Topology and its Applications 94 (1999) 127–145

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Stratifications of mapping cylinders

Bruce Hughes [1,2]

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

Received 29 July 1997; received in revised form 10 October 1997

Abstract

We characterize those maps between homotopically stratified spaces whose mapping cylinders are also homotopically stratified spaces. Two applications are offered. The first concerns locally flat submanifolds of topological manifolds, and the second concerns algebraic maps between algebraic varieties. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Stratified space; Homotopically stratified space; Stratified approximate fibration; Mapping cylinder; Manifold embedding; Stratified collection of bundles; Algebraic variety

AMS classification: Primary 57N80; 55R65, Secondary 57N40; 58A35

1. Introduction

A basic technique in topology is to convert a map (i.e., a continuous function) into a space by taking its mapping cylinder. When doing this it is useful to remain within a given category. For example, the mapping cylinder of a piecewise linear map between polyhedra can be given the structure of a polyhedron.

This paper establishes an analogous result for maps between homotopically stratified spaces. Homotopically stratified spaces were introduced by Quinn [23] and are the homotopy-theoretic analogues of Whitney stratified spaces. In fact, homotopically stratified spaces with manifold strata (or *manifold stratified spaces*, for short) appear to be the topological analogue of polyhedra (see [16]).

What plays the role of piecewise linear maps in this rarefied homotopy-theoretic context? It is the *stratified approximate fibrations*, maps with the property that homotopies

[1] E-mail: hughes@math.vanderbilt.edu.
[2] Supported in part by NSF Grant DMS-9504759.

PII: S0166-8641(98)00028-5

respecting strata in the range can be approximately lifted to homotopies respecting strata in the domain. These maps were introduced in [10].

The main result of this paper is that the mapping cylinder of a proper map between homotopically stratified spaces is itself a homotopically stratified space if and only if the map is a stratified approximate fibration. See Theorem 5.11 for a precise statement.

In Section 6 the techniques developed in this paper are applied to manifold embeddings. It is proved that the mapping cylinder of a map between manifolds is itself a manifold with the base a locally flat submanifold if and only if the map is a manifold approximate fibration with spherical homotopy fibre (see Section 6 for dimensional restrictions).

A different perspective on a recent result of Cappell and Shaneson [2] appears in Section 7. They proved that mapping cylinders of certain maps ('stratified maps' in their terminology) between Whitney stratified spaces are homotopically stratified even though the mapping cylinders need not be Whitney stratified. This is reproved by first formulating a result (Theorem 7.3) for homotopy stratifications (instead of Whitney stratifications) and then deriving the Cappell and Shaneson result as a corollary. As a consequence of the main result it is proved that a proper algebraic map between algebraic varieties is a stratified approximate fibration (Corollary 7.5).

A generalized Tubular Neighborhood Theorem for manifold stratified spaces was announced in [9] and a proof for the two strata case was given in [13]. The present paper along with [10] provides a good deal of the stratified homotopy theory needed for the proof of the general case to appear in [11].

2. Definitions

In this section we gather some background material on spaces with stratifications, homotopy links, mapping cylinders and controlled homotopy equivalences.

Spaces with stratifications. The basic definitions from the theory of stratifications are presented. For a fuller treatment see [10].

Definition 2.1.

(1) A *partition* of a space X consists of an index set $\mathcal{I}$ and a collection $\{X_i\}_{i\in\mathcal{I}}$ of pairwise disjoint subspaces of X such that $X=\bigcup_{i\in\mathcal{I}} X_i$.

(2) A *stratification* of a space X consists of an index set $\mathcal{I}$ and a locally finite partition $\{X_i\}_{i\in\mathcal{I}}$ of locally closed subspaces of X.

(3) In either case, for $i\in\mathcal{I}$, X_i is the i*-stratum* and

$$X^i=\bigcup\bigl\{X_k \mid X_k\cap \mathrm{cl}(X_i)\neq\emptyset\bigr\}$$

is the i*-skeleton*.

For a space X with a stratification $\{X_i\}_{i\in\mathcal{I}}$, define a relation $\leqslant$ on the index set $\mathcal{I}$ by

$$i\leqslant j \quad\text{if and only if}\quad X_i\subseteq \mathrm{cl}(X_j).$$

The stratification satisfies the *Frontier Condition* if for every $i, j \in \mathcal{I}$,

$$X_i \cap \mathrm{cl}(X_j) \neq \emptyset \quad \text{implies} \quad X_i \subseteq \mathrm{cl}(X_j).$$

Remark 2.2 [10]. If $\{X_i\}_{i\in\mathcal{I}}$ is a stratification of X, then the Frontier Condition holds if and only if $\leqslant$ is a partial ordering of $\mathcal{I}$ and for each $i \in \mathcal{I}$, $X^i = \mathrm{cl}(X_i)$.

Definition 2.3. Let X and Y be spaces with partitions with index sets $\mathcal{I}_X$ and $\mathcal{I}_Y$, respectively.

(1) A map $f : X \to Y$ is *filtered* if for every $i \in \mathcal{I}_X$ there exists $j \in \mathcal{I}_Y$ such that $f(X^i) \subseteq Y^j$.
(2) The map f is *stratified* if for every $i \in \mathcal{I}_X$ there exists $j \in \mathcal{I}_Y$ such that $f(X_i) \subseteq Y_j$.

Proposition 2.4. *If X and Y are spaces with stratifications satisfying the Frontier Condition and $f : X \to Y$ is a stratified map, then f is filtered and the induced function $f : \mathcal{I}_X \to \mathcal{I}_Y$ defined by*

$$f(i) = j \quad \textit{if and only if} \quad f(X_i) \subseteq Y_j$$

is order preserving.

Proof. To see that f is filtered it suffices to show that if $f(X_i) \subseteq Y_j$, then $f(X^i) \subseteq Y^j$. Since $X^i = \mathrm{cl}(X_i)$, $Y^j = \mathrm{cl}(Y_j)$ by Remark 2.2, this is immediate. To see that $f : \mathcal{I}_X \to \mathcal{I}_Y$ is order preserving, assume that $X_i \subseteq \mathrm{cl}(X_j)$ and show that $Y_{f(i)} \subseteq \mathrm{cl}(Y_{f(j)})$. Since $f(X_i) \subseteq f(\mathrm{cl}(X_j)) \subseteq \mathrm{cl}(f(X_j)) \subseteq \mathrm{cl}(Y_{f(j)})$ and $f(X_i) \subseteq Y_{f(i)}$, it follows that $Y_{f(i)} \cap \mathrm{cl}(Y_{f(j)}) \neq \emptyset$. Thus the Frontier Condition implies that $Y_{f(i)} \subseteq \mathrm{cl}(Y_{f(j)})$. $\square$

Definition 2.5. Let X be a space with a partition.

(1) A map $f : Z \times A \to X$ is *stratum preserving along A* if for each $z \in Z$, $f(\{z\} \times A)$ lies in a single stratum of X.
(2) A map $f : Z \times I \to X$ is a *stratum preserving homotopy* if f is stratum preserving along I.

Definition 2.6. Let X and Y be spaces with partitions. A stratified map $f : X \to Y$ is a *stratified homotopy equivalence* if there exist a stratified map $g : Y \to X$ and stratified homotopies $F : \mathrm{id}_X \simeq gf$ and $G : \mathrm{id}_Y \simeq fg$.

Note that a stratified homotopy equivalence $f : X \to Y$ induces a one-to-one correspondence between the collections of strata of X and of Y.

Homotopy links. A homotopy model for the normal space of a subspace is provided by the homotopy link (cf. [23]).

Definition 2.7. Let X be a space with a partition $\{X_i\}_{i\in\mathcal{I}}$ and $Y \subseteq X$.

(1) The *homotopy link* of Y in X is defined by

$$\mathrm{holink}(X, Y) = \left\{\omega \in X^I \mid \omega(t) \in Y \text{ if and only if } t = 0\right\}.$$

(2) The *stratified homotopy link* of Y in X consists of all ω in $\mathrm{holink}(X, Y)$ such that $\omega((0, 1])$ lies in a single stratum of X:

$$\mathrm{holink}_s(X, Y) = \bigl\{\omega \in \mathrm{holink}(X, Y) \mid \text{for some } i,\ \omega(t) \subseteq X_i \text{ for all } t \in (0, 1]\bigr\}.$$

(3) Evaluation at 0 defines maps

$$q : \mathrm{holink}(X, Y) \to Y \quad \text{and} \quad q : \mathrm{holink}_s(X, Y) \to Y; \quad q(\omega) = \omega(0),$$

both called *holink evaluation*.

(4) The stratified homotopy link has a natural partition with i-stratum

$$\mathrm{holink}_s(X, Y)_i = \bigl\{\omega \in \mathrm{holink}_s(X, Y) \mid \omega(1) \in X_i\bigr\}.$$

Mapping cylinders. Notation is established for mapping cylinders and various maps associated with them.

Definition 2.8. The *mapping cylinder* of a map $p : X \to Y$ is the space

$$\mathrm{cyl}(p) = \bigl((X \times I) \amalg Y\bigr)/\bigl\{(x, 1) \sim p(x) \in Y \mid x \in X\bigr\}$$

with the *teardrop topology*; that is, the minimal topology such that:

(1) the inclusion $X \times [0, 1) \to \mathrm{cyl}(p)$ is an open embedding,

(2) the map

$$c : \mathrm{cyl}(p) \to Y \times I; \qquad \begin{cases} (x, t) \mapsto (p(x), t), & \text{if } (x, t) \in X \times [0, 1), \\ y \mapsto (y, 1), & \text{if } y \in Y \end{cases}$$

is continuous.

If $p : X \to Y$ is a proper map between locally compact Hausdorff spaces, then the teardrop topology agrees with the usual quotient space topology on the mapping cylinder. See [12, Chapter 12] and [13, §3] for further remarks on the teardrop topology.

We will have occasion to use the following three maps:

$$p_X : \mathrm{cyl}(p) \setminus Y \to X; \qquad [x, t] \mapsto x, \quad (x, t) \in X \times [0, 1),$$

$$p_Y : \mathrm{cyl}(p) \to Y; \qquad \begin{cases} [x, t] \mapsto p(x), & (x, t) \in X \times I, \\ [y] \mapsto y, & y \in Y, \end{cases}$$

$$p_I : \mathrm{cyl}(p) \to I; \qquad \begin{cases} [x, t] \mapsto t, & (x, t) \in X \times I, \\ [y] \mapsto 1, & y \in Y. \end{cases}$$

The *open mapping cylinder* $\overset{\circ}{\mathrm{cyl}}(p)$ is $\mathrm{cyl}(p) \setminus (X \times \{0\})$.

Controlled homotopy equivalence. We generalize the notions of controlled maps and controlled homotopy equivalences as presented in [14] to the stratified setting. See [13, §3] for other interpretations of controlled maps.

Definition 2.9. Let X and Y be spaces with partitions $\{X_i\}_{i \in \mathcal{I}}$ and $\{Y_j\}_{j \in \mathcal{J}}$, respectively, and let $p : X \to B$ and $q : Y \to B$ be a maps.

(1) A *controlled map* $f = \{f_t\}$ from p to q consists of a family of maps $f_t : X \to Y$, $0 \leqslant t < 1$, such that the induced maps:

$$f : X \times [0,1) \to Y; \qquad (x,t) \mapsto f_t(x),$$
$$\hat{f} : X \times I \to B; \qquad (x,t) \mapsto \begin{cases} qf_t(x), & \text{if } t < 1,\\ p(x), & \text{if } t = 1\end{cases}$$

are continuous.

(2) A controlled map $f = \{f_t\}$ from p to q is a *controlled homotopy equivalence* provided there exist controlled maps $g = \{g_t\}$ from q to p, $F = \{F_t\}$ from $X \times I \xrightarrow{\text{proj}} X \xrightarrow{p} B$ to $X \xrightarrow{p} B$, and $G = \{G_t\}$ from $Y \times I \xrightarrow{\text{proj}} Y \xrightarrow{q} B$ to $Y \xrightarrow{q} B$ such that

$$F_t(x,0) = x \quad \text{and} \quad F_t(x,1) = g_t f_t(x) \quad \text{for } x \in X,$$
$$G_t(y,0) = y \quad \text{and} \quad G_t(y,1) = f_t g_t(y) \quad \text{for } y \in Y.$$

(3) A controlled map f from p to q as in (1) above is a *stratified controlled map* provided for each $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ such that $f(X_i \times [0,1)) \subseteq Y_j$.

(4) A controlled map f from p to q as in (1) above is a *stratified controlled homotopy equivalence* provided there exist a stratified controlled map $g : Y \times [0,1) \to X$ and controlled maps $F : X \times I \times [0,1) \to X$ and $G : Y \times I \times [0,1) \to Y$ as in (2) above such that F and G are stratum preserving along $I \times [0,1)$.

3. The homotopy link of a mapping cylinder

For notation, let X and Y be spaces with partitions $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively, and let $p : X \to Y$ be a map. The mapping cylinder $\mathrm{cyl}(p)$ is naturally partitioned by

$$\big\{X_i \times [0,1)\big\}_{i\in\mathcal{I}} \cup \{Y_j\}_{j\in\mathcal{J}}.$$

Theorem 3.1. *$p : X \to Y$ and $q : \mathrm{holink}_s(\mathrm{cyl}(p), Y) \to Y$ are stratified controlled homotopy equivalent.*

Proof. For each $x \in X$ let $\omega_x : I \to \mathrm{cyl}(p)$ be the path defined by $\omega_x(t) = [x, 1-t]$. Thus $\omega_x \in \mathrm{holink}_s(\mathrm{cyl}(p), Y)$ and there is a function

$$f : X \to \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big); \qquad x \mapsto \omega_x.$$

To verify that f is continuous, since path spaces are given the compact-open topology, one only needs to observe that the adjoint $f^* : X \times I \to \mathrm{cyl}(p)$; $(x,t) \mapsto [x, 1-t]$ is continuous. Define

$$\Phi : X \times [0,1) \to \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big); \qquad (x,t) \mapsto f(x).$$

Since $p = qf$, $\Phi = \{\Phi_t\}$ is a stratified controlled map from p to q. Define

$$\Psi : \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big) \times [0,1) \to X; \qquad (\omega, t) \mapsto p_X\big(\omega(1-t)\big).$$

Evaluation maps are continuous in the compact-open topology; hence, Ψ is continuous. In fact, $\Psi = \{\Psi_t\}$ is a controlled map from q to p because of the continuity of

$$\widehat{\Psi} : \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big) \times I \to Y; \qquad (\omega, t) \mapsto \begin{cases} p\omega(1-t), & \text{if } t < 1, \\ \omega(0), & \text{if } t = 1. \end{cases}$$

Also observe that Ψ is a stratified controlled map. We claim that Φ and Ψ are stratified controlled homotopy inverses of each other as in Definition 2.9(4). Since $\Psi_t \circ \Phi_t : X \to X$ is the identity for each t, it remains to investigate

$$\Phi_t \circ \Psi_t : \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big) \to \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big).$$

To this end note that $\omega_{\Psi(\omega,t)}(u) = [\Psi(\omega, t), 1-u]$ and $\Phi_t\Psi_t(\omega) = \omega_{\Psi(\omega,t)}$. Now define

$$\begin{aligned} &\beta : \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big) \times [0,1) \times I \to \mathrm{holink}_s(\mathrm{cyl}(p), Y); \\ &\beta(\omega, t, s)(u) \\ &\quad = \begin{cases} \big[\Psi\big(\omega, st + (1-s)(1-u)\big), s(1-u) + (1-s)p_I\omega(u)\big], & \text{if } u > 0, \\ \big[p_Y\omega\big(s(1-t)\big)\big], & \text{if } u = 0. \end{cases} \end{aligned}$$

Note that β is continuous, $\beta(\omega, t, 0)(u) = \omega(u)$, and $\beta(\omega, t, 1)(u) = [\Psi(\omega, t), 1-u]$. It follows that

$$\begin{aligned} &\gamma : \mathrm{holink}_s(\mathrm{cyl}(p), Y) \times [0,1) \times I \to \mathrm{holink}_s(\mathrm{cyl}(p), Y) \times [0,1); \\ &\gamma(\omega, t, s) \mapsto (\beta(\omega, t, s), t) \end{aligned}$$

is a stratum preserving homotopy from the identity to $\{\Phi_t\Psi_t\}$. We just need to check that it is a controlled homotopy. This amounts to verifying the continuity of the function

$$\gamma_+ : \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big) \times I \times I \to Y; \qquad (\omega, t, s) \mapsto \begin{cases} q\beta(\omega, t, s), & \text{if } t < 1, \\ q(\omega), & \text{if } t = 1. \end{cases}$$

One checks that γ_+ is given by

$$\begin{aligned} (\omega, t, s) \mapsto &\begin{cases} \big[\Psi(\omega, st - 1 - s), 1\big], & \text{if } t < 1, \\ \omega(0), & \text{if } t = 1, \end{cases} \\ &= \begin{cases} p_Y\omega(s - st), & \text{if } t < 1, \\ p_Y\omega(0), & \text{if } t = 1, \end{cases} \\ &= p_Y\omega(s - st). \end{aligned}$$

Once again the properties of the compact-open topology show that γ_+ is continuous. $\square$

4. Stratified fibrations and stratified approximate fibrations

Definitions are recalled from [9,10], and a mapping cylinder criterion for a map to be a stratified approximate fibration is established (Proposition 4.2). Let X and Y be spaces with partitions $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively.

Definition 4.1.

(1) A map $p: X \to Y$ is a *stratified fibration* provided given any space Z and any commuting diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & Y \end{tikzcd}$$

with F a stratum preserving homotopy, there exists a *stratified solution*; i.e., a stratum preserving homotopy $\widetilde{F}: Z \times I \to X$ such that $\widetilde{F}(z,0) = f(z)$ for each $z \in Z$ and $F = p\widetilde{F}$.

(2) A map $p: X \to Y$ is a *stratified approximate fibration* provided given any commuting diagram as in (1), there exists a *stratified controlled solution*; i.e., a map $\widetilde{F}: Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\widetilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\overline{F}: Z \times I \times I \to Y$ defined by $\overline{F}|Z \times I \times [0,1) = p\widetilde{F}$ and $\overline{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous and stratum preserving along $I \times I$.

Of course, an *approximate fibration* is a map $p: X \to Y$ which is a stratified approximate fibration when X and Y are stratified with a single stratum. For more background on approximate fibrations from the point of view used here see [14]. The original definition of an approximate fibration is due to Coram and Duvall [5].

Proposition 4.2. *Let $p: X \to Y$ be a map between spaces with partitions. If*

$$q: \mathrm{holink}_s(\mathrm{cyl}(p), Y) \to Y$$

is a stratified fibration, then $p: X \to Y$ is a stratified approximate fibration.

Proof. Define

$$\begin{aligned} &f: X \to \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big), \\ &\Phi: X \times [0,1) \to \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big), \\ &\Psi: \mathrm{holink}_s\big(\mathrm{cyl}(p), Y\big) \times [0,1) \to X \end{aligned}$$

as in the proof of Theorem 3.1 and recall that $p = qf$, $\Psi_t\Phi_t = \mathrm{id}_X$ and $\Phi_t = f$ for all $t \in [0,1)$. Given a commuting diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ Z \times I \arrow[r, "H"] & Y \end{tikzcd}$$

with H is a stratum preserving homotopy, there is an induced commuting diagram

$$\begin{tikzcd} Z \arrow[r, "fh"] \arrow[d, "\times 0"'] & \mathrm{holink}_s(\mathrm{cyl}(p), Y) \arrow[d, "q"] \\ Z \times I \arrow[r, "H"] & Y. \end{tikzcd}$$

Since q is a stratified fibration, there is a stratified solution

$$G : Z \times I \to \operatorname{holink}_s\bigl(\operatorname{cyl}(p), Y\bigr)$$

of the second problem. Finally define a stratified controlled solution

$$\widetilde{H} : Z \times I \times [0,1) \to X; \qquad (x,s,t) \mapsto \Psi\bigl(G(z,s),t\bigr)$$

of the original problem. $\square$

5. Homotopically stratified spaces

Some definitions from Quinn [23] are recalled (see also [9,10,13]) and the main result Theorem 5.11 is stated and proved.

Definition 5.1. A subset $Y \subseteq X$ is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I$, $h_1(U) = Y$, and $h((U \setminus Y) \times [0,1)) \subseteq X \setminus Y$.

Definition 5.2. A space X with a stratification satisfying the Frontier Condition is a *homotopically stratified space* if the following two conditions are satisfied:

(i) *Forward Tameness.* For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.

(ii) *Normal Fibrations.* For each $k > i$, the holink evaluation

$$q : \operatorname{holink}(X_i \cup X_k, X_i) \to X_i$$

is a fibration.

Definition 5.3. Let X be a space with a partition and let $Y \subseteq X$.

(i) Y is *stratified forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I$, $h_1(U) = Y$, $h((U \setminus Y) \times [0,1)) \subseteq X \setminus Y$, and h is stratum preserving along $[0,1)$. We say that h is a *nearly stratum preserving deformation of U to Y in X* rel Y.

(ii) The *space of nearly stratum preserving paths* is

$$\begin{aligned} &\mathrm{P}_{\mathrm{nsp}}(X,Y) \\ &\quad = \bigl\{\omega \in \operatorname{holink}(X,Y) \mid \text{there exists a stratum } X_i \text{ with } \omega((0,1]) \subseteq X_i\bigr\} \end{aligned}$$

and is partitioned by

$$\mathrm{P}_{\mathrm{nsp}}(X,Y)_i = \bigl\{\omega \in \mathrm{P}_{\mathrm{nsp}}(X,Y) \mid \omega(1) \in X_i\bigr\}.$$

Evaluation at 0 is denoted $q : \mathrm{P}_{\mathrm{nsp}}(X,Y) \to Y$.

We will need the following result from [10].

Proposition 5.4. *Let X be a homotopically stratified metric space with finitely many strata and let $Y \subseteq X$ be a closed union of some of the strata of X. Then*

(1) *Y is stratified forward tame in X, and*
(2) *$q : \mathrm{P}_{\mathrm{nsp}}(X, Y) \to Y$ is a stratified fibration.*

The following is part of the main result.

Proposition 5.5. *Let $p : X \to Y$ be a map between homotopically stratified metric spaces each with only finitely many strata. If the mapping cylinder $\mathrm{cyl}(p)$ with the natural partition is a homotopically stratified space, then p is a stratified approximate fibration.*

Proof. It follows from a variation of Proposition 5.4(2) established in [10], that $q : \mathrm{holink}_s(\mathrm{cyl}(p), Y) \to Y$ is a stratified fibration (one needs to observe here that $\mathrm{cyl}(p)$ is metrizable—see [13, §3]). Now apply Proposition 4.2. □

The following deformation extension property will be needed in verifying forward tameness (Proposition 5.7).

Lemma 5.6. *Let X, Y be locally compact metric spaces with partitions and let $p : X \to Y$ be a proper stratified approximate fibration. Suppose $A \subseteq Y$ is a closed union of strata for which there is an open neighborhood U of A in Y and a nearly stratum preserving deformation $h : U \times I \to Y$ of U to A in Y rel A. Let $p_U = p| : p^{-1}(U) \to U$. Then there exists a homotopy*

$$H : \mathrm{cyl}(p_U) \times I \to \mathrm{cyl}(p)$$

such that
(1) *$H_0 = \mathrm{inclusion} : \mathrm{cyl}(p_U) \to \mathrm{cyl}(p)$,*
(2) *$H|(U \times I) = h : U \times I \to Y$,*
(3) *$H((\mathrm{cyl}(p_U) \setminus U) \times I) \subseteq \mathrm{cyl}(p) \setminus Y$,*
(4) *$H| : (\mathrm{cyl}(p_U) \setminus U) \times I \to \mathrm{cyl}(p)$ is a stratum preserving homotopy.*

Proof. Define maps

$$F : p^{-1}(U) \times I \times I \to Y; \qquad (x, s, t) \mapsto h\big(p(x), st\big),$$
$$F' = F| : p^{-1}(U) \times [0, 1) \times I \to Y,$$
$$f : p^{-1}(U) \times [0, 1) \to X; \qquad (x, s) \mapsto x.$$

Then

$$\begin{array}{ccc} p^{-1}(U) \times [0,1) & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \times 0} & & \downarrow{\scriptstyle p} \\ p^{-1}(U) \times [0,1) \times I & \xrightarrow{F'} & Y \end{array}$$

is a stratified lifting problem. Thus, there is a stratified controlled solution

$$\widetilde{F} : p^{-1}(U) \times [0, 1) \times I \times [0, 1) \to X.$$

In particular, $\widetilde{F}$ is stratum preserving along $I \times [0,1)$, $\widetilde{F}(x,s,0,u) = x$ for each $(x,s,u) \in p^{-1}(U) \times [0,1) \times [0,1)$, and, by [13, §3], the function

$$\overline{F} : p^{-1}(U) \times [0,1) \times I \times I \to \mathrm{cyl}(p);$$
$$(x,s,t,u) \mapsto \begin{cases} \big[\widetilde{F}(x,s,t,u), u\big], & \text{if } u < 1, \\ F(x,s,t) = h(p(x), st), & \text{if } u = 1 \end{cases}$$

is continuous. Define a function

$$G : \big(\mathrm{cyl}(p_U) \times [0,1) \times I\big) \cup \big(U \times \{1\} \times I\big) \to \mathrm{cyl}(p);$$
$$\begin{cases} G([x,s],t,u) = \overline{F}(x,t,u,s), & \text{if } (x,s) \in p^{-1}(U) \times I, \\ G([y],t,u) = h(y,tu), & \text{if } (y,t) \in U \times [0,1), \\ G(y,1,u) = h(y,u), & \text{if } y \in U. \end{cases}$$

Even though G need not be continuous, it is true that

$$G| : \mathrm{cyl}(p_U) \times [0,1) \times I \to \mathrm{cyl}(p)$$

is continuous. To see this, recall that $\mathrm{cyl}(p_U)$ has the quotient topology. Let $q : (p^{-1}(U) \times I) \amalg Y \to \mathrm{cyl}(p_U)$ be the quotient map. It follows that

$$q \times \mathrm{id}_{[0,1)\times I} : \big[(p^{-1}(U) \times I) \amalg Y\big] \times [0,1) \times I \to \mathrm{cyl}(p_U) \times [0,1) \times I$$

is also a quotient map (e.g., [18, Theorem 20.1]). The continuity of $\overline{F}$ easily implies that $G| \circ (q \times \mathrm{id}_{[0,1)\times I})$ is continuous and the continuity of $G|$ follows from the standard transgression lemma. Since $\mathrm{cyl}(p_U)$ is metrizable (e.g., [13, §3]), we can use a partition of unity argument to construct a map $\varphi : \mathrm{cyl}(p_U) \to I$ such that

(1) $\varphi^{-1}(1) = U$,
(2) $s \leqslant \varphi([x,s] \leqslant 1$ for each $[x,s] \in \mathrm{cyl}(p_U) \setminus U$,
(3) $\mathrm{diam}\{G([x,s'],s,t) \mid \varphi([x,s]) \leqslant s' \leqslant 1\} < 1 - s$ for each $[x,s] \in \mathrm{cyl}(p_U) \setminus U$ and $t \in I$.

Then the function

$$H : \mathrm{cyl}(p_U) \times I \to \mathrm{cyl}(p);$$
$$\begin{cases} H([x,s],t) = G([x,\varphi([x,s])],s,t), & \text{if } (x,s) \in p^{-1}(U) \times I, \\ H([y],t) = G(y,1,t) = h(y,t), & \text{if } y \in U \end{cases}$$

is continuous and is the desired extension of h. $\square$

Proposition 5.7. *Let Y be a locally compact homotopically stratified metric space with a finite number of strata, let X be a locally compact metric space with a partition, and let $p : X \to Y$ be a proper stratified approximate fibration. If $A \subseteq Y$ is a closed union of strata of Y, then A is stratified forward tame in* $\mathrm{cyl}(p)$.

Proof. By Proposition 5.4(1), A is stratified forward tame in Y, so let U be an open neighborhood of A in Y with a nearly stratum preserving deformation $h : U \times I \to Y$ of U

to A in Y rel A and let $H : \mathrm{cyl}(p_U) \times I \to \mathrm{cyl}(p)$ be the extension of h given by Lemma 5.6. Define the map

$$\pi : \mathrm{cyl}(p) \times I \to \mathrm{cyl}(p);$$
$$\begin{cases} ([x,s],t) \mapsto [x,(1-t)s+t], & \text{if } (x,s) \in X \times I, \\ [y] \mapsto [y], & \text{if } y \in Y. \end{cases}$$

Define

$$G : \mathrm{cyl}(p_U) \times I \to \mathrm{cyl}(p); \qquad (z,t) \mapsto H(\pi(z,t),t).$$

Then G is a nearly stratum preserving deformation of $\mathrm{cyl}(p_U)$ to A in $\mathrm{cyl}(p)$ rel A. $\square$

Corollary 5.8. *Let $p : X \to Y$ be a proper stratified approximate fibration between locally compact homotopically stratified metric spaces each with only finitely many strata and suppose the strata of Y are path connected. Then the natural partition of* $\mathrm{cyl}(p)$ *is a stratification satisfying the Frontier Condition and Forward Tameness condition* 5.2(i).

Proof. The partition of $\mathrm{cyl}(p)$ is obviously locally finite (in fact, it is finite). That each stratum is locally closed (i.e., the intersection of an open set and a closed set) follows easily from the fact that p maps closed sets onto closed sets. For both the Frontier Condition and Forward Tameness there is only one nontrivial case to consider: suppose X_i, Y_j are strata of X, Y, respectively, and $Y_j \cap \mathrm{cl}(X_i \times [0,1)) \neq \emptyset$ where the closure is taken in $\mathrm{cyl}(p)$. Since $Y^j = \mathrm{cl}\, Y_j$ is stratified forward tame in Y by Proposition 5.4(1), it follows from Proposition 5.7 that Y^j is stratified forward tame in $\mathrm{cyl}(p)$. Hence, let U be an open neighborhood of Y^j in $\mathrm{cyl}(p)$ with a nearly stratum preserving deformation $h : U \times I \to \mathrm{cyl}(p)$ of U to Y^j in $\mathrm{cyl}(p)$ rel Y^j. Since $Y_j \cap \mathrm{cl}(X_i \times [0,1)) \neq \emptyset$ and Y_j is open in Y^j, there exists $(x_0,t_0) \in (X_i \times [0,1)) \cap U$ such that $p(x_0) \in Y \cap U$ and $h_1(p(x_0)) \in Y_j$. Say $p(x_0) \in Y_k$. Then for every $s \in [0,1)$, $\omega_s : [0,s] \to Y$; $\omega_s(t) = h(p(x_0),t)$ is a stratum preserving path with initial lift $x_0 \in X$. Since p is a stratified approximate fibration, there are stratum preserving paths $\widetilde{\omega}_s : [0,s] \to X$ with $\widetilde{\omega}_s(0) = x_0$ and $p\widetilde{\omega}_s$ as close to ω_s as desired. Then $(\widetilde{\omega}_s(s),s) \in X_i \times [0,1)$ are points arbitrarily close to $h_1(p(x_0))$. This shows that $h_1(p(x_0)) \in \mathrm{cl}(X_i \times [0,1))$. Since h is nearly stratum preserving, we also have $h_1(p(x_0)) \in \mathrm{cl}\, Y_k$. Since the stratification of Y satisfies the Frontier Condition, $Y_j \subseteq \mathrm{cl}(Y_k)$. Now suppose $y_0 \in Y_j$ is another point. Let $\sigma : I \to Y_j$ be a path from $h_1(p(x_0))$ to y_0. Since $\omega : I \to Y$; $\omega(t) = h(p(x_0),t)$ is a nearly stratum preserving path from $p(x_0)$ to $h_1(p(x_0))$ and $q : \mathrm{holink}(Y_k \cup Y_j, Y_j) \to Y_j$ is a fibration, it follows that there exists a nearly stratum preserving path $\tilde{\sigma} : I \to Y_k \cup Y_j$ from $p(x_0)$ to y_0. Now the same argument given above for $h_1(p(x_0))$ shows that $y_0 \in \mathrm{cl}(X_i \times [0,1))$. Hence $Y_j \subseteq \mathrm{cl}(X_i \times [0,1))$ and the Frontier Condition is verified.

To show that Y_j is forward tame in $\mathrm{cl}(X_i \times [0,1)) \cup Y_j$, simply let

$$V = h_1^{-1}(Y_j) \cap \left[\mathrm{cl}\left(X_i \times [0,1)\right) \cup Y_j\right].$$

Then $h|(V \times I)$ is a nearly stratum preserving deformation of V to Y_j in $\mathrm{cl}(X_i \times [0,1)) \cup Y_j$ rel Y_j. $\square$

Lemma 5.9. *Let X be a homotopically stratified metric space with finitely many strata and let $Y \subseteq X$ be a closed union of some of the strata of X. Suppose Z is a metric space for which there is a stratified lifting problem*:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & \mathrm{P}_{\mathrm{nsp}}(X, Y) \arrow[d, "q"] \\ Z \times I \arrow[r, "F"] & Y \end{tikzcd}$$

and let $Z_0 = \{z \in Z \mid f(z)(t) = f(z)(0)$ for all $t \in I\}$. Then there exists a stratified solution $\widetilde{F} : Z \times I \to \mathrm{P}_{\mathrm{nsp}}(X, Y)$ such that $\widetilde{F}(z, s)(t) = F(z, s)$ for all $(z, s, t) \in Z_0 \times I \times I$.

Proof. Let $G : (Z \setminus Z_0) \times I \to \mathrm{P}_{\mathrm{nsp}}(X, Y)$ be a stratified solution of the restricted problem (which exists by Proposition 5.4(1)). Using a partition of unity define a map $\varphi : (Z \setminus Z_0) \times I \to [0, 1)$ such that

$$\operatorname{diam}\bigl\{G(z, s)(t) \mid 0 \leqslant t \leqslant \varphi(z, s)\bigr\} \leqslant 2 \operatorname{diam}\bigl\{f(z)(I)\bigr\}$$
$$\text{for each } (z, s) \in (Z \setminus Z_0) \times I.$$

Define

$$\widetilde{F}(z, s)(t) = \begin{cases} G(z, s)(t\varphi(z, s)), & \text{if } z \in Z \setminus Z_0, \\ F(z, s), & \text{if } z \in Z_0. \end{cases} \qquad \square$$

The following proposition is the remaining part of the main result.

Proposition 5.10. *Let $p : X \to Y$ be a proper stratified approximate fibration between locally compact homotopically stratified metric spaces each with only finitely many strata and suppose the strata of Y are path connected. Then* $\mathrm{cyl}(p)$ *with the natural partition is a homotopically stratified space.*

Proof. Given Corollary 5.8 it only remains to verify the Normal Fibrations condition 5.2(ii). There is only one nontrivial case: suppose X_i, Y_j are strata of X, Y, respectively, and $Y_j \subseteq \mathrm{cl}(X_i \times [0, 1))$ where the closure is taken in $\mathrm{cyl}(p)$. We need to show that $q : \mathrm{holink}((X_i \times [0, 1)) \cup Y_j, Y_j) \to Y_j$ is a fibration. For this we will apply the machinery of [13, §4] with which we assume the reader is familiar. By Proposition 5.7, $Y^j = \mathrm{cl}\, Y_j$ is stratified forward tame in $\mathrm{cyl}(p)$. Let U be an open neighborhood of Y^j in $\mathrm{cyl}(p)$ with a nearly stratum preserving deformation $h : U \times I \to \mathrm{cyl}(p)$ of U to Y^j in $\mathrm{cyl}(p)$ rel Y^j. Let

$$V = h_1^{-1}(Y_j) \cap \bigl(X_i \times [0, 1)\bigr) \cap p_Y^{-1}(U \cap Y).$$

Then V is a neighborhood of Y_j in $(X_i \times [0, 1)) \cup Y_j$ and $r = h_1| : V \to Y_j$ is a retraction. By [13, §4] it suffices to verify the $W(r)$-lifting property for the pair $((X_i \times [0, 1)) \cup Y_j, Y_j)$. Recall the definitions:

$$W_1(r) = \bigl\{(z, \omega) \in Y_j \times Y_j^I \mid z = \omega(1)\bigr\},$$
$$W_2(r) = \bigl\{(z, \omega) \in (V \setminus Y_j) \times Y_j^I \mid r(z) = \omega(1)\bigr\},$$
$$W(r) = W_1(r) \cup W_2(r).$$

The goal is to define a map $\alpha : W(r) \to ((X_i \times [0,1)) \cup Y_j)^I$ such that

(1) $\alpha(z,\omega)(0) = \omega(0)$ for each $(z,\omega) \in W(r)$,

(2) $\alpha(z,\omega)(1) = z$ for each $(z,\omega) \in W(r)$,

(3) $\alpha(z,\omega) = \omega$ for each $(z,\omega) \in W_1(r)$,

(4) $\alpha(z,\omega) \in \mathrm{holink}((X_i \times [0,1)) \cup Y_j, Y_j)$ for each $(z,\omega) \in W_2(r)$.

Define a stratified lifting problem

$$\begin{tikzcd} W(r) \arrow[r, "f"] \arrow[d, "\times 0"'] & \mathrm{P_{nsp}}(Y, Y_j) \arrow[d, "q"] \\ W(r) \times I \arrow[r, "F"] & Y_j \end{tikzcd} \tag{1}$$

by $f(z,\omega)(s) = h(p_Y(z), 1-s)$ and

$$F\big((z,\omega), s\big) = \begin{cases} h_1(p_Y h(z, 2s)), & \text{if } 0 \leqslant s \leqslant 1/2, \\ \omega(2-2s), & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

Note that if $(z,\omega) \in W_1(r)$, then $f(z,\omega)$ is the constant path at $z \in Y_j$. Thus, Lemma 5.9 implies that there exists a stratified solution of (1) $\widetilde{F} : W(r) \times I \to \mathrm{P_{nsp}}(Y, Y_j)$ such that $\widetilde{F}((z,\omega), s)(t) = F((z,\omega), s)$ for all $((z,\omega), s, t) \in W_1(r) \times I \times I$. Now define a stratified lifting problem

$$\begin{tikzcd} W_2(r) \times [0,1) \arrow[r, "g"] \arrow[d, "\times 0"'] & X \arrow[d, "p"] \\ W_2(r) \times [0,1) \times I \arrow[r, "G"] & Y \end{tikzcd} . \tag{2}$$

Note that there are continuous extensions of G to $G' : W(r) \times I \times I \to Y$ and g to $g' : W_2(r) \times I \to X$; it is just that G' is no longer stratum preserving and g' does not extend to $W(r)$. Let $\widetilde{G} : W_2(r) \times [0,1) \times I \times [0,1) \to X$ be a stratified controlled solution of (2). In particular,

$$\overline{G} : W_2(r) \times [0,1) \times I \times I \to \mathrm{cyl}(p);$$
$$\big((z,\omega), s, tu\big) \mapsto \begin{cases} \big[\widetilde{G}((z,\omega), s, t, u), u\big], & \text{if } u < 1, \\ \big[G((z,\omega), s, t)\big], & \text{if } u = 1 \end{cases}$$

is continuous [13, §3]. Use a partition of unity to construct a map $\varphi : W_2(r) \times [0,1) \to I$ such that

$$\operatorname{diam}\big\{\overline{G}\big((z,\omega), s, s, s'\big) \mid \varphi\big((z,\omega), s\big) \leqslant s' \leqslant 1\big\} \leqslant 1 - s \quad \text{and}$$
$$s \leqslant \varphi\big((z,\omega), s\big) < 1 \quad \text{for each } \big((z,\omega), s\big) \in W_2(r) \times [0,1).$$

Define

$$\beta : W_2(r) \times I \to \mathrm{cyl}(p); \qquad \big((z,\omega), s\big) \mapsto \overline{G}\big((z,\omega), s, s, \varphi((z,\omega), s)\big).$$

Then β extends continuously to $\widehat{\alpha} : W(r) \times I \to \mathrm{cyl}(p)$ by setting $\widehat{\alpha}((z,\omega), s) = F((z,\omega), s)$ if $(z,\omega) \in W_1(r)$. The adjoint of $\widehat{\alpha}$ is a map $\alpha : W(r) \to \mathrm{cyl}(p)^I$ with the desired properties. $\square$

The main result of this paper can now be established.

Theorem 5.11. *Let $p : X \to Y$ be a proper map between locally compact homotopically stratified metric spaces each with only finitely many strata and suppose the strata of Y are path connected. Then p is a stratified approximate fibration if and only if the mapping cylinder* $\mathrm{cyl}(p)$ *with the natural partition is a homotopically stratified space.*

Proof. If p is a stratified approximate fibration, then Proposition 5.10 implies that $\mathrm{cyl}(p)$ is homotopically stratified. The converse follows from Proposition 5.5. □

Remark 5.12. Connolly and Vajiac [4] have recently used this result in their work on ends of stratified spaces.

6. Locally flat submanifolds

In this section we specialize to the case in which the stratified spaces are (unstratified) manifolds. A proper approximate fibration between manifolds is called a *manifold approximate fibration*. It follows from Theorem 5.11 that the mapping cylinder of a manifold approximate fibration is a homotopically stratified space with two strata. This fact also follows from the results in [13] where homotopically stratified spaces with two strata were studied in more detail.

Here we are interested in knowing when the mapping cylinder of a manifold approximate fibration is actually a manifold with the base a locally flat submanifold. The answer (Theorem 6.1) is not surprising and should be considered part of the folklore (see Remark 6.3). Nevertheless, we include it here in order to make the point that this local flatness result follows from the general machinery of stratified spaces (see also [16]). Moreover, stratified techniques should have applications to certain nonlocally flat embeddings. We hope to explore nonlocally flat submanifolds in a future paper.

Theorem 6.1 (Folklore). *Let $p : M \to N$ be a proper map between manifolds with* $\dim M = m \geqslant 5$ *and* $\dim N = n$. *Then* $\mathring{\mathrm{cyl}}\,(p)$ *is a manifold with N a locally flat submanifold if and only if p is a manifold approximate fibration with homotopy fibre homotopy equivalent to the $(m-n)$-sphere.*

Proof. Suppose first that N is a locally flat submanifold of $\mathring{\mathrm{cyl}}\,(p)$. Then Fadell [7] proved that $q : \mathrm{holink}(\mathrm{cyl}(p), N) \to N$ is a fibration with fibre S^{m-n}. Theorem 3.1 implies that p is controlled homotopy equivalent to q. Hence p is an approximate fibration [14, 12.8] (or use Theorem 5.11 since $\mathrm{cyl}(p)$ is homotopically stratified). Moreover, the associated Hurewicz fibration of p is fibre homotopy equivalent to q [13, 4.8] so the homotopy fibre of p is S^{m-n}.

Conversely, assume that p is a manifold approximate fibration with homotopy fibre S^{m-n}. Let $p_0 : V \to \mathbb{R}^n$ be the fibre germ of p (cf. [14]). Then the homotopy fibre of p_0 is S^{m-n} and there is a controlled homotopy equivalence from p_0 to the projection

$\mathrm{proj}: S^{m-n} \times \mathbb{R}^n \to \mathbb{R}^n$ [14, 12.5, 12.15]. By [3] $p_0 : V \to \mathbb{R}^n$ and $\mathrm{proj}: S^{m-n} \times \mathbb{R}^n \to \mathbb{R}^n$ are controlled homeomorphic. This means that $\mathring{\mathrm{cyl}}\,(p)$ is locally homeomorphic to the mapping cylinder of $\mathrm{proj}: S^{m-n} \times \mathbb{R}^n \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is a locally flat submanifold of the manifold $\mathring{\mathrm{cyl}}\,(\mathrm{proj})$, the result follows. □

Corollary 6.2 (Folklore). *Let N be a submanifold of a manifold K with $\dim K = k \geqslant 6$ and $\dim N = n \geqslant 5$. Then N is locally flat in K if and only if N has a mapping cylinder neighborhood $\mathrm{cyl}(p)$ where $p : M \to N$ is a manifold approximate fibration, $\dim M = k - 1$, and the homotopy fibre of p is homotopy equivalent to the $(k - n - 1)$-sphere.*

Proof. The only new information needed here to apply Theorem 6.1 is the fact that if N is locally flat in K, then N has a mapping cylinder neighborhood in K. This result is due to Edwards [6] (for a published proof see [15]). □

Remark 6.3. Some of the papers which establish closely related local flatness results (and which likely could be used to give proofs of Theorem 6.1 and Corollary 6.2 as stated) are those of Chapman [3], Edwards [6], and Quinn [19,21,22].

7. Stratified collections of bundles

In this section we show that mapping cylinders of certain stratified maps between homotopically stratified spaces are themselves homotopically stratified spaces. This gives a different perspective on a result due to Cappell and Shaneson [2].

Definition 7.1. Let X and Y be spaces with partitions $\{X_i\}_{i \in \mathcal{I}}$ and $\{Y_j\}_{j \in \mathcal{J}}$, respectively. A map $p : X \to Y$ is a *stratified collection of bundles* if for every stratum Y_j of Y, $p^{-1}(Y_j)$ is a union of strata of X, and $p| : X_i \to Y_j$ is a fibre bundle projection for each stratum $X_i \subseteq p^{-1}(Y_j)$.

Examples 7.2.

(1) Let X, Y be Whitney stratified subsets of smooth manifolds M, N, respectively. Let $p : X \to Y$ be a proper map which is the restriction of a smooth map $M \to N$ such that for each stratum Y_j of Y, the inverse image $p^{-1}(Y_j)$ is a union of strata of X, each of which is mapped submersively onto Y_j. (In the literature, this is often the definition of a 'stratified map'.) It follows from Thom's First Isotopy Lemma [17, 24] that p is a stratified collection of fibre bundles.

(2) If $p : X \to Y$ is a proper algebraic map between real or complex algebraic varieties, then the conditions in (1) are satisfied (cf. [8] and Corollary 7.5 below) so that p is a stratified collection of bundles.

Theorem 7.3. *If X and Y are locally compact homotopically stratified metric spaces each with only finitely many strata, the strata of Y are path connected, and $p : X \to Y$ is a proper* *stratified collection of bundles, then p is a stratified approximate fibration and* $\mathrm{cyl}(p)$ *with the natural partition is a homotopically stratified space.*

Proof. By Theorem 5.11 it suffices to show that $\mathrm{cyl}(p)$ is homotopically stratified. That the partition is locally finite and the strata are locally closed is obvious. It remains to check the Frontier Condition, Forward Tameness and Normal Fibrations. As in the proofs of Corollary 5.8 and Proposition 5.10, there is only one nontrivial case to consider: suppose X_i, Y_j are strata of X, Y, respectively, and $Y_j \cap \mathrm{cl}(X_i \times [0,1)) \neq \emptyset$ where the closure is taken in $\mathrm{cyl}(p)$. Let Y_k be the stratum of Y such that $p(X_i) \subseteq Y_k$. Either $Y_k = Y_j$ or $Y_j \subseteq Y_k$.

For the Frontier Condition note that since Y_k is path connected, $p(X_i) = Y_k$. Since $Y_j \subseteq \mathrm{cl}(Y_k)$ it follows that $Y_j \subseteq \mathrm{cl}(X_i \times [0,1))$.

To verify that Y_j is forward tame in $(X_i \times [0,1)) \cup Y_j$ assume $Y_k \neq Y_j$ as the case $Y_k = Y_j$ is trivial (push down the mapping cylinder segments). Use the fact that Y_j is forward tame in $Y_k \cup Y_j$ to get a neighborhood U of Y_j in $Y_k \cup Y_j$ and a nearly stratum preserving deformation $h : U \times I \to Y_k \cup Y_j$ of U to Y_j in $Y_k \cup Y_j \operatorname{rel} Y_j$. Then there is a commuting diagram

$$\begin{tikzcd}
p^{-1}(U) \cap X_i \arrow[r, "\text{inclusion}"] \arrow[d] & X_i \arrow[d, "p|"] \\
{[p^{-1}(U) \cap X_i] \times [0,1)} \arrow[r, "F"] & Y_k
\end{tikzcd},$$

where $F(x,s) = h(p(x),s)$. Since $p| : X_i \to Y_k$ is a bundle projection, this 'half-open' homotopy lifting problem has a solution $\widetilde{F} : [p^{-1}(U) \cap X_i] \times [0,1) \to X_i$. Define $H : ([p^{-1}(U) \cap X_i] \cup Y_j) \times I \to (X_i \times [0,1)) \cup Y_j$ by

$$H(x,s) = \begin{cases} \big(\widetilde{F}(x,s), s\big) \in X_i \times [0,1), & \text{if } x \in p^{-1}(U) \cap X_i,\ 0 \leqslant s < 1, \\ h\big(p(x),1\big) \in Y_j, & \text{if } x \in p^{-1}(U) \cap X_i,\ s = 1, \\ x, & \text{if } x \in Y_j,\ 0 \leqslant s \leqslant 1. \end{cases}$$

Then H is a nearly stratum preserving deformation of $[p^{-1}(U) \cap X_i] \cup Y_j$ to Y_j in $[X_i \times [0,1)] \cup Y_j \operatorname{rel} Y_j$.

To show that $q : \mathrm{holink}((X_i \times [0,1)) \cup Y_j, Y_j) \to Y_j$ is a fibration, consider first the case that $Y_k = Y_j$. Note that $([X_i \cup [0,1)] \cup Y_j, Y_j) = (\mathrm{cyl}(p| : X_i \to Y_j), Y_j)$ and Theorem 3.1 implies that $q : \mathrm{holink}(\mathrm{cyl}(p| : X_i \to Y_j), Y_j)$ is controlled homotopy homotopy equivalent to a bundle projection. It follows that q is an approximate fibration [14, 12.8] and hence, a fibration [10, 8.6] (cf. [13, 4.8]). Now consider the case $Y_j \subseteq Y_k$, $Y_j \neq Y_k$ and suppose given a lifting problem:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "\times 0"'] & \mathrm{holink}((X_i \times [0,1)) \cup Y_j, Y_j) \arrow[d, "q"] \\
Z \times I \arrow[r, "F"] & Y_j
\end{tikzcd}.$$

There is an induced map

$$p_* : \mathrm{holink}\big((X_i \times [0,1)) \cup Y_j, Y_j\big) \to \mathrm{holink}(Y_k \cup Y_j, Y_j);$$
$$p_*(\omega)(t) = p_Y p\big(\omega(t)\big)$$

and a stratified lifting problem

$$\begin{tikzcd} Z \arrow[r, "p_* f"] \arrow[d, "\times 0"'] & \mathrm{holink}((Y_k \cup Y_j, Y_j) \arrow[d, "q"] \\ Z \times I \arrow[r, "F"] & Y_j \end{tikzcd}$$

which has a solution $G : Z \times I \to \mathrm{holink}(Y_k \cup Y_j, Y_j)$ (since Y is homotopically stratified). In turn, this gives rise to a commuting diagram

$$\begin{tikzcd} Z \times (0,1] \arrow[r, "\hat f"] \arrow[d, "\times 0"'] & X_i \arrow[d, "p|"] \\ Z \times I \times (0,1] \arrow[r, "\widehat G"] & Y_k \end{tikzcd},$$

where $\hat f(z,s) = p_X(f(z)(t))$ and $\widehat G(z,s,t) = G(z,s)(t)$. Since $p| : X_i \to Y_k$ is a fibre bundle, there exists a solution $H : Z \times I \times [0,1)$ of this 'half-open' homotopy lifting problem. Finally define a solution $\widetilde F : Z \times I \to \mathrm{holink}((X_i \times [0,1)) \cup Y_j, Y_j)$ of the original problem by

$$\widetilde F(z,s)(t) = \begin{cases} (H(z,s,t), c_I(f(z)(t))), & \text{if } 0 < t \leqslant 1, \\ F(z,s), & \text{if } t = 0. \end{cases} \qquad \square$$

Corollary 7.4 (Cappell and Shaneson [2]). *Let X, Y be Whitney stratified subsets of smooth manifolds M, N, respectively. Let $p : X \to Y$ be a proper map which is the restriction of a smooth map $M \to N$ such that for each stratum Y_j of Y, the inverse image $p^{-1}(Y_j)$ is a union of strata of X, each of which is mapped submersively onto Y_j. Then* $\mathrm{cyl}(p)$ *with the natural stratification is a homotopically stratified space.*

Proof. Combine Remark 7.2(1) with Theorem 7.3. $\square$

Corollary 7.5. *Suppose X and Y are*

$$\left\{\begin{array}{l} \textit{subanalytic} \\ \textit{real semialgebraic} \\ \textit{complex analytic} \\ \textit{complex algebraic} \end{array}\right. \textit{subsets of} \left\{\begin{array}{l} \textit{real analytic} \\ \textit{real algebraic} \\ \textit{complex analytic} \\ \textit{complex algebraic} \end{array}\right. \textit{smooth manifolds } M \textit{ and } N,$$

respectively. If $p : X \to Y$ is a proper

$$\left\{\begin{array}{l} \textit{subanalytic} \\ \textit{real algebraic} \\ \textit{complex analytic} \\ \textit{complex algebraic} \end{array}\right. \textit{map},$$

then there are Whitney stratifications of X and Y such that p becomes a stratified approximate fibration.

Proof. There are Whitney stratifications of X and Y so that p satisfies the hypothesis of Theorem 7.3. See the references in [8, Part I, 1.7]. □

Remarks 7.6.

(1) Cappell and Shaneson point out [2] that in the setting of Corollaries 7.4 and 7.5, $\operatorname{cyl}(p)$ need not be Whitney stratified. In particular, homotopically stratified spaces naturally arise in the setting of algebraic maps between algebraic varieties. The new point here is that stratified approximate fibrations also arise naturally.

(2) Cappell and Shaneson proved more in [2] than in Corollary 7.4, namely that $\operatorname{cyl}(p)$ is a 'manifold homotopy link-stratified space'. However, one of their main steps was showing that $\operatorname{cyl}(p)$ is homotopically stratified. Our proof of this step might be considered more elementary than the one offered in [2] because our proof relies only on Thom's First Isotopy Lemma whereas [2] uses both the First and Second Isotopy Lemmas (the Second Isotopy Lemma is usually reserved for situations where Thom regularity holds [17,24]).

(3) A topological treatment of Thom's Isotopy Lemmas is expected to appear in [11], cf. [9].

(4) The assumption of path connectivity of the strata in Theorem 7.3 can usually be avoided by passing to finer stratifications, cf. [1] and [10, §10].

(5) There is an obvious definition of 'stratified collection of fibrations' for which the proof of Theorem 7.3 generalizes.

Stratified collections are different from stratified systems (cf. [20,10]) as the final example shows.

Example 7.7. Let X be the unit square $\{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$ in $\mathbb{R}^2$, and let $Y = [0,1]$ be the unit interval. Stratify X and Y so that they each have exactly two strata with lower strata $X_0 = \{(0,y) \mid 0 \leqslant y \leqslant 1\}$ and $Y_0 = \{0\}$, respectively. Define a map $p : X \to Y$ with the following properties. First, $p^{-1}(Y_0) = X_0$ and $p| : X \setminus X_0 \to Y \setminus Y_0$ is a fibre bundle projection with fibre the closed interval. However, p is not to be first coordinate projection. Instead insist for each $n = 2,3,4,\dots$ that $p^{-1}(1/n)$ is a smooth arc in X running from the bottom edge of X to the top edge of X. The arc $p^{-1}(1/n)$ is to be the graph of a smooth map f_n defined on a small closed neighborhood $[a_n, b_n]$ of $1/n$ with image $[0,1]$ such that $f_n(a_n) = 0$, $f_n(b_n) = 1$ and f_n has exactly two local extrema in (a_n, b_n). The local extrema occur at c_n and d_n with $a_n < c_n < d_n < b_n$, $f_n(c_n) = 2/3$, and $f_n(d_n) = 1/3$. Then there is no deformation of a neighborhood U of Y_0 to Y_0 in $Y \operatorname{rel} Y_0$ which is covered by a deformation of $p^{-1}(U)$ to X_0 in $X \operatorname{rel} X_0$. Thus, p is a stratified collection of bundles, but not a stratified system of fibrations (or bundles).

References

[1] A. Beshears, G-isovariant structure sets and stratified structure sets, Ph.D. thesis, Vanderbilt Univ. (1997).

[2] S. Cappell and J. Shaneson, The mapping cone and cylinder of a stratified map, in: F. Quinn, ed., Prospects in Topology; Proc. Conf. in Honor of William Browder, Ann. of Math. Stud. 138 (Princeton Univ. Press, Princeton, 1995) 58–66.
[3] T.A. Chapman, Approximation results in topological manifolds, Mem. Amer. Math. Soc. 34 (251) (1981).
[4] F. Connolly and B. Vajiac, An end theorem for stratified spaces, Preprint.
[5] D. Coram and P. Duvall, Approximate fibrations, Rocky Mountain J. Math. 7 (1977) 275–288.
[6] R.D. Edwards, TOP regular neighborhoods, Handwritten manuscript (1973).
[7] E. Fadell, Generalized normal bundles for locally-flat embeddings, Trans. Amer. Math. Soc. 114 (1965) 488–513.
[8] M. Goresky and R. MacPherson, Stratified Morse Theory, Ergeb. Math. Grenzgeb. Ser. 3, Vol. 14 (Springer, New York, 1988).
[9] B. Hughes, Geometric topology of stratified spaces, Electron. Res. Announc. Amer. Math. Soc. 2 (1996) 73–81.
[10] B. Hughes, Stratified path spaces and fibrations, Preprint.
[11] B. Hughes, The geometric topology of stratified spaces, in preparation.
[12] B. Hughes and A. Ranicki, Ends of Complexes, Cambridge Tracts in Math. 123 (Cambridge Univ. Press, Cambridge, 1996).
[13] B. Hughes, L. Taylor, S. Weinberger and B. Williams, Neighborhoods in stratified spaces with two strata, Preprint.
[14] B. Hughes, L. Taylor and B. Williams, Bundle theories for topological manifolds, Trans. Amer Math. Soc. 319 (1990) 1–65.
[15] B. Hughes, L. Taylor and B. Williams, Manifold approximate fibrations are approximately bundles, Forum Math. 3 (1991) 309–325.
[16] B. Hughes and S. Weinberger, Surgery and stratified spaces, Preprint.
[17] J. Mather, Notes on Topological Stability (Harvard Univ., Cambridge, 1970) (photocopied).
[18] J.R. Munkres, Elements of Algebraic Topology (Addison-Wesley, New York, 1984).
[19] F. Quinn, Ends of maps I, Ann. of Math. 110 (1979) 275–331.
[20] F. Quinn, Ends of maps II, Invent. Math. 68 (1982) 353–424.
[21] F. Quinn, Resolutions of homology manifolds, and the topological characterization of manifolds, Invent. Math. 72 (1983) 267–284; Corrigendum, Invent. Math. 85 (1986) 653.
[22] F. Quinn, An obstruction to the resolution of homology manifolds, Michigan Math. J. 34 (1987) 285–291.
[23] F. Quinn, Homotopically stratified sets, J. Amer. Math. Soc. 1 (1988) 441–499.
[24] A. Verona, Stratified Mappings—Structure and Triangulability, Lecture Notes in Math. 1102 (Springer, New York, 1984).