>0$ so small that $N_{2\delta}(C)$ has compact closure contained in U. Then $V=N_\delta(C)$, $K=L\cap\mathrm{cl}(N_{2\delta}(C))$ and $g=h|(V\setminus A)\times I$ satisfy the conclusion. □

6. Adjunctions of manifold stratified spaces

This section contains the main result of this paper in Theorem 6.2 below. An application to subdividing manifold stratified spaces is given in Corollary 6.4. First a special case of the main result is established.

Lemma 6.1 (Attaching an external collar). *Let X be a manifold stratified space with only finitely many strata and let Y be a pure subset of X. The adjunction space*

$$Z=X\bigcup_{Y=Y\times\{0\}}Y\times[0,1)$$

obtained by attaching an external open collar to Y is a manifold stratified space with the natural stratification (the strata of Z are $\{X_i\mid X_i$ is a stratum of $X\}\cup\{X_i\times(0,1)\mid X_i$ is a stratum of $X, X_i\subseteq Y\}$).

Proof. The only condition in Definition 2.3 which is not immediately obvious is the Compactly Dominated Local Holinks condition for a point $x_0 \in Y = Y \times \{0\}$. We will verify Lemma 5.1(3). Let Y_i be the stratum of Y containing x_0 and let U be a neighborhood of x_0 in Z. We may assume that $U \cap (\mathrm{cl}(Y_i) \setminus Y_i) = \emptyset$. Let $U_X = U \cap X$ and $U_Y = U_X \cap Y$. We may assume that $U \cap (Y \times [0,1)) = U_Y \times [0, \varepsilon)$ for some $\varepsilon > 0$. Since the local holink of x_0 in X is compactly dominated it follows from Lemma 5.1(3) that there is a stratum preserving deformation $h : X \setminus (U_X \cap Y_i) \times I \to X$ such that

(1) $h_t|(X \setminus U_X) = \text{inclusion} : X \setminus U_X \to X$ for every $t \in I$,
(2) $h_t(U \setminus Y_i) \subseteq U$ for every $t \in I$,
(3) there exists a neighborhood W of x_0 in X such that $W \subseteq U_X$ and $\mathrm{cl}\, h_1(W \setminus Y_i) \cap Y_i = \emptyset$.

Let $\rho : Y \to [0, \varepsilon/2]$ be a map such that $\rho^{-1}(0) = \mathrm{cl}(Y_i)$. Define

$$g : \big([Y \times [0,1)] \setminus [(U_Y \cap Y_i) \times \{0\}]\big) \times I \to Y \times [0,1)$$

by

$$g(y,s,t) = \begin{cases} (y,s) & 0 \leqslant \rho(y) \leqslant s, \\ \big(h\big(y, \big(\rho(y) - s\big) \cdot t\big), s\big) & s \leqslant \rho(y) \leqslant 2s, \\ \big(h(y,t), s\big) & 2s \leqslant \rho(y) \leqslant 1. \end{cases}$$

Then h and g piece together to define a stratum preserving deformation

$$j : Z \setminus (U \cap Y_i) \times I \to Z.$$

Define $\sigma : Y \to [0, \varepsilon/2]$ such that $\sigma^{-1}(\varepsilon/2)$ is a neighborhood of x_0 in U_Y which misses $\mathrm{cl}\, h_1(W \setminus Y_i)$. Define

$$k : Y \times [0,1) \to Y \times [0,1)$$

by

$$k(y,s,t) = \begin{cases} \big(y, (1-t)s + t\sigma(y)\big) & 0 \leqslant s \leqslant \sigma(y), \\ (y,s) & \sigma(y) \leqslant s \leqslant 1. \end{cases}$$

Thus, k is a deformation which pushes up to the graph of σ and is stratum preserving when restricted to $Y \times (0,1) \times I$. Finally, define

$$H : Z \setminus (U \cap Y_i) \times I \to Z$$

by

$$H(z,t) = \begin{cases} j(z, 2t) & 0 \leqslant t \leqslant 1/2, \\ k\big(j(z,1), 2t-1\big) & 1/2 \leqslant t \leqslant 1,\ z \in Y \times [0,1), \\ j(z,1) & 1/2 \leqslant t \leqslant 1,\ z \in X. \end{cases}$$

Then H satisfies the conditions of Lemma 5.1(3). □

Theorem 6.2 (Adjunctions of manifold stratified spaces). *Let X and Y be manifold stratified spaces with only finitely many strata, let A be a pure subset of X and let $f : A \to Y$ be an MSAF. Then the adjunction space $W = X \cup_f Y$ obtained by attaching*

X to Y via f is a manifold stratified space with the natural stratification (*the strata of W are $\{X_i \mid X_i$ is a stratum of X, $X_i \cap A = \emptyset\} \cup \{Y_j \mid Y_j$ is a stratum of $Y\}$*).

Proof. For the Forward Tameness condition recall that A is stratified forward tame in X [5]. It follows that Y is stratified forward tame in W, from which the condition follows immediately.

For the Normal Fibrations condition, the only non-trivial case to check is the evaluation $q : \mathrm{holink}(X_i \cup Y_j, Y_j) \to Y_j$ where Y_j is a stratum of Y and X_i is a stratum of X such that $X_i \cap A = \emptyset$ and $\mathrm{cl}(X_i) \cap A \neq \emptyset$. It must be shown that q is a fibration. Recall from Corollary 3.5 that the mapping cylinder $\mathrm{cyl}(f)$ is a manifold stratified space with the natural stratification. We use the convention that $A \times \{0\}$ is the top of the cylinder and that $A \times \{1\}$ is identified with Y to form the bottom of the cylinder. Now form the space

$$Z = X \bigcup_{A = A \times \{0\}} \mathrm{cyl}(f)$$

by identifying $A \subseteq X$ with the top of $\mathrm{cyl}(f)$. It follows from Lemma 6.1 that Z is a manifold stratified space. Let

$$\begin{array}{ccc} B & \xrightarrow{\;g\;} & \mathrm{holink}(X_i \cup Y_j, Y_j) \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow{\;G\;} & Y_j \end{array} \tag{6.2.1}$$

be a homotopy lifting problem. There is an induced map $g^* : B \times (0,1] \to X_i \subseteq Z$ defined by $g^*(b,s) = g(b)(s) \in X_i$. Since A is stratified forward tame in X [5], there exists a neighborhood U of A in X and a nearly stratum preserving deformation $r : U \times I \to X$ of U to A in X (in particular, $r_0 =$ inclusion and $r_1(U) = A$). Because of the local nature of the problem we are trying to solve, we may assume that the image of g^* is contained in U [14]. Consider the homotopy lifting problem

$$\begin{array}{ccc} B & \xrightarrow{\;\hat{g}\;} & \mathrm{holink_s}(\mathrm{cyl}(f), Y) \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow{\;G\;} & Y \end{array} \tag{6.2.2}$$

where

$$\hat{g}(b)(t) = \begin{cases} \left[r_1 g^*(b,t), 1-t\right] \in \mathrm{cyl}(f) & \text{if } t > 0 \\ G(b,0) \in Y & \text{if } t = 0 \end{cases}$$

and $\mathrm{holink_s}$ denotes the stratified holink [5].

We must show that $\hat{g}$ is continuous. For this it suffices to show that $\hat{\hat{g}} : B \times I \to \mathrm{cyl}(f)$ defined by $\hat{\hat{g}}(b,t) = \hat{g}(b)(t)$ is continuous. Here are some auxiliary maps which will also be used below. The mapping cylinder collapse $\mathrm{cyl}(f) \to Y \times I$ is given by $c([z,t]) = (f(z), t)$ if $(z,t) \in A \times I$ and $c([z]) = (z,1)$ if $z \in Y$. The projection $\pi_Y : \mathrm{cyl}(f) \to Y$ is the composition

$$\pi_Y : \mathrm{cyl}(f) \xrightarrow{\;c\;} Y \times I \xrightarrow{\;\mathrm{proj}\;} Y.$$

Let $\rho : X \amalg Y \to W$ be the quotient map and let $\hat{r}_1 : U \cup_f Y \to Y$ be the retraction induced by r_1; that is, $\hat{r}_1 \circ \rho| : U \amalg Y \to Y$ is $fr_1 \amalg \mathrm{id}_Y$. We use the continuity criterion of [10] to verify that $\hat{\hat{g}}$ is continuous. First, $\hat{\hat{g}}| : (\hat{\hat{g}})^{-1}(\mathrm{cyl}(f) \setminus Y) \to \mathrm{cyl}(f) \setminus Y$ is obviously continuous. Second, the composition

$$B \times I \xrightarrow{\hat{\hat{g}}} \mathrm{cyl}(f) \xrightarrow{c} Y \times I \xrightarrow{\mathrm{proj}} I$$

is given by $(b,t) \mapsto 1 - t$ which is continuous. Third, the composition

$$B \times I \xrightarrow{\hat{\hat{g}}} \mathrm{cyl}(f) \xrightarrow{c} Y \times I \xrightarrow{\mathrm{proj}} Y$$

is given by

$$\begin{aligned}(b,t) \mapsto \begin{cases} \pi_Y\big([r_1 g^*(b,t), 1-t]\big) \\ \pi_Y G(b,0) \end{cases} &= \begin{cases} fr_1 g^*(b,t) \\ G(b,0) \end{cases} \\ &= \begin{cases} \hat{r}_1 \rho g^*(b,t) \\ \hat{\rho} G(b,0) \end{cases} = \begin{cases} g(b,t) & \text{if } t > 0, \\ G(b,0) & \text{if } t = 0. \end{cases}\end{aligned}$$

This is $(b,t) \mapsto g(b)(t)$ which is continuous.

Now because $q : \mathrm{holink}_s(\mathrm{cyl}(f), Y) \to Y$ is a stratified fibration [5], there exists a stratum preserving solution

$$H : B \times I \to \mathrm{holink}_s\big(\mathrm{cyl}(f), Y\big)$$

of the problem (6.2.2). Let $\pi_X : \mathrm{cyl}(f) \setminus Y \to X$ denote the natural projection $\pi_X([z,u]) = z$ where $[z,u] \in \mathrm{cyl}(f) \setminus Y$; that is, $z \in A$ and $u \in (0,1]$ (in words, π_X is the projection of $\mathrm{cyl}(f) \setminus Y$ to the top A followed by the inclusion of A into X). Define $\widehat{H} : B \times (0,1] \times I \to X$ by

$$\widehat{H}(b,s,t) = \pi_X\big(H(b,t)(s)\big).$$

Define $\tilde{g} : B \times (0,1] \to \mathrm{holink}_s(X, A)$ by

$$\tilde{g}(b,s)(t) = r_{(1-t)} g^*(b,s)$$

and note that there is a stratified homotopy lifting problem:

$$\begin{tikzcd} B \times (0,1] \arrow[r, "\tilde{g}"] \arrow[d, "\times 0"'] & \mathrm{holink}_s(X, A) \arrow[d, "q"] \\ B \times (0,1] \times I \arrow[r, "\widehat{H}"] & A \end{tikzcd} \tag{6.2.3}$$

Let $H^* : B \times (0,1] \times I \to \mathrm{holink}_s(X, A)$ be a stratum preserving solution of (6.2.3). Consider the commuting diagram

$$\begin{tikzcd} B \times I \arrow[r, "\beta"] \arrow[d, "\times 0"'] & W^I \arrow[d, "q"] \\ B \times I \times I \arrow[r, "\alpha"] & W \end{tikzcd} \tag{6.2.4}$$

where $\alpha(b,s,t) = \pi_Y(H(b,t)(s))$ and

$$\beta(b,s)(t) = \begin{cases} \rho\tilde{g}(b,s)(t) & \text{if } s > 0, \\ G(b,0) & \text{if } s = 0. \end{cases}$$

There is a "partial" solution $\gamma : B \times (0,1] \times I \to W^I$ of (6.2.4) defined by

$$\gamma(b,s,t)(u) = \rho\big(H^*(b,s,t)(u)\big).$$

It follows from [5, §8] that there exists a map $u : B \times (0,1] \times I \to (0,1]$ such that $u(B \times (0,1] \times \{0\}) = 1$ and $\widetilde{G} : B \times I \to W^I$ defined by

$$\widetilde{G}(b,s)(t) = \begin{cases} \gamma(b,t,s)\big(u(b,t,s)\big) & \text{if } t > 0, \\ G(b,s) & \text{if } t = 0, \end{cases}$$

is continuous. It is easy to check that $\widetilde{G}$ actually defines a solution of the original problem (6.2.1).

It only remains to show that W has compactly dominated local holinks. Let $y_0 \in Y$ and let Y_0 be the stratum of Y containing y_0. We are going to verify condition (2) of Lemma 5.1. To this end, let U'' be an open neighborhood of y_0 in W. Assume that U'' has compact closure and let U' be another open neighborhood of y_0 in W such that $\mathrm{cl}(U') \subseteq U''$. Let $U'_Y = U' \cap Y$. Apply condition (3) of Lemma 5.1 to $y_0 \in Y_0 \subseteq Y$ with U'_Y the given neighborhood of y_0. Thus, there exists a stratum preserving deformation

$$k : \big[Y \setminus \big(U'_Y \cap Y_0\big)\big] \times I \to Y$$

such that

(1) $k_t|(Y \setminus U'_Y) = \text{inclusion} : Y \setminus U'_Y \to Y$ for every $t \in I$,
(2) $k_t(U'_Y \setminus Y_0) \subseteq U'_Y$ for every $t \in I$,
(3) there exists a neighborhood T of y_0 in Y such that $\mathrm{cl}(T) \subseteq U'_Y$ and

$$\mathrm{cl}\, k_1(T \setminus Y_0) \cap Y_0 = \emptyset.$$

Moreover, we may assume that

(4) $y_0 \notin \mathrm{cl}(\mathrm{Image}(k_1))$.

To see why this last claim can be made, let $\tau : Y \to I$ be a map such that $\tau^{-1}(1)$ is a neighborhood of y_0, $\tau^{-1}(1) \subseteq T$ and $Y \setminus T \subseteq \tau^{-1}(0)$. Define a new deformation by $(x,t) \mapsto k(x, \tau(x) \cdot t)$. Then this deformation has all the properties of k above (with T replaced by $\tau^{-1}(1)$) as well as item (4).

We have already proved that W is a homotopically stratified metric space with only finitely many strata. Thus, $W \setminus Y_0$ is also such a space. We can use the Stratum Preserving Deformation Extension Theorem [5] to get a stratum preserving deformation

$$\tilde{k} : (W \setminus Y_0) \times I \to W \setminus \big(U' \cap Y_0\big)$$

such that

(1) $\tilde{k}|(Y \setminus Y_0) \times I = k|(Y \setminus Y_0)$.

In fact, since $U' \setminus Y_0$ is homotopically stratified and k is supported on $U' \setminus Y_0$, we may assume that

(ii) $\tilde{k}_t(U \setminus Y_0) \subseteq U'$ for all $t \in I$, and
(iii) $\tilde{k}_t|(W \setminus U') = \text{inclusion} : W \setminus U' \to W$ for all $t \in I$.

Let T' be an open set in W such that $T' \cap Y = T$ and $\mathrm{cl}(T') \subseteq U'$. Let S' be an open neighborhood of y_0 in W such that $\mathrm{cl}(S') \subseteq T'$ and

$$\mathrm{cl}\big(S'\big) \cap \mathrm{cl}\big(\mathrm{Image}(k_1)\big) = \emptyset.$$

Let $\pi : X \to W$ denote the restriction of the quotient map $\rho : X \amalg Y \to W$. Let $C = f^{-1}(Y_0 \cap \mathrm{cl}(U'_Y))$ and let $U = \pi^{-1}(U'')$. Then C is a compact subset of A and $C \subseteq U \subseteq X$. Thus we can apply Lemma 5.2 to find an open neighborhood V of C in X with $V \subseteq U$, a compact subset K of $U \setminus A$ and a stratum preserving deformation

$$g : (V \setminus A) \times I \to U \setminus A$$

such that $g_1(V \setminus A) \subseteq K$. It will be convenient to extend g to $X \setminus A$. To this end, let $\sigma : X \to I$ be a map such that $X \setminus V \subseteq \sigma^{-1}(0)$ and $\sigma^{-1}(1)$ is a neighborhood of C. Define $\tilde{g} : (X \setminus A) \times I \to X \setminus A$ by

$$\tilde{g}(x,t) = \begin{cases} g\big(x, \sigma(x) \cdot t\big) & \text{if } x \in V \setminus A \\ x & \text{if } x \in X \setminus (V \cup A). \end{cases}$$

Now $\tilde{g}$ induces a deformation of $W \setminus Y$,

$$g' : (W \setminus Y) \times I \to W \setminus Y,$$

defined by $g'(x,t) = \pi\tilde{g}(\pi^{-1}(x), t)$.

We need one more auxiliary map. Let $\beta : W \to I$ be a map such that:

(1) $\beta^{-1}(1)$ is a compact neighborhood of $Y_0 \cap \mathrm{cl}(U'_Y)$ in U'',

(2) $\mathrm{cl}(\beta^{-1}((0,1])) \subseteq U''$,

(3) $\mathrm{cl}(\beta^{-1}((0,1])) \cap \mathrm{cl}(\mathrm{Image}(k_1)) = \emptyset$.

We are finally in a position to define a deformation which will satisfy the conditions of Lemma 5.1(2). Define $h(S' \setminus Y_0) \times I \to U''$ by

$$h(x,t) = \begin{cases} \tilde{k}(x, 2t) & \text{if } 0 \leqslant t \leqslant 1/2 \text{ and } x \in S' \setminus Y_0, \\ \tilde{k}(x,1) & \text{if } 1/2 \leqslant t \leqslant 1 \text{ and } x \in (S' \cap Y) \setminus Y_0, \\ g'\big(\tilde{k}(x,1), (2t-1) \cdot \beta\big(\tilde{k}(x,1)\big)\big) & \text{if } 1/2 \leqslant t \leqslant 1 \text{ and } x \in S' \setminus Y. \end{cases}$$

In order to verify that h is continuous, the only questionable case concerns a point $(x_0, t_0) \in (S' \cap Y) \setminus Y_0 \times [1/2, 1]$ with a sequence $\{(x_n, t_n)\}_{n=1}^{\infty}$ in $(S' \setminus Y) \times I$ converging to (x_0, t_0). Then $h(x_0, t_0) = \tilde{k}(x_0, 1)$ and

$$h(x_n, t_n) = g'\big(\tilde{k}(x_n, 1), (2t_n - 1)\beta\big(\tilde{k}(x_n, 1)\big)\big).$$

Since $x_n \to x_0$, it follows that $\tilde{k}_1(x_n) \to \tilde{k}_1(x_0) = k_1(x_0)$. Since $k_1(x_0) \in \mathrm{Image}(k_1)$, it follows that $\tilde{k}_1(x_n) \notin \mathrm{cl}(\beta^{-1}((0,1]))$ for sufficiently large n; that is to say, $\beta(\tilde{k}(x_n, 1)) = 0$. Hence, $g'(\tilde{k}(x_n, 1)(2t_n - 1)\beta(\tilde{k}(x_n 1))) = \tilde{k}(x_n, 1)$ for sufficiently large n. Thus, $h(x_n, t_n) \to h(x_0, t_0)$ as desired.

To see that $h_1(S' \setminus Y_0)$ is contained in some compact subset of $U'' \setminus Y_0$, use the fact that

$$h_1\big(S' \setminus Y_0\big) = h_1\big(\big(S' \cap Y\big) \setminus Y_0\big) \cup h_1\big(S' \setminus Y\big).$$

Then note that $h_1((S' \cap Y) \setminus Y_0) = k_1((S' \cap Y) \setminus Y_0) \subseteq \mathrm{cl}(k_1(T \setminus Y_0))$ which is compact and misses Y_0. For $x \in S' \setminus Y$, then $h_1(x) = g'(\tilde{k}_1(x), \beta(\tilde{k}_1(x)))$. This can be used to finish the proof. $\square$

We now give an application to subdividing manifold stratified spaces.

Definition 6.3. Let X be a manifold stratified space with only finitely many strata and let A be a closed union of strata of X. Suppose A' is a manifold stratified space with only finitely many strata such that as spaces $A' = A$ and each stratum of A is a union of strata of A'. Let X' be the space with stratification obtained from X by replacing the strata of A by those of A' (i.e., $X' = X$; the strata of X' are the strata of $X \setminus A$ and the strata of A'). Then X' is called the *subdivision of X obtained by subdividing A into A'*.

Corollary 6.4 (Subdivision). *If X' is a subdivision of a manifold stratified space X, then X' is a manifold stratified space.*

Proof. If X' is obtained from X by subdividing A into A', then the identity map $\mathrm{id}: A \to A'$ is obviously an MSAF and $X' = X \cup_{\mathrm{id}} A'$. Hence, Proposition 6.2 applies. $\square$

7. Partial MSAFs, local MSAFs and restrictions of MSAFs over open subsets

This section contains some applications of the mapping cylinder characterization Theorem 3.4 and the adjunction Theorem 6.2 which will be used in [9]. Propositions 7.3 and 7.4 are stratified analogues of results known for approximate fibrations (cf. [2]).

Let X and Y be manifold stratified spaces with only finitely many strata, let $f: X \to Y$ be an MSAF and let $K \subseteq Y$ be a pure subset of Y. Define the quotient space $Y' = X/{\sim}$ obtained from the equivalence relation generated by $x \sim y$ if $f(x) = f(y) \in K$. The quotient map $q: X \to Y'$ is called a *partial manifold stratified approximate fibration*. Let $p: Y' \to Y$ be the unique map such that $f = p \circ q$.

Proposition 7.1. *Y' is a manifold stratified space with strata*

$$\{p^{-1}(S) \mid S \text{ is a stratum of } Y\}$$

and the partial manifold stratified approximate fibration $q: X \to Y'$ is a manifold stratified approximate fibration.

Proof. Since Y' is the adjunction space

$$X \cup_{f|: f^{-1}(K) \to K} K,$$

it follows from Proposition 6.2 that Y' is a manifold stratified space. According to Theorem 3.4 $\mathrm{cyl}(f)$ is homotopically stratified and it suffices to show that $\mathrm{cyl}(q)$ is a homotopically stratified. But pairs of strata in $\mathrm{cyl}(q)$ for which the Forward Tameness and Normal Fibrations conditions need to be checked correspond to pairs of strata in $\mathrm{cyl}(f)$ or in $\mathrm{cyl}(\mathrm{id}_X)$ for which the conditions hold. $\square$

Here is a closely related result.

Proposition 7.2 (Unions of MSAFs). *Let X and Y be manifold stratified spaces written as the union of pure subsets, $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. If $p : X \to Y$ is a proper map such that $p(X_i) = Y_i$ and $p| : X_i \to Y_i$ is an MSAF for $i = 1, 2$, then p is an MSAF.*

Proof. As in the proof of Proposition 7.1, the verification is easily made using the characterization in Theorem 3.4. □

Proposition 7.3. *Let $p : X \to Y$ be a proper map between manifold stratified spaces with only finitely many strata. If there exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$, $p| : p^{-1}(U) \to U$ is an MSAF, then p is an MSAF.*

Proof. According to Theorem 3.4 it suffices to verify that the mapping cylinder $\mathrm{cyl}(p)$ is homotopically stratified. But Theorem 3.4 also implies that each mapping cylinder $\mathrm{cyl}(p|p^{-1}(U))$ is homotopically stratified. Since these restricted mapping cylinders form an open cover of $\mathrm{cyl}(p)$ and the property of being homotopically stratified is a local one [14], the result follows. □

Proposition 7.4. *If $p : X \to Y$ is an MSAF between manifold stratified spaces with only finitely many strata and U is an open subset of Y, then $p| : p^{-1}(U) \to U$ is also an MSAF.*

Proof. According to Theorem 3.4 $\mathrm{cyl}(p)$ is homotopically stratified and it suffices to verify that $\mathrm{cyl}(p|p^{-1}(U))$ is homotopically stratified. But $\mathrm{cyl}(p|p^{-1}(U))$ is an open subset of $\mathrm{cyl}(p)$ so the result follows. □

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