Surgery and stratified spaces

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0. Introduction

The past couple of decades has seen significant progress in the theory of stratified spaces through the application of controlled methods as well as through the applications of intersection homology. In this paper we will give a cursory introduction to this material, hopefully whetting your appetite to peruse more thorough accounts.

In more detail, the contents of this paper are as follows: the first section deals with some examples of stratified spaces and describes some of the different categories that have been considered by various authors. For the purposes of this paper, we will work in either the PL category or a very natural topological category introduced by Quinn [Q4]. The next section discusses intersection homology and how it provides one with a rich collection of self dual sheaves. These can be manipulated by ideas long familiar to surgery theorists who have exploited Poincaré duality from the start. We will give a few applications of the tight connection between an important class of stratified spaces (Witt spaces), self dual sheaves, and K-theory; one last application will appear in the final section of the paper (where we deal with the classification of "supernormal" spaces with only even codimensional strata).

Section three begins an independent direction, more purely geometric. We describe the local structure of topological stratified spaces in some detail, in particular explaining the teardrop neighborhood theorem ([HTWW], [H2]) and giving applications to isotopy theorems and the like. The last three sections describe the theory of surgery on stratified spaces, building on our understanding of teardrop neighborhoods, and some applications to classification problems (other applications can also be found in the survey [CW4]).

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1. Definitions and Examples of Stratified Spaces

A *stratification* $\Sigma = \{X_i\}$ of a space X is a locally finite decomposition of X into pairwise disjoint, locally closed subsets of X such that each $X_i \in \Sigma$ is a topological manifold. We always assume that X is a locally compact, separable metric space and that Σ satisfies the *Frontier Condition*: $\operatorname{cl} X_i \cap X_j \neq \emptyset$ if and only if $X_j \subseteq \operatorname{cl} X_i$. The index set is then partially ordered by $j \leq i$ if and only if $X_j \subseteq \operatorname{cl} X_i$. The set $X_i \in \Sigma$ is called a *stratum* and $X^i = \operatorname{cl} X_i = \cup\{X_j \mid j \leq i\}$ is a *skeleton* (or *closed stratum* in the terminology of [W4]).

Partitioning non-manifold spaces into manifold pieces is a very old idea — one has only to consider polyhedra in which the strata are differences between successive skeleta. However, it was not until relatively recently that attention was paid to how the strata should fit together, or to the geometry of the neighborhoods of strata. In 1962 Thom [T1] discussed stratifications in which the strata have neighborhoods which fibre over that stratum and which have "tapis" maps (the precursor to the tubular maps in Mather's formulation in 1.2 below). It was also in this paper that Thom conjectured that the topologically stable maps between two smooth manifolds are dense in the space of all smooth maps. In fact, it was Thom's program for attacking that conjecture which led him to a study of stratifications [T2]. The connection between stratifications and topological stability (and, more generally, the theory of singularities of smooth maps) is outside the scope of this paper, but the connections have continued to develop (for a recent account, see the book of du Plessis and Wall [dPW].)

Here we review the major formulations of the conditions on neighborhoods of strata. These are due to Whitney, Mather, Browder and Quinn, Siebenmann, and Quinn. The approaches of Whitney, Mather, Browder and Quinn are closely related to Thom's original ideas. These types of stratifications are referred to as *geometric stratifications*. The approaches of Siebenmann and Quinn are attempts at finding an appropriate topological setting.

1.1 Whitney stratifications. In two fundamental papers [Wh1],[Wh2], Whitney clarified some of Thom's ideas on stratifications and introduced his Conditions A and B. To motivate these conditions consider a real algebraic set $V \subseteq \mathbb{R}^n$, the common locus of finitely many real polynomials. The singular set ΣV of all points where V fails to be a smooth manifold is also an algebraic set. There is then a finite filtration $V = V^m \supseteq V^{m-1} \supseteq \cdots \supseteq V^0 \supseteq V^{-1} = \emptyset$ with $V^{i-1} = \Sigma V^i$. One obtains a stratification of V by considering the strata $V_i = V^i \setminus V^{i-1}$. However, with this naive construction the strata need not have geometrically well-behaved neighborhoods; that is, the local topological type need not be locally constant

along strata. For example, consider the famous Whitney umbrella which is the locus of $x^2 = zy^2$, an algebraic set in $\mathbb{R}^3$. The singular set ΣV is the z-axis and is a smooth manifold, so one obtains just two strata, $V \setminus \Sigma V$ and ΣV. However, there is a drastic change in the neighborhood of ΣV in V as one passes through the origin: for negative z small neighborhoods meet only ΣV whereas this is not the case for positive z.

If X, Y are smooth submanifolds of a smooth manifold M, then X is *Whitney regular over* Y if whenever $x_i \in X$, $y_i \in Y$ are sequence of points converging to some $y \in Y$, the lines $l_i = \overline{x_i y_i}$ converge to a line l, and the tangent spaces $T_{x_i}X$ converge to a space τ, then

(A) $T_y Y \subseteq \tau$ and
(B) $l \subseteq \tau$.

A stratification $\Sigma = \{X_i\}$ of X is a *Whitney stratification* if whenever $j \leq i$, X_i is Whitney regular over X_j.

In the Whitney umbrella V, $V \setminus \Sigma V$ is not Whitney regular over ΣV at the origin. However, the stratification can be modified to give a Whitney stratification and a similar construction works for a class of spaces more general than algebraic sets: a subset $V \subseteq \mathbb{R}^n$ is a *semi-algebraic set* if it is a finite union of sets which are the common solutions of finitely many polynomial equations and inequalities. Examples include real algebraic sets and polyhedra. In fact, the class of semi-algebraic sets is the smallest class of euclidean subsets containing the real algebraic sets and which is closed under images of linear projections. If V is semi-algebraic, then there is a finite filtration as in the case of an algebraic set discussed above obtained by considering iterated singular sets. This filtration can be modified by removing from the strata the closure of the set of points where the Whitney conditions fail to hold. In this way, semi-algebraic sets are given Whitney stratifications (see [GWdPL]).

In fact, Whitney [Wh2] showed that any real or complex analytic set admits a Whitney stratification. These are sets defined analogously to algebraic sets with analytic functions used instead of polynomials. Łojasiewicz [Lo] then showed that semi-analytic sets (the analytic analogue of semi-algebraic sets) have Whitney stratifications. An even more general class of spaces, namely the *subanalytic sets*, were shown by Hardt [Hr] to admit Whitney stratifications. For a modern and thorough discussion of stratifications for semi-algebraic and subanalytic sets see Shiota [Shi].

1.2 Mather stratifications: tube systems. Mather clarified many of the ideas of Thom and Whitney and gave complete proofs of the isotopy lemmas of Thom. He worked with a definition of stratifications closer to Thom's original ideas than to Whitney's, but then proved that spaces with Whitney stratifications are stratified in his sense.

Definition. For $0 \leq k \leq +\infty$, a *Mather C^k-stratification* of X is a triple $(X, \Sigma, \mathbf{T})$ such the following hold:

(1) $\Sigma = \{X_i\}$ is a stratification of X such that each stratum $X_i \in \Sigma$ is a C^k-manifold.
(2) $\mathbf{T} = \{T_i, \pi_i, \rho_i\}$ is called a *tube system* and T_i is an open neighborhood of X_i in X, called the *tubular neighborhood* of X_i, $\pi_i : T_i \to X_i$ is a retraction, called the *local retraction* of T_i, and $\rho_i : T_i \to [0, \infty)$ is a map such that $\rho_i^{-1}(0) = X_i$.
(3) For each $X_i, X_j \in \Sigma$, if $T_{ij} = T_i \cap X_j$ and the restrictions of π_i and ρ_i to T_{ij} are denoted π_{ij} and ρ_{ij}, respectively, then the map

$$(\pi_{ij}, \rho_{ij}) : T_{ij} \to X_i \times (0, \infty)$$

is a C^k-submersion.
(4) If $X_i, X_j, X_k \in \Sigma$, then the following compatibility conditions hold for $x \in T_{jk} \cap T_{ik} \cap \pi_{jk}^{-1}(T_{ij})$:

$$\pi_{ij} \circ \pi_{jk}(x) = \pi_{ik}(x),$$

$$\rho_{ij} \circ \pi_{jk}(x) = \rho_{ik}(x).$$

When $k = 0$ above, a C^0-submersion, or topological submersion, means every point in the domain has a neighborhood in which the map is topologically equivalent to a projection (see [S2]).

Mather [Ma1], [Ma2] proved that Whitney stratified spaces have Mather C^∞-stratifications.

The Thom isotopy lemmas mentioned above are closely related to the geometric structure of neighborhoods of strata. For example, the first isotopy lemma says that if $f : X \to Y$ is a proper map between Whitney stratified spaces with the property that for every stratum X_i of X there exists a stratum Y_j of Y such that $f| : X_i \to Y_j$ is a smooth submersion, then f is a fibre bundle projection (topologically — not smoothly!) and has local trivializations which preserve the strata. Mather applied this to the *tubular maps*

$$\pi_i \times \rho_i : T_i \to X_i \times [0, \infty)$$

defined on the tubular neighborhoods of the strata of a Whitney stratified space X in order to show that every stratum X_i has a neighborhood N such that the pair (N, X_i) is homeomorphic to $(\mathrm{cyl}(f), X_i)$ where $\mathrm{cyl}(f)$ is the mapping cylinder of some fibre bundle projection $f : A \to X_i$. The existence of these mapping cylinder neighborhoods was abstracted by Browder and Quinn as is seen next.

1.3 Browder-Quinn stratifications: mapping cylinder neighborhoods. In order to classify stratified spaces Browder and Quinn [BQ] isolated the mapping cylinder structure as formulated by Mather. The mapping cylinder was then part of the data that was to be classified. More will be said about this in §4 below. Here we recall their definition.

Let $\Sigma = \{X_i\}$ be a stratification of a space X such that each stratum X_i is a C^k-manifold. The *singular set* ΣX_i is $\operatorname{cl} X_i \setminus X_i = \cup\{X_j \mid j < i\}$. (This is in general bigger than the singular set as defined in 1.1.)

Definition. Σ is a C^k *geometric stratification* of X if for every i there is a closed neighborhood N_i of ΣX_i in $X^i = \operatorname{cl} X_i$ and a map $\nu_i : \partial N_i \to \Sigma X_i$ such that

(1) ∂N_i is a codimension 0 submanifold of X_i,
(2) N_i is the mapping cylinder of ν_i (with ∂N_i and ΣN_i corresponding to the top and bottom of the cylinder),
(3) if $j < i$ and $W_j = X_j \setminus \operatorname{int} N_j$, then $\nu_i| : \nu_i^{-1}(W_j) \to W_j$ is a C^k-submersion.

The complement of $\operatorname{int} N_i$ in X^i is called a *closed pure stratum* and is denote $\overline{X}^i$.

Note this definition incorporates a topological theory by taking $k = 0$. Browder and Quinn also pointed out that by relaxing the condition on the maps v_i other theories can be considered. For example, one can insist that the strata be PL manifolds and the v_i be PL block bundles with manifold fibers.

1.4 Siebenmann stratifications: local cones. In the late 1960s Cernavskii [Ce] developed intricate geometric techniques for deforming homeomorphisms of topological manifolds. In particular, he proved that the group of self homeomorphisms of a compact manifold is locally contractible by showing that two sufficiently close homeomorphisms are canonically isotopic. The result was reproved by Edwards and Kirby [EK] by use of Kirby's torus trick. Siebenmann [S2] developed the technique further in order to establish the local contractibility of homeomorphism groups for certain nonmanifolds, especially, compact polyhedra.

Siebenmann's technique applied most naturally to stratified spaces and a secondary aim of [S2] was to introduce a class of stratified spaces that he thought might "come to be the topological analogues of polyhedra in the piecewise-linear realm or of Thom's stratified sets in the differentiable realm." These are the *locally conelike TOP stratified sets* whose defining property is that strata are topological manifolds and for each point x in the n–stratum there is a compact locally conelike TOP stratified set L (with fewer strata — the definition is inductive) and a stratum-preserving

homeomorphism of $\mathbb{R}^n \times \mathring{c}L$ onto an open neighborhood of x where $\mathring{c}L$ denotes the open cone on L and the homeomorphism takes $0 \times \{\text{vertex}\}$ to x. Simple examples include polyhedra and the topological (C^0) versions

It is important to realize that Siebenmann didn't just take the topological version of Mather's stratified space, but he did have Mather's C^0-tubular maps *locally* at each point. The reason he was able to work in this generality was that the techniques for proving local contractibility of homeomorphism groups were purely local.

As an example, consider a pair (M, N) of topological manifolds with N a locally flat submanifold of M. With the two strata, N and $M \setminus N$, the local flatness verifies that this is a locally conelike stratification. However, Rourke and Sanderson [RS] showed that N need not have a neighborhood given by the mapping cylinder of a fibre bundle projection. Thus, Siebenmann's class is definitely larger than the topological version of the Thom-Whitney-Mather class.

On the other hand, Edwards [E] did establish that locally flat submanifolds of high dimensional topological manifolds do, in general, have mapping cylinder neighborhoods. However, the maps to the submanifold giving the mapping cylinder need not be a fibre bundle projection. It turns out that the map is a *manifold approximate fibration*, a type of map which figures prominently in the discussion of the geometry of homotopically stratified spaces below.

Later, Quinn [Q2,II] and Steinberger and West [StW] gave examples of locally conelike TOP stratified sets in which the strata do not have mapping cylinder neighborhoods of any kind. In fact, their examples are orbit spaces of finite groups acting locally linearly on topological manifolds. Such orbit spaces are an important source of examples of locally conelike stratified sets and many of advances in the theory of stratified spaces were made with applications to locally linear actions in mind. These examples were preceded by an example mentioned by Siebenmann [S3] of a locally triangulable non-triangulable space.

Milnor's counterexamples to the Hauptvermutung [M1] give non-homeomorphic polyhedra whose open cones are homeomorphic. As Siebenmann observed, these show that the links in locally conelike stratified sets are not unique up to homeomorphism. Siebenmann does prove that the links are stably homeomorphic after crossing with a euclidean space of the dimension of the stratum plus 1. The non-uniqueness of links points to the fact that Siebenmann's stratified spaces are too rigid to really be the topological analogue of polyhedra and smoothly stratified sets, whereas the stable uniqueness foreshadows the uniqueness up to controlled homeomorphism of fibre germs of manifold approximate fibrations [HTW1].

The main applications obtained by Siebenmann, namely local contracti-

bility of homeomorphism groups, isotopy extension theorems, and the fact that many proper submersions are bundle projections, can all be generalized to the setting of homotopically stratified sets discussed below.

Siebenmann himself experimented with a less rigid class of stratified spaces, called *locally weakly conelike*. In order to include in this class stratified spaces with isolated singularities which arise as the one-point compactifications of manifolds with nonvanishing Siebenmann obstruction [S], he no longer required the existence of links. However, neighborhoods around strata of dimension n were still required to split off a factor of $\mathbb{R}^n$ locally. In other words, in a locally conelike set X a point in the n-dimensional stratum X_n has a neighborhood U in X with $U \setminus X_n$ homeomorphic to $L \times \mathbb{R}^{n+1}$ with L the compact link. In a weakly conelike set, $U \setminus X_n$ is homeomorphic to $C \times \mathbb{R}^n$ with C a noncompact space with a certain tameness property at infinity. While this generalization was a move in the right direction, the role of the weak link C prevented further developments and it was left to Quinn to make a bolder generalization.

1.5 Quinn stratifications: homotopy mapping cylinders. Quinn [Q5] introduced a class of spaces which we will call *manifold homotopically stratified sets*. His objective was to "give a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds." As has been pointed out above, the previously defined topologically stratified spaces were inadequate. On one hand, the geometrically stratified spaces (that is, the topological version of Thom's spaces as formulated by Mather or Browder and Quinn) require too strong of a condition on neighborhoods of strata (namely, mapping cylinder neighborhoods) ruling out important examples (like locally flat submanifolds and orbit spaces of locally linear group actions). On the other hand, the locally conelike stratified sets of Siebenmann require a very strong local condition which need not propagate to the entire neighborhood of the strata. Without an understanding of the geometry of neighborhoods of strata, topological stratified versions of surgery, transversality, and h-cobordism theorems were missing.

Quinn formulated his definition to be equivalent to saying that for $j < i$, $X_i \cup X_j$ is homotopy equivalent near X_j to the mapping cylinder of some fibration over X_j. This has two pleasant properties. First, besides the geometric condition that the strata be manifolds, the definition is giving in homotopy theoretic terms. Second, the condition concerns neighborhoods of strata rather than closed strata, so that, in particular, there are no complicated compatibility conditions where more than two strata meet. The links are now defined only up to homotopy.

Even without a geometric condition on neighborhoods of strata, Quinn was able to derive geometric results. These will be discussed in §3 below

along with a theorem of Hughes, Taylor, Weinberger and Williams which says that neighborhoods of strata do carry a weak geometric structure. One thing that Quinn did not do was to develop a surgery theory for these manifold homotopically stratified sets. That piece of the picture was completed by Weinberger [W4] (see §5 below).

1.6 Group actions. Suppose that G is a finite group acting on a topological manifold M. One attempts to study the action by studying the orbit space M/G and the map $M \to M/G$. For example, if G acts freely, then M/G is a manifold and $M \to M/G$ is a covering projection. Moreover, the surgery theoretic set of equivariant manifold structures on M is in 1-1 correspondence with the set of manifold structures on M/G via the pull-back construction.

When the action is not free, M/G must be viewed as a space with singularities and $M \to M/G$ as a collection of covering projections. The prototypical example occurs when M is a closed Riemann surface and G is a finite cyclic group acting analytically. Then $M \to M/G$ is a *branched covering*.

More generally, if M is a smooth manifold and G acts differentiably, then M has a Whitney stratification with the strata $M_{(H)}$ indexed by conjugacy classes of subgroups of G and consisting of all points with isotropy subgroup conjugate to H. This induces a Whitney stratification of M/G. The standard reference is Lellmann [Le], but Dovermann and Schultz [DS] provide a more accessible proof. In the more general setting of a compact Lie group G, Davis [Dv1] showed how to view $M \to M/G$ as a collection of fibre bundle projections based on the fact that each $M_{(H)} \to M_{(H)}/G$ is a smooth fibre bundle projection with fibre G/H.

Now if the action of the finite group G on the topological manifold M is locally linear (also called locally smooth), then the examples of Quinn and Steinberger and West show (as mentioned above) that M/G need not have a geometric stratification, but it is a locally conelike TOP stratified set, and so Siebenmann's results can be applied. Lashof and Rothenberg [LR] used stratification theory of the orbit space to classify equivariant smoothings of locally smooth G-manifolds. Hsiang and Pardon [HsP] and Madsen and Rothenberg [MR] used stratifications for the classification of linear representations up to homeomorphism (see also [CSSW], [CSSWW], [HP]). Stratifications also played an important role in the work of Steinberger and West [StW] on equivariant s-cobordism theorems and equivariant finiteness obstructions.

The stratification theory of the orbit space actually corresponds with the isovariant, rather than the equivariant, theory of the manifold.

Locally linear actions on topological manifolds have the property that

fixed sets are locally flat submanifolds. It is natural to consider all such actions. These are essentially the actions whose orbit space is a manifold homotopically stratified set. After being introduced by Quinn [Q5], Yan [Y] applied Weinberger's stratified surgery (see §5 below) to study equivariant periodicity. More recently, Beshears [Bs] made precise the properties of the map $M \to M/G$ and proved that the isovariant structures on M are in 1-1 correspondence with the stratum preserving structures on M/G.

1.7 Mapping cylinders. Mapping cylinders provide examples of spaces with singularities. The mapping cylinder $\mathrm{cyl}(p)$ of a map $p : M \to N$ between manifolds has a natural stratification with three strata: the top M, the bottom N and the space in between $M \times (0,1)$. The properties of the stratification depend on the map p. With this stratification $\mathrm{cyl}(p)$ is geometrically stratified if and only if $p \times \mathrm{id}_{\mathbb{R}}$ can be approximated arbitrarily closely by fibre fibre bundle projections. On the other hand, $\mathrm{cyl}(p)$ is a manifold homotopically stratified set if and only if p is a manifold approximate fibration. The cylinder is nonsingular; i.e., a manifold with N a locally flat submanifold if and only if p is a manifold approximate fibration with spherical homotopy fibre. (Here and elsewhere in this section, we ignore problems with low dimensional strata.)

More generally, one can consider the mapping cylinder of a map $p : X \to Y$ between stratified spaces which take each stratum of X into a stratum of Y. The natural collection of strata of $\mathrm{cyl}(p)$ contains the strata of X and Y. Cappell and Shaneson [ChS4] observed that even if one considers maps between smoothly stratified spaces which are smooth submersions over each stratum of X, then $\mathrm{cyl}(p)$ need not be smoothly stratified (they refer to an example of Thom [T1]). However, Cappell and Shaneson [CS5] proved that such cylinders are manifold homotopically stratified sets, showing that the stratifications of Quinn arise naturally in the theory of smoothly stratified spaces.

Even more generally, the mapping cylinder $\mathrm{cyl}(p)$ of a stratum preserving map between manifold homotopically stratified sets is itself a manifold homotopically stratified set (with the natural stratification) if and only if p is a manifold stratified approximate fibration [H2].

2. Intersection Homology and Surgery Theory

In the mid 70's Cheeger and Goresky-MacPherson, independently and by entirely different methods, discovered that there is a much larger class of spaces than manifolds that can be assigned a sequence of "homology groups" that satisfy Poincaré duality. Given the central role that Poincaré duality plays in surgery theory, it was inevitable that this would lead to a new environment for the applications of surgery.

2.1. Let X be a stratified space where $X^i \setminus X^{i-1}$ is an i-dimensional F-homology manifold, for a field F. We shall assume that the codimension one stratum is of codimension at least two and that $X \setminus X^{n-1}$ (the nonsingular part) is given an F-orientation; for simplicity we will also mainly be concerned with the case of $F = \mathbb{Q}$. It pays to think PL, as we shall, but see [Q3] for an extension to homotopically stratified sets. A *perversity* p is a nondecreasing function from the natural numbers to the nonnegative integers, with $p(1) = p(2) = 0$, and for each i, $p(i+1) \leq p(i) + 1$. The *zero perversity* is the identically 0 function and the *total perversity* t has $t(i) = i - 2$ for $i \geq 2$. Two perversities, p and q are *dual* if $p + q = t$.

We say that X is *normal* if the link of any simplex of codimension larger than 1 is connected. This terminology is borrowed from algebraic geometry. It is not hard to "normalize" "abnormal" spaces by an analogue of the construction of the orientation two-fold cover of a manifold.

A *chain* is just what it always was in singular homology: we say it is *p-transverse*, or *p-allowable*, if for every simplex in the chain $\Delta \cap X^{n-i}$ has dimension at most i larger than what would be predicted by transversality and the same is true for the simplices in its boundary that have nonzero coefficient.

Note: It is not always the case that the chain complex of p-transverse chains with coefficients in $\mathbb{R}$ is just the complex for $\mathbb{Z}$ tensored with $\mathbb{R}$, as it would be in ordinary homology, because a non-allowable chain can become allowable after tensoring when some simplex in the boundary gets a 0-coefficient.

2.2. $IH_p(X)$ is the homology obtained by considering p-allowable chains. It is classical for normal spaces that IH^t is just ordinary homology; a bit more amusing is the theorem of McCrory that IH^0 is cohomology in the dual dimension. The forgetful map is capping with the fundamental class.

Note that IH is **not** set up to be a functor. It turns out to be functorial with respect to *normally nonsingular* or (*homotopy*) *transverse* maps. (We'll discuss these in a great deal more details in §§4,5.) Thus, it **is** functorial with respect to (PL) homeomorphisms and inclusions of open subsets and collared boundaries.

Note also, that one can give "cellular" versions of IH, which means that one can define perverse finiteness obstructions and Reidemeister and Whitehead torsions in suitable circumstances. (See [Cu, Dr].) Here one would usually want to build in refinements to integer coefficients that we will not discuss till 2.10 below.

2.3. The main theorems of [GM1] are that (1) IH is stratification independent (indeed it is a topological invariant, even a stratified homotopy invariant) and (2) for dual perversities the groups in dual dimensions are

dual. The latter boils down to Poincaré duality in case X is a manifold, however it is much more general.

2.4. What is important in many applications is that one can often get a self duality. Unfortunately, there is no self dual perversity function (what should $p(3)$ equal?). However, we have two *middle perversities* $0,0,1,1,2,2,\dots$ and $0,0,0,1,1,2\dots$; note that these differ only on the condition of intersections with odd codimensional strata. Consequently, for spaces with only even codimensional strata, the middle intersection homology groups are self dual.

2.5. It turns out that the middle perversity groups have many other amazing properties. Cheeger independently discovered the "De Rham" version of these. He gave a polyhedral X as above a piecewise flat metric (i.e. flat on the simplices, and conelike) and observed that the L^2 cohomology of the incomplete manifold obtained by removing the singular set was very nice. Under a condition that easily holds when one has even codimensional strata, the $*$ operator takes L^2 forms to themselves, and one formally obtains Poincaré duality. A consequence of this is that the **Künneth formula holds**.

In addition, Goresky and MacPherson [GM3] proved that Morse theory takes a very nice form for stratified spaces when you use intersection homology. This leads to a proof of the Lefschetz hyperplane section theorem. (A sheaf theoretic proof appears in [GM2].) [BBD] proved hard Lefschetz for the middle perversity intersection homology of a singular variety using the methods of characteristic p algebraic geometry. This requires the sheaf theoretic reformulation to be discussed below. Finally Saito [Sa] gave an analytic proof of this and a Hodge decomposition for these groups.

2.6. Let us return to pure topology by way of example. Consider a manifold with boundary $W, \partial W$, and the singular space obtained by attaching a cone to ∂W. Normality would correspond to the assumption that ∂W is connected.

What are the intersection homology groups in this case? Fix p. We would ordinarily not expect any chain of dimension less than n to go through the cone point. Once $i + p(i)$ is at least n, we begin allowing all chains to now go through the cone point, so one gets above that dimension all of the reduced homology. Below that dimension, we are insisting that our chains miss the cone point, so one gets $H_*(W)$. There is just one critical dimension where the chain can go through and the boundary cannot: here one gets the image of the ordinary homology in the reduced homology.

Using these calculations, one can reduce the Goresky-MacPherson duality theorem to Poincaré-Lefschetz duality for the manifold with boundary.

If the dimension of W is even, one gets in the middle dimension (for the middle perversity) the usual intersection pairing on $(W, \partial W)$ modulo its torsion elements.

Note though that if W is odd dimensional the failure of self duality is caused by the middle dimensional homology of ∂W. If its homology vanished, we'd still get Poincaré duality.

2.7. Of course, one immediately realizes that one can now define signatures for spaces with even codimensional singularities (that lie in the Witt group $W(F)$ of the ground field.) We'll, for now, only pay attention to $F = \mathbb{R}$ and ordinary signature.

Thom and Milnor's work on PL L-classes and Sullivan's work on $KO[\frac{1}{2}]$ orientations for PL manifolds all just depend on a cobordism invariant notion of signature that is multiplicative with respect to products with closed smooth manifolds. Thus, as observed in [GM I] it is possible to define such invariants lying in ordinary homology and $KO[\frac{1}{2}]$ of any space with even codimensional strata.

2.8. It is very natural to sheafify. Nothing prevents us from considering the intersection homology of open subsets and one sees that for each open set one has duality between locally finite homology and cohomology. It turns out that the usual algebraic apparatus of surgery theory mainly requires self dual sheaves rather than manifolds. So we can define symmetric signatures that take the fundamental group into account, which are just the assemblies (in the sense of assembly maps) of the classes in 2.7.

2.9. The original motivations to sheafify were rather different. Firstly, using sheaf theory there are simple Eilenberg-Steenrod type axioms that can be used to characterize IH; these are useful for calculational purposes and for things like identifying the Cheeger description with the geometric definition of Goresky and MacPherson.

Secondly, using various constructions in the derived category of sheaves, push forwards and proper push forwards and truncations of various sorts, it is possible to give a direct abstract definition of IH without using chains. This definition is appropriate to characteristic p algebraic varieties.

Finally, there is a very simple sheaf theoretic statement, Verdier duality, that can be used to express locally the self duality of the intersection homology of all open subsets of a given X. It says that IC^m is a self-dual sheaf for spaces with even codimensional singularities. We will see below that this is quite a powerful statement.

2.10. We can ask for which spaces is IC self dual? We know that all spaces with even codimensional strata have this property, but they are not all of them, for we saw in 2.6 that if we have an isolated point of odd

codimension one still gets Poincaré duality in middle perversity IH if (and only if) the middle dimensional homology of the link – which is a manifold – vanishes. One can generalize this observation to see that if the link of each simplex of odd codimension in X has vanishing middle IH^m, then IC is self dual on X. (Indeed this condition is necessary and sufficient.) Such X's were christened by Siegel [Si], *Witt spaces*. Actually they were introduced somewhat earlier by Cheeger as being the set of spaces for which the $*$ operator on L^2 forms on the nonsingular part behaves properly.

The main point of Siegel's thesis, though, was to compute the bordism of Witt spaces. Obviously the odd dimensional bordism groups vanish, because the cone on an odd dimensional Witt space is a Witt nullcobordism. For even dimensional Witt spaces this only works if there is no middle dimensional IH^m. By a surgery process on middle dimensional cycles, he shows that you can cobord a Witt space to one of those if and only if the quadratic form in middle $IH^m(\ ;\mathbb{Q})$ is hyperbolic – so there is no obstruction in 2 mod 4, but there's an obstruction in $W(\mathbb{Q})$ in 0 mod 4. Moreover, aside from dimension 0, where all that can arise is $\mathbb{Z} \subseteq W(\mathbb{Q})$ given by signature, in all other multiples of 4 all the other (exponent 4 torsion, computed in [MH]) elements can be explicitly constructed, essentially by plumbing. The isomorphism of the bordism with $W(\mathbb{Q})$ is what gives these spaces their name.

However, making use of the natural transformations discussed above, we actually see that Witt spaces form a nice cycle theory for the (connective) spectrum $L(\mathbb{Q})$ if we ignore dimension 0. Siegel phrases it by inverting 2:

Theorem. *Witt spaces form a cycle theory for connective $KO \otimes \mathbb{Z}[1/2]$, i.e.*

$$\Omega^{\mathrm{Witt}}(X) \otimes \mathbb{Z}[1/2] \to KO(X) \otimes \mathbb{Z}[1/2]$$

is an isomorphism.

Pardon, [P] building on earlier work of Goresky and Siegel, [GS], showed that the spaces with **integrally self dual** IC form a class of spaces (which does not include all spaces with even codimensional strata: one needs an extra condition on the torsion of the one off the middle dimension IH group) whose cobordism groups agree with $L^*(\mathbb{Z})$ and then give a cycle theoretic description for the connective version of this spectrum.

Other interesting bordism calculations for classes of singular spaces can be found in [GP].

2.11 (Some remarks of Siegel). The fact that one has a bordism invariant signature for Witt spaces contains useful facts about signature for manifolds. For instance, using the identification of signature for manifolds with boundary with the intersection signature of the associated singular space

with an isolated singular point, it is easy to write down a Witt cobordism (the pinch cobordism) which proves Novikov's additivity theorem [AS].

Also, the mapping cylinder of a fiber bundle is not always a Witt cobordism: there is a condition on the middle homology of the fiber. One could have thought that one can still define signature for singular spaces where the links have signature zero (obviously one can't introduce a link type with nonzero signature and have a cobordism invariant signature). However, Atiyah's example of nonmultiplicativity of signature gives a counterexample to this [A]. It is thus quite interesting that having no middle homology **is** enough for doing this.

2.12. Siegel's theorem has had several interesting applications. The first is a purely topological disproof of the integral version of the Hodge conjecture (already disproven by analytic methods in [AH]) on the realization of all (p,p) homology classes of a nonsingular variety by algebraic cycles. If one were looking for nonsingular cycles, then one can use failure of Steenrod representability, or better, Steenrod representability by unitary bordism!, but here we allow singular cycles. Thanks to Hironaka, we could apply resolution of singularities to make the argument work anyway. However, even without resolution one sees that these homology classes have a refinement to K-homology, which is a nontrivial homotopical condition (as in [AH] which develops explicit counterexamples along these lines).

Another application stems from the fact that the bordism theory is homology at the prime 2. Since one can define a signature operator for Witt spaces which is bordism invariant [PRW], one can view the signature operator from the point of view of Witt bordism and thus obtain a refinement at the prime 2 of the K-homology class of the signature operator to ordinary homology [RW]. This, then implies that the K-homology class of the signature operator is a homotopy invariant for, say, $\mathbb{R}P^n$.

Yet other applications concern "eta type invariants". The basic idea for these applications is that if one knows the Novikov conjecture for a group π, then by Siegel's theorem $\Omega^{\mathrm{Witt}}(B\pi) \to L(\mathbb{Q}\pi)$ rationally injects. This means that one can get geometric coboundaries from homotopical hypotheses. Thus, for instance, homotopy equivalent manifolds should be rationally Witt cobordant (preserving their fundamental group).

In particular, then, the invariant of Atiyah-Patodi-Singer [APS] associated to an an odd dimensional manifold with a unitary representation of its fundamental group can only differ, for homotopy equivalent manifolds, that a twisted signature of the cobounding Witt space, e.g. a rational number. In [W3], known results regarding the Novikov conjecture and the deformation results of [FL] are used to prove this unconditionally.

A similar application is made in [W6] to define "higher rho invariants"

for various classes of manifolds. For instance, say that a manifold is antisimple if its chain complex is chain equivalent to one with 0 in its middle dimension (this can be detected homologically). Then its symmetric signature vanishes and, therefore, assuming the Novikov conjecture, it is Witt nullcobordant. By gluing together the Witt nullcobordism and the algebraic nullcobordism one obtains a closed object one dimension higher, whose symmetric signature (modulo suitable indeterminacies) is an interesting invariant of such manifolds. It can be used to show that the homeomorphism problem is undecidable even for manifolds which are given with homotopy equivalences to each other [NW].

2.13 (Dedicated to the Cheshire cat). Associated to any Witt space one has a self dual sheaf, namely IC^m. Actually, the cobordism group of self dual sheaves over a space X (assuming the self duality is symmetric) can be identified with $H_*(X; L^*(\mathbb{Q}))$, (see [CSW] for a sketch, and [Ht] for a more general statement including some more general rings[1]).

This statement has some immediate implications: Since IC^m is topologically invariant, all of the characteristic classes introduced for singular spaces in this way are topologically invariant. (This is basically the topological invariance of rational Pontrjagin classes extended to Witt spaces.)

We thus have, away from 2, three rather different descriptions of K-homology: Witt space bordism, homotopy classes of abstract elliptic operators (see [BDF, K]), and bordism of self dual sheaves (and, not so different from this one: controlled surgery obstruction groups).

A number of applications to equivariant and stratified surgery come from these identifications (and generalizations of them). We will return to some of these in §6.

2.14. A very nice application of cobordism of the self dual sheaves associated to IH and its various pushforwards is given in [CS2]. The goal is to extend the usual multiplicativity of signature in fiber bundles (with no monodromy) to stratified maps. We will not give a precise definition of a stratified map, but it is the intuitive notion, e.g. a fiber bundle has just one stratum.

Theorem. *Let $f : X \to Y$ be a stratified map between spaces with even codimensional strata, and suppose that all the strata of f are of even codimension and the pure strata are simply connected. We then have*

$$f_*(L(X)) = \sum \operatorname{sign}(c(\operatorname{star}_f(V)))L(V)\,,$$

[1] In general there are algebraic K-theoretic difficulties with identifying the Witt group self dual sheaves, at 2, with a homology theory. However, as Hutt himself was aware, one can certainly include many more rings than included in that paper.

where V runs over the strata of f (which is a substratification of Y). Here $c(\cdot)$ stands for a compactification – in this case it means the following. If $V = Y$ it is just the generic fiber. If V is a proper stratum, then one can consider $f^{-1}(\mathrm{cone}(L))$, where L is the link of a generic top simplex of V, and then one-point compactify it (i.e, cone off its boundary).

One can deal with nonsimply connected open strata by putting in a correction term for the monodromy action of $\pi_1(\mathrm{int}(V))$ on $IH(c(\mathrm{star}_f(V)))$.

The proof of the theorem in [CS2] is very pretty; it makes use of the machinery on perverse sheaves found in [BBD] but in explicit cases essentially produces an explicit cobordism between $f_*IC(X)$ and an explicit sum of other intersection sheaves: one for each stratum of the map.

Remarks.

(1) To get a feeling for the theorem it is worth considering a few special cases. Firstly, the case of a fiber bundle just reduces to [CHS]. As a second special case, if one considers a pinch map from a union of two manifolds along their common boundary, the formula boils down to Novikov additivity, and the cobordism implicit in the proof is the pinch cobordism of 2.11. As a final example, one can consider the case of a circle action on a manifold. Aside from some fundamental group points, there is a similar cobordism between M and some projective space bundles over the fixed set components of the circle action, and the formula of the theorem generalizes by considering projection to the quotient – with some slight modifications for 0 mod 4 components of fixed set, which lead to non-Witt singularities – (or specializes to) the formula in [W2] that identifies the higher signature of a manifold and that of its fixed point set of any circle action with nonempty fixed point set. The cobordism (discussed in both [W2] and [CS2]) is then the bubble quotient cobordism of [CW3].

(2) In the case of an **algebraic map**, one could directly apply [BBD] which gives a beautiful and deep decomposition theorem for $f_*IC(X)$ and the general machinery on self dual sheaves to prove the existence of a formula like the one in the theorem. However, it is not so clear what the coefficients are.

(3) Still in the algebraic case, it is important to realize that there are many additional characteristic classes that can be defined for singular varieties beyond just the L-classes, for instance, MacPherson Chern classes and Todd classes. In [CS3], there are announced generalizations of the basic formula where the meaning of c is different: one must use projective completion to get a variety, and then the formula must be rewritten to account for the extra topology (think

about the case of intersection Euler characteristic classes, which can be dealt with by the proof of the usual Hurwitz formula for Euler characteristic of branched cover, together with the sheaf version of intersection Künneth). To prove such formulae one uses deformation to the normal cone (see [Fu2]) to replace the cobordism theory.

2.15. It is worth mentioning but beyond the purview of this survey to describe in any detail the applications of 2.14 given in [CS3, CS4, Sh2] to lattice point problems. The connection goes via the theory of toric varieties for which there are several excellent surveys [Od, Da, Fu1], which gives an assignment of a (perhaps singular) variety to every convex polygon with lattice point vertices on which a complex torus acts. (See also [Gu] for a discussion of the purely symplectic aspects of this situation.) Problems like counting numbers of lattice points inside such a polytope (= computation of the Erhart polynomial) and Euler – Maclaurin summation formulae can be reduced to calculations of the Todd class, which are studied in tandem with L-classes using the projection formulae. These, in turn, have substantial number theoretic implications.

3. The geometry of homotopically stratified spaces

One of the strengths of Quinn's formulation of manifold homotopically stratified spaces is that the defining conditions are homotopic theoretical (except, obviously, the geometric condition that the strata be manifolds). This, of course, makes it easier to verify the conditions, but harder to establish geometric facts about manifold homotopically stratified spaces. Nevertheless, Quinn was able to prove two important geometric results: homogeneity and stratified h-cobordism theorems.

Quinn's homogeneity result says that if x, y are two points in the same component of a stratum (with adjacent strata of dimension at least 5) of a manifold homotopically stratified space X, then there is a self-homeomorphism (in fact, isotopy) of X carrying x to y. Quinn obtains this as a consequence of an stratified isotopy extension theorem (an isotopy on a closed union of strata can be extended to a stratum preserving isotopy on the whole space). In turn, Quinn proves the isotopy extension theorem by using the full force of his work on "Ends of maps" (see [Q2,IV]).

As an example of the usefulness of the homogeneity result, consider a finite group acting on a manifold M. Even though the action need not be locally linear, the quotient M/G is often a manifold homotopically stratified space. Thus, the homogeneity result can be used to verify local linearity by establishing local linearity at a single point of each stratum component. Quinn first used this technique to construct locally linear actions whose fixed point set does not have an equivariant mapping cylinder neighborhood

[Q2,II]. Weinberger [W1] and Buchdahl, Kwasik and Schultz [BKS] have also used this result to verify that certain actions were locally linear.

It turns out that there is an alternative way to prove Quinn's homogeneity theorem which is based on engulfing (the classical way that homotopy information is converted into homeomorphism information in manifolds). In fact, this alternative method uses a description of neighborhoods of strata together with MAF (manifold approximate fibration) technology, and is useful for many other geometric results.

We have seen in §1 that in certain formulations of conditions on a stratification $\Sigma = \{X_i\}$ of a space X one considers tubular maps

$$\tau_i : U_i \to X_i \times [0, +\infty)$$

where U_i is a neighborhood of X_i and τ_i restricts to the identity $U_i \setminus X_i \to X_i \times (0, +\infty)$. For Whitney stratifications, the tubular maps are submersions on each stratum and fibre bundles over $X \times (0, +\infty)$. Since strata of manifold homotopically stratified spaces need not have mapping cylinder neighborhoods, such a result is too much to hope for in general. However, there is the following result which was proved by Hughes, Taylor, Weinberger and Williams [HTWW] in the case of two strata and by Hughes [H3] in general.

Theorem. *For manifold homotopically stratified spaces in which all strata have dimension greater than or equal to* 4*, tubular maps exist which are manifold stratified approximate fibrations.*

The neighborhoods of the strata which are the domains of these MSAF (manifold stratified approximate fibration) tubular maps are called *teardrop neighborhoods*. They are effective substitutes for mapping cylinder neighborhoods, and the result should be thought of as a *tubular neighborhood theorem* for stratified spaces. The point is that even though Quinn's definition does not postulate neighborhoods with any kind of reasonable tubular maps, one is able to derive their existence. The situation is optimal: minimal conditions in the definition with much stronger conditions as a consequence. This makes the surgery theory conceptually easier than for geometrically stratified sets for which the geometric neighborhood structure must be part of the data.

As mentioned above, these teardrop neighborhoods can be used to give a different proof of Quinn's isotopy extension theorem. Manifold approximate fibrations have the approximate isotopy covering property [H1]. This property holds in the stratified setting and is used inductively to extend isotopies from strata to their teardrop neighborhoods. In fact, parametric isotopy extension is now possible whereas Quinn's methods only work for a single isotopy.

Similarly, other results in geometric topology can be extended to manifold homotopically stratified sets by using MAF technology. For example, the homeomorphism group of a manifold homotopically stratified set is locally contractible, and a stratified version of the Chapman and Ferry [ChF] α-approximation theorem holds. In short, the case can be made that manifold homotopically stratified sets are the correct topological version of polyhedra and Thom's stratified sets.

4. Browder-Quinn theory

In [BQ], Browder and Quinn introduced an interesting, elegant, and useful general classification theory for strongly stratified spaces. The setting is a category where one has a fixed choice of strong stratification as part of the data one is interested in.

4.1. Let X be a strongly stratified space (e.g. a geometrically stratified space as in §1.3) with *closed pure strata* $\overline{X}^i$ (see §1.3). An h-cobordism with boundary X is a stratified space Z with boundary $X \cup X'$ where the inclusions of X and X' are stratified homotopy equivalences, and the neighborhood data for the strata of Z are the pullbacks with respect to the retractions of the data for X (and of X'). This condition is automatic in the PL and Diff categories when one is dealing with something like the quotient of a group action stratified by orbit types.

Theorem. *The h-cobordisms with boundary X (ignoring low dimensional strata) are in a 1–1 correspondence with a group* $\mathrm{Wh}^{BQ}(X)$. *There is an isomorphism* $\mathrm{Wh}^{BQ}(X) \cong \bigoplus \mathrm{Wh}(\overline{X}^i)$.

The map $\mathrm{Wh}^{BQ}(X) \to \bigoplus \mathrm{Wh}(\overline{X}^i)$ is given by sending

$$(Z, X) \to (\tau(\overline{Z}^i, \overline{X}^i)).$$

One proves the isomorphism (and theorem) inductively, using the classical h-cobordism theorem to begin the induction, and using the strong stratifications to pull up product structures to deal with one more stratum.

4.2. The surgery theory of Browder and Quinn deals with the problem of turning a degree one normal map into a stratified homotopy equivalence which is **transverse**, i.e. one for which the strong stratification data in the domain is the pullback of the data from the range.

This transversality is, in practice, the fly in the ointment. When one is interested in classifying embeddings or group actions usually the bundle data is something one is interested in understanding rather than *a priori* assuming. Still, in some problems (e.g. those mentioned in 6.3) one can

sometimes prove that transversality is automatic. Also, of course, if one uses the machinery to construct examples, it is certainly fine if one produces examples that have extra restrictions on the bundle data.

4.3. Either by induction or by mimicking the usual identification of normal invariants, one can prove that $NI^{BQ}(X) \cong [X; F/Cat]$.

4.4. Define $S^{BQ}(X)$ to be the strongly stratified spaces with a transverse stratified simple homotopy equivalence to X up to Cat-strongly stratified simple isomorphism (note this implicitly is keeping track of "framings"). Then, one has groups $L^{BQ}(X)$ and a long exact surgery sequence:

$$\cdots \to L^{BQ}(X \times I \text{ rel } \partial) \to S^{BQ}(X) \to [X; F/Cat] \to L^{BQ}(X).$$

4.5. Note that unlike the Whitehead theory $L^{BQ}(X)$ does not decompose into a sum of L-groups of the closed strata. Indeed, for a manifold with boundary S^{BQ} is just the usual structure set (existence and uniqueness of collars gives the strong stratification structures) and the L-group is the usual L-group of a manifold with boundary, i.e. is a relative L-group, not a sum of absolute groups.

However, there is a connection between the L-groups of the pure strata and $L^{BQ}(X)$. This exact sequence generalizes the exact sequence of a pair in usual L-theory, and expresses the fact that as a space $L^{BQ}(X)$ is the fiber of the composition

$$L(X_0) \to L(\partial \,\text{Neighborhood}(X_0)) \to L(\text{cl}(X \setminus X_0))$$

where the first map is a transfer and the second is an inclusion.

4.6. The proof of this theorem is by the method of chapter 9 of [Wa]: one need only verify the π-π theorem. This is done by induction.

5. Homotopically stratified theory

If one does not want to insist on the transversality condition required in the Browder-Quinn theory, or if one is only dealing with homotopically stratified spaces, it is necessary to proceed somewhat differently. For more complete explanations, see [W4], [W5]. We will only discuss the topological version. The PL version is simpler but slightly more complicated.

5.1 The h-cobordism theorem. That new phenomena would arise in any systematic study of Whitehead torsion on nonmanifolds was clear from the start. Milnor's original counterexamples to the Hauptvermutung for polyhedra were based on torsion considerations [M1]. Siebenmann gave examples of locally triangulable nontriangulable spaces – not at all due to

Kirby-Siebenmann considerations, but rather K_0. More pieces came forward in the work on Anderson-Hsiang [AnH1, AnH2] and then in [Q2], which showed that under appropriate K-theoretic hypotheses one can triangulate, and therefore apply the straightforward PL torsion theory.

Real impetus came from the theory of group actions. Cappell and Shaneson [CS1] gave the striking examples of equivariantly homeomorphic representation spaces, which laid down the gauntlet to the topological community at large to deal with the issue of equivariant classification. h-cobordism theorems suitable for the equivariant category were produced by Steinberger (building on joint work with West) [St] and by Quinn [Q4] in the generality of homotopically stratified spaces (although the theorem in that paper does not include realization of all torsions in an h-cobordism[2]).

The ultimate theorem asserts, as usual, that (ignoring low dimensional issues) h-cobordisms on a stratified space X are classified by an abelian group $\mathrm{Wh}^{\mathrm{top}}(X)$.

Theorem. $\mathrm{Wh}^{\mathrm{top}}(X) \cong \bigoplus \mathrm{Wh}^{\mathrm{top}}(X^i, X^{i-1})$, *and we have an exact sequence*

$$\cdots \to H_0(X^{i-1}; \mathbf{Wh}(\pi_1(\text{holink}))) \to \mathrm{Wh}(\pi_1(X^i \setminus X^{i-1})) \to$$
$$\mathrm{Wh}^{\mathrm{top}}(X^i, X^{i-1}) \to H_0(X^{i-1}; \mathbf{K}_0(\pi_1(\text{holink}))) \to K_0(\pi_1(X^i \setminus X^{i-1}))\,.$$

Boldface terms are spectra. This decomposition of $\mathrm{Wh}^{\mathrm{top}}$ into a direct sum does not respect the involution obtained by turning h-cobordisms upside down, which is a pleasant descendant of the analogous fact in the Browder-Quinn theory. It does not have an analogue in L-theory.

5.2 Stable classification. Ranicki (following a sketch using geometric Poincaré complexes in place of algebraic ones, by Quinn) has elegantly reformulated the usual Browder-Novikov-Sullivan-Wall surgery exact sequence in the topological manifold setting as the assertion that there is a fibration:

$$\mathbf{S}(M) \to \mathbf{H}_n(M; \mathbf{L}(e)) \to \mathbf{L}_n(\pi_1(M))$$

where $\mathbf{X}$ means a space (or better a spectrum) whose homotopy groups are those of the group valued functor ordinarily denoted by X. $S(M)$ is the *structure set* of M, which essentially[3] consists of the manifolds simple homotopy equivalent to M up to homeomorphism. The map $\mathbf{H}_n(M; \mathbf{L}(e)) \to$

[2] As pointed out in [HTWW], the teardrop neighborhood theorem can be used to complete the proof of realization.

[3] In actuality, for our purposes it is best to use the finite dimensional ANR homology manifolds, and the equivalence relation is s-cobordism. See Mio's paper [Mi] in Volume 1 for a discussion of the difference this makes. (It is at most a single $\mathbb{Z}$ if M is connected.)

$\mathbf{L}_n(\pi_1(M))$ is called the *assembly map* and can be defined purely algebraically. Geometrically it has several interpretations: most notably, as the map from normal invariants to surgery obstructions in the topological category, or as a forgetful map from some variant of controlled surgery to uncontrolled surgery.

Since the assembly map has a purely algebraic definition, one can ask whether it computes anything interesting if X is not a manifold? and alternatively, if X is just a stratified space, what is the analogue of this sequence?

Cappell and the second author gave an answer to the first question in [CW2] where it is shown (under some what more restrictive hypotheses) if X is a manifold with singularities, i.e. X contains a subset Σ whose complement is a manifold, and suppose further that Σ is 1-LCC embedded[4] in X, then $S^{\text{alg}}(X) \cong S(X \text{ rel } \Sigma)$ where $S^{\text{alg}}(X)$ denotes the fiber of the algebraically defined assembly map $H_*(X; L(e)) \to L(X)$ and $S(X \text{ rel } \Sigma)$ means

$$\{\varphi : X' \to X \mid \varphi \text{ is a stratified simple}^5 \text{ homotopy equivalence with } \varphi|\Sigma(X') \text{ already a homeomorphism}\}.$$

The answer to the second question is a bit more complicated, and actually requires two fibrations in general. The first is a stable generalization of the surgery exact sequence:

$$S^{-\infty}(X \text{ rel } Y) \to \mathbf{H}_0(X; \mathbf{L}^{BQ-\infty}(—, \text{ rel } Y)) \to \mathbf{L}^{BQ-\infty}(X \text{ rel } Y).$$

Here the superscripts $-\infty$ denote that we are using a stable version of structure theory: we will soon explain that it only differs from the usual thing at the prime 2, and the phenomenon is governed by algebraic K-theory. The coefficients of the homology is with respect to a cosheaf of L-spectra: to each open set U of X one assigns the spectrum $\mathbf{L}(U \text{ rel } U \cap Y$with compact support). The BQ superscripts are a slight generalization of the theory discussed in §4.

To complete the theory one needs a destabilization sequence. This is given by the following:

$$S(X) \to S^{-\infty}(X) \to \hat{H}(\mathbb{Z}_2; \text{Wh}^{\text{top}}(X)^{\leq 1})$$

Here $S(X)$ is the geometric structure set, and $S^{-\infty}(X)$ is the stabilized version, which differ by a Tate cohomology term. An analogous sequence

[4]This means that maps of 2-complexes into X can be deformed slightly to miss Σ.
[5]The material of 5.1 can be used to make sense of this.

developed for a quite similar purpose appears in [WW]. Indeed in [HTWW] the theory outlined in this subsection is deduced from the [WW] results using blocked surgery [Q1, BLR, CW2] and [HTW1,2] (the classification of manifold approximate fibrations) and the teardrop neighborhood theorem. On the other hand, there are different approaches to all this using controlled end and/or surgery theorems that are sketched in [W4], [W5].

6. Some applications of the stratified surgery exact sequence

In practice the application of the theory of the previous section requires additional input for the calculations to be either possible or comprehensible. See [CW4] for the application to topological group actions. The last 100 pages of [W4] also give more applications than we can hope to discuss here.

6.1. Probably the simplest interesting and illustrative example of the classification theorem is to locally flat topological embeddings. The first point is that this problem is susceptible to study by these methods: Every topological locally flat embedding gives a two stratum homotopically stratified space where the holink is a homotopy sphere *and conversely*. This last is essentially Quinn's characterization of local flatness in [Q2,I].

Things are very different in codimensions one and two from codimension three and higher. We will defer to 6.3 the low codimensional discussion and restrict our attention here to the last of these cases.

Lemma. *If (W, M) is a manifold pair with $cod(M) > 2$, then $L^{BQ}(W, M) \cong L(W) \times L(M)$ where the map is the forgetful map.*

The proof is straightforward. Note that the lemma implies the analogous statement holds at the level of cosheaves of spectra ($\cong$ being quasi-isomorphism). It is quite straightforward in this case to work out the Whitehead theory: there are no surprises. Thus, we obtain:

Corollary. $S(W, M) \cong S(W) \times S(M)$.

Note that the discussion makes perfect sense even if (W, M) is just a Poincaré pair (see [Wa]), and then the corollary boils down to the statement that isotopy classes of embeddings of one topological manifold in another (in codimension at least 3) are in a 1-1 correspondence with the Poincaré embeddings (see [Wa]).

(Actually, a bit more work enables one to prove the same thing for M an ANR homology manifold.)

6.2. Using the material from §2 we can also analyze, away from 2, $S(X)$ for a very interesting class of spaces that have even codimensional strata.

We continue to let $S^{\mathrm{alg}}(X)$ denote the fiber of the classical assembly map $H_*(X; L(e)) \to L(X)$. It is what the structure set of X would be if X were a manifold.

Theorem([CW2]**).** *If X is a space with even codimensional strata and all holinks of all strata in one another simply connected, then there is an isomorphism $\otimes[1/2]$*

$$S(X) \cong \bigoplus S^{\mathrm{alg}}(V)$$

where the sum is over closed strata.

The proof consists of building an isomorphism $L^{BQ}(X) \cong \bigoplus L(V)[1/2]$ for arbitrary X satisfying the hypotheses. It is obvious enough how to introduce $\mathbb{Q}$ coefficients into L^{BQ}. Ranicki [R] has shown that introducing coefficients does not change L at the odd primes, but with $\mathbb{Q}$-coefficients one can make forgetful maps to $L(V; \mathbb{Q})$ using the intersection chain complexes.

6.3. To give an example where things work out differently, we shall assume that the holinks are aspherical, and that the Borel conjecture holds for the fundamental groups of these holinks. (This example is a special case of the theory of "crigid holinks" in [W4].)

In this case there is nothing good that can be said about the global L^{BQ} term, in general. However, the assumptions are enough to imply that $H_*(X; L^{BQ}) \cong [X; L(e)]$. (See [W4], [BL] for a discussion.) In particular, for locally flat embeddings in codimension 1 and 2, one sees that the fiber of $S(W, M) \to S(W)$ only involves fundamental group data, not, say, the whole homology of the manifold and submanifolds. This, too, reflects phenomena already found in Wall's book [Wa].

Another amusing example is $X =$ simplicial complex, stratified by its triangulation. Then one gets $L^{BQ}(X) \cong [X; L(e)]$.

There are other interesting examples that display similar phenomena that come up from toric varieties. The theory of multiaxial actions (see §2 and [Dv2]) is another situation where the local cosheaves tend to decompose into pieces that can be written in simple terms involving skyscraper $L(e)$-cosheaves. Not all holinks are crigid and consequently different phenomena appear: indeed signatures 0 and 1 alternate in the simply connected holinks, with quite interesting implications. As a simple exercise, one can see that while $S^{6n-1}/U(n)$ is contractible, its structure set[6] $S(S^{6n-1}/U(n)) \cong \mathbb{Z}_2$. Similarly, $S^{12n-1}/Sp(n)$ is contractible, but its structure set[7] is $\mathbb{Z}$.

[6]Actually, the structure set is $\mathbb{Z} \times \mathbb{Z}_2$ with the extra $\mathbb{Z}$ corresponding to actions on nonresolvable homology manifolds that are homotopy spheres.

[7]Same caveat as above.

Remark. If all holinks are simply connected (as in the case of multiaxial actions of $U(n)$ and $Sp(n)$) one always has a spectral sequence computing $S(X)$ in terms of the $S^{\mathrm{alg}}(X_i)$. For instance, if there are just two strata $X \supset \Sigma$, there is an exact sequence:

$$\cdots \to S^{\mathrm{alg}}(\Sigma \times I) \to S^{\mathrm{alg}}(X) \to S(X) \to S^{\mathrm{alg}}(\Sigma) \to \dots$$

The sequence continues to the left in the most obvious way. On the right it continues via deloops of the algebraic structure spaces. The map $S^{\mathrm{alg}}(\Sigma \times I) \to S^{\mathrm{alg}}(X)$ depends on the symmetric signature of the holink (and on the monodromy of the holink fibration). The case where the simply connected holink is rigid is essentially that of manifolds with boundary. The normal invariant term here is $[X; L(e)]$, but thought of here as $H(X, \partial X; L(e))$. On the other hand, in 6.2 we gave an important case where this spectral sequence degenerates (at least away from the prime 2).

6.4. As our final example, let us work out in detail a case that is somewhat opposite to the one of the previous paragraph: X = the mapping cylinder of even a PL block bundle $V \to N$, with fiber F, where N is a sufficiently good aspherical manifold. (Sufficiently good is a function of the reader's knowledge. Even the circle is a case not devoid of interest.) We are interested in understanding what the general theory tells us about $S(X \text{ rel } V)$.

Firstly, there is the calculation of the Whitehead group. (Or even pseudoisotopy spaces ...). In this case, the sequence boils down to:

$$H_0(N; \mathbf{Wh}_1(F)) \to \mathrm{Wh}_1(V) \to \mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V) \\ \to H_0(N; \mathbf{K}_0(F)) \to K_0(V)\,.$$

In a totally ideal world, the assembly maps $H_0(N; \mathbf{Wh}_1(F)) \to \mathrm{Wh}_1(V)$ and $H_0(N; \mathbf{K}_0(F)) \to K_0(V)$ would be isomorphisms, and $\mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V)$ would vanish. However, even in the case of $N = S^1$ where the assembly map (for the product bundle) was completely analyzed by [BHS], this is not true. In that case, there is an extra piece called *Nil* that obstructs this; however, *Nil* is a split summand. Thus, the assembly maps are still injections, and one obtains an isomorphism of $\mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V)$ with a sum of *Nil*s. In general, the pattern discovered by Farrell and Jones [FJ] shows that the cokernel of these assembly maps is at least reasonably conjectured to be a "sum" of *Nil*s.

The splitting of the K-theory assembly map essentially boils down to the assertion that $\mathrm{Wh}^{BQ}(X \text{ rel } V) \to \mathrm{Wh}^{\mathrm{top}}(X \text{ rel } V)$ has a section. There are fairly direct proofs of this fact when N is a nonpositively curved Riemannian

manifold in [FW] and in [HTW3]. The first approach notes that putting a PL structure on stratified spaces can be viewed (essentially following [AnH1, AnH2]) as a problem of reducing the tangent microbundle to the group of block bundle maps: but in the presence of curvature assumptions this can be done in the large by the methods of controlled topology.

The approach in [HTW3] depends on the same controlled topology, but its focus is showing that one can associate a MAF structure to any map whose homotopy fiber is finitely dominated. The teardrop neighborhood theorem of course provides the relation between these approaches.

The same analyses can be done for the (stable) structure set $S(X \text{ rel } V)$. In this case one does often have the vanishing of the analogue of *Nil* (although if there's orientation reversal or complicated monodromy in the bundle, this might not be the case). The structure set is here described as the fiber of the assembly map, and thus it often vanishes.

This has an interesting interpretation. Let us suppose that the fiber is K-flat, i.e. that $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$ for all k to avoid any potential end obstructions. In this case one also knows that all MAF's are equivalent to block bundle projections.

The vanishing of $S(X \text{ rel } V)$ means that $S(X) \cong S(V)$ by the "obvious" fibration: $S(X \text{ rel } V) \to S(X) \to S(V)$. (We'll discuss the " " marks in a moment.) Now $S(X)$ is basically the same thing as the F-block bundles on N with fiber a manifold homotopy equivalent to F. Thus we have a generalized fibration theorem for manifolds with maps to N. (Indeed, the Farrell fibration theorem [Fa] is all that is necessary to feed into the machinery to get out the calculation of L-groups: that's the content of Shaneson's thesis [Sh1]!)

Without the K-flatness, we see that there are still only *Nil* obstructions to obtaining MAF structures (but genuine K-theory obstructions to getting block structures).

To return to the "obvious" fibration, a little thought shows that it is not at all obvious. What is obvious is that it is a fibration over the components of $S(V)$ in the image of the map $S(X) \to S(V)$. We are asserting, after the arguments given above, that this image is all the components, but prima facie, the argument in whole is circular.

However, that is not the case as a consequence of the complete general theory. The map $S(X) \to S(V)$ is actually an infinite loop map, isomorphic to its own 4^{th} loop space (see [CW1, We5]). Thus, the fact that we knew exactness at the π_i level for $i = 3, 4$ gives us everything we want for any such ad hoc component problem. (This is exactly the same point involved in continuing the exact sequence of 6.3 further to the right.)

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