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Topology with Control and Approximate Fibrations

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§1. Introduction. Since approximate fibrations were first introduced in the mid-1970s, they have appeared both as the object of study and as a tool in that modern branch of geometric topology called "controlled topology." The purpose of this paper is to illuminate that connection by giving a survey of the recent classification theorem for approximate fibrations between manifolds. This classification theorem represents joint work with Larry Taylor and Bruce Williams.

Approximate fibrations were first defined by Coram and Duvall [2] (see [1] for a survey). Their intention was to generalize Hurewicz fibrations as cell-like maps generalize homeomorphisms. They defined a map $p : E \to B$ to be an *approximate fibration* if given a space X, a map $G : X \times I \to B$, a map $g : X \to E$ such that $pg(x) = G(x, 0)$ for each x in X, and an open cover ϵ of B, there exists a map $\tilde{G} : X \times I \to E$ such that $g(x) = \tilde{G}(x, 0)$ for each x in X and $p\tilde{G}$ is ϵ-close to G. When approximate fibrations arise in geometric applications, they are often proper maps between manifolds. In that case they are called *manifold approximate fibrations* and those are the maps that are classified in [7]. In order to find a model for a classification theorem, we will first review the related classical theories of bundles and fibrations.

§2. Classical Theories. Fibre bundles and fibrations have both been classified in some sense and we want to review those theories here. A map $p : E \to B$ is a *fibre bundle over* B *with fibre* F if for each x in B there is a neighborhood U of x in B and a homeomorphism

$h : U \times F \to p^{-1}(U)$ such that

$$\begin{array}{ccc} U \times F & \xrightarrow{h} & p^{-1}(U) \\ & \searrow_{p_1} \quad \swarrow_{p|} & \\ & U & \end{array}$$

commutes where p_1 is projection onto the first coordinate. The following theorem is due to Milnor [10] and is based on work of Steenrod [12].

Theorem (Milnor). *There exists a "classifying space" $BTOP(F)$ such that equivalence classes of fibre bundles over B with fibre F are in one-to-one correspondence with $[B, BTOP(F)]$, the set of homotopy classes of maps of B into $BTOP(F)$.*

The significance of this theorem is that it reduces a geometric problem (classification of fibre bundles) to a problem in algebraic topology (understand the homotopy type of $BTOP(F)$).

Of course, one needs to know which equivalence relation is referred to in the statement of the theorem. Two fibre bundles $p : E \to B$ and $p' : E' \to B$ are *equivalent* if there exists a homeomorphism $h : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ & \searrow_{p} \quad \swarrow_{p'} & \\ & B & \end{array}$$

commutes.

It is also important to notice that the fibre F is only well-defined up to homeomorphism and is homeomorphic to $p^{-1}(x)$ for each x in B. The classifying space $BTOP(F)$ is constructed from the (simplicial) group $TOP(F)$ of self-homeomorphisms of F.

Next recall Stasheff's classification theorem for fibrations [11]. A map $p : E \longrightarrow B$ is a *fibration* if given a space X, a map $G : X \times I \to B$, a map $g : X \to E$ such that

$pg(x) = G(x,0)$ for each x in X, there exists a map $\tilde{G} : X \times I \to E$ such that $g(x) = \tilde{G}(x,0)$ for each x in X and $p\tilde{G} = G$. If B is path-connected, (as we will always assume it to be), then the *homotopy fibre* F of p is defined to be $p^{-1}(x)$ for some x in B. It is a theorem that the homotopy fibre is well-defined up to homotopy. One can consider the (simplicial) monoid $G(F)$ of self-homotopy equivalences of F and form a "classifying space" $BG(F)$. In order to classify fibrations, one needs the correct notion of equivalence. Two fibrations $p : E \to B$ and $p' : E' \to B$ are *fibre homotopy equivalent* provided there exist maps $f : E \to E'$, $g : E' \to E$ and homotopies $gf \simeq id_E$, $fg \simeq id_{E'}$, such that all maps and homotopies commute over B. By a result of Dold, this is equivalent to requiring a homotopy equivalence $f : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow_p \quad \swarrow_{p'} & \\ & B & \end{array}$$

commutes.

Theorem (Stasheff). *Fibre homotopy equivalence classes of fibrations over B with homotopy fibre F are in one-to-one correspondence with $[B, BG(F)]$, the set of homotopy classes of maps of B into $BG(F)$.*

In both of these theorems it is important to have the appropriate definitions of "fibre" and "equivalence." In addition, fibrations and bundles both satisfy a pull-back property which is crucial in the proofs of these theorems. For example, in order to go from a map $f : B \to BTOP(F)$ to a bundle over B with fibre F, one forms a pull-back diagram

$$\begin{array}{ccc} E & \longrightarrow & ETOP(F) \\ p\downarrow & & \downarrow \\ B & \xrightarrow{f} & BTOP(F) \end{array}$$

where $ETOP(F) \longrightarrow BTOP(F)$ is a "universal" fibre bundle with fibre F. The pull-back property referred to above asserts that $p : E \to B$ is indeed a fibre bundle with fibre F. To show that the equivalence class of p is determined by the homotopy class of f, one needs to know that if $\tilde{p} : \tilde{E} \longrightarrow B \times I$ is a fibre bundle, then $\tilde{p} \mid : \tilde{p}^{-1}(B \times 0) \longrightarrow B$ is equivalent to $\tilde{p} \mid : \tilde{p}^{-1}(B \times 1) \longrightarrow B$.

Fibrations satisfy analogous properties. In classifying manifold approximate fibrations, we will look for similar properties. However, a more fundamental problem is to find the appropriate category in which to work.

§3. The controlled category. For a space B we now define a category C_B, called the *controlled category over* B. The objects of C_B are spaces over B; that is, an object consists of a space E and a map $p : E \longrightarrow B$. A morphism of C_B from $p : E \longrightarrow B$ to $p' : E' \longrightarrow B$ is called a *controlled map*, denoted f^c, and consists of a continuous family of maps $f_t : E \longrightarrow E'$, $0 \leq t < 1$, (that is,F the induced map $f : E \times [0,1) \longrightarrow E' \times [0,1)$ is continuous) such that

$$\tilde{f} = (p' \times id_{[0,1)}) f \cup p : E \times [0,1] \longrightarrow B \times [0,1]$$

is continuous.

To understand what a controlled map is note that if B is compact metric, $\tilde{f}$ is continuous if and only if $\lim_{t \to 1} p' f_t = p$. Thus we don't require that

$$\begin{array}{ccc} E & \xrightarrow{f_t} & E' \\ & \searrow^{p} \qquad \swarrow^{p'} & \\ & B & \end{array}$$

commutes for any value of t, but we do require that the degree of noncommutativity goes to 0 as t goes to 1. Also, under mild conditions on p' (or a change of topology for mapping cylinders), $\tilde{f}$ is continuous if and only if the induced map on mapping cylinders $f \cup id_B : M(p) \longrightarrow M(p')$ is continuous.

Note that if $f : E \longrightarrow E'$ is a fibre map (i.e., $p'f = p$), then $f \times id_{[0,1)}$ is a controlled map. But not all controlled maps arise this way. For example, let E, E' and B all be the disk B^2, let $p = id$, and let p' be a cell-like map with a nondegenerate point inverse. Then there does not exist a fibre map from E to E', but there is a controlled map.

A controlled map f^c is a *controlled homeomorphism* if the map $f : E \times [0,1) \longrightarrow E' \times [0,1)$ is a homeomorphism and if f^{-1} defines a controlled map from p' to p. Thus, controlled homeomorphisms are the isomorphisms in the category $\mathcal{C}_B$.

In order to talk about *controlled homotopy equivalences* we simply have to note that if $p : E \longrightarrow B$ is a space over B, then $E \times I$ becomes a space over B by considering the composition

$$E \times I \xrightarrow{p_1} E \xrightarrow{p} B.$$

Using the notion of controlled map, we can give a new definition of approximate fibrations. We say $p : E \longrightarrow B$ is an *approximate fibration* if given a space X, a map $G : X \times I \longrightarrow B$, and a map $g : X \to E$ such that $pg(x) = G(x,0)$ for each x in X, there exists a controlled map $\tilde{G}^c$ from $G : X \times I \longrightarrow B$ to $p : E \longrightarrow B$ such that $\tilde{G}_t(x,0) = g(x)$ for each x in X and $0 \leq t < 1$. It is shown in [7] that this definition is often equivalent to the original one of Coram and Duvall. The advantage of the new definition is that there are not different lifts corresponding to each open cover of B.

§4. Controlled homeomorphisms of manifold approximate fibrations. Recall that we are looking for the appropriate equivalence relation to put on manifold approximate fibrations in order to prove a classification theorem. According to the comments in §2 we should examine 1-parameter families of manifold approximate fibrations. For this we have the following theorem from [7].

Theorem (Hughes-Taylor-Williams). *Let $p_0 : M_0 \longrightarrow B$ and $p_1 : M_1 \longrightarrow B$ be manifold approximate fibrations. Assume M_0, M_1 and B are closed and that $\dim M_i \geq 5$, $i = 0, 1$. The following are equivalent:*

1) *There exists a manifold approximate fibration $p : E \longrightarrow B \times I$ such that p equals p_i over $B \times \{i\}$ for $i = 0, 1$ and the composition $E \xrightarrow{p} B \times I \xrightarrow{p_2} I$ is a fibre bundle.*

2) *For every $\epsilon > 0$ there exists a homeomorphism $h : M_0 \longrightarrow M_1$ such that $p_1 h$ is ϵ-close to p_0.*

3) *There exists a controlled homeomorphism from p_0 to p_1.*

It follows that the reasonable equivalence relation is that of controlled homeomorphism. The equivalence of conditions 1) and 2) follows from earlier work on approximate fibrations [5]. A proof that 2) and 3) are equivalent can be found in [7]. The following result is the basic principle which is used in the proof of this theorem.

Theorem (Hughes). *If M and B are closed manifolds and $\dim M \geq 5$, then the space of all approximate fibrations from M to B in the compact-open topology is locally n-connected for every $n \geq 0$.*

This theorem was proved in [4] for M a Hilbert cube manifold and in [5] for finite dimensional manifolds.

§5. Fibre germs for manifold approximate fibrations. In order to get an analogue of the classification theorems for bundles and fibrations, we need the appropriate notion of fibre of a manifold approximate fibration. First recall the result of Coram and Duvall which says that if $p : E \longrightarrow B$ is a proper approximate fibration between ANRs and B

is connected, then $p^{-1}(x)$ and $p^{-1}(y)$ are shape equivalent whenever x and y are in B. At first it might seem that the fibre should be defined to be a point inverse up to shape equivalence. However, it turns out that if one fattens up the point inverses then one gets an object well-defined up to homeomorphism, in fact, controlled homeomorphism. To make this precise, let $p : M \longrightarrow B$ be a manifold approximate fibration and assume that M and B are closed and B is connected and of dimension i. For convenience we will only consider the case where B is oriented. Let $\mathbb{R}^i \subset B$ be an orientation-preserving embedding. By restricting p over $\mathbb{R}^i$ we get a manifold approximate fibration $q : V \longrightarrow \mathbb{R}^i$ ($V = p^{-1}(\mathbb{R}^i)$ and $q = p\,|$). Then $q : V \longrightarrow \mathbb{R}^i$ is called the *fibre germ* of p.

Theorem (Hughes-Taylor-Williams). *If* $\dim M \geq 5$, *then the fibre germ is well-defined up to controlled homeomorphism.*

The proof of this theorem appears in [7] and is based on engulfing and the annulus theorem.

§6. The classification theorem. Now that we have the notion of controlled homeomorphism and fibre germ, we can state the main result. However, there is one more problem that we need to discuss. Namely, approximate fibrations may fail to pull-back to approximate fibrations. This fact prevents the existence of a simple classifying space. Nevertheless, approximate fibrations do pull back to approximate fibrations over open embeddings [1] (this was used implicitly in defining the fibre germ). It follows that, unlike the bundle and fibration cases, we can construct a classifying space twisted by the tangent bundle of B.

Let B be a closed manifold of dimension i and assume that B is oriented. We want to classify all manifold approximate fibrations over B with a given fibre germ. Thus, let $q : V \longrightarrow \mathbb{R}^i$ be a manifold approximate fibration where V is without boundary and of dimension greater than 4. Let $M(q)$ denote the mapping cylinder of q and let $TOP^{\text{level}}(q)$ denote the (simplicial) group of all self-homeomorphisms of $M(q)$ which preserve the levels of the mapping cylinder (i.e., preserve the $[0,1]$-coordinates) and which restrict to an orientation-preserving homeomorphism of the base $\mathbb{R}^i$. Then restriction defines a homomorphism $TOP^{\text{level}}(q) \longrightarrow TOP_i$ where TOP_i is the (simplicial) group of orientation-preserving homeomorphisms of $\mathbb{R}^i$. The kernel of this homomorphism is $TOP^c(q)$, the (simplicial) group of controlled homeomorphisms from q to q. Thus, on the level of classifying spaces, there is a fibration

$$\begin{array}{ccc} BTOP^c(q) & \longrightarrow & BTOP^{\text{level}}(q) \\ & & \downarrow \\ & & BTOP_i \end{array}$$

Now B has a topological tangent bundle which is classified by a map $\tau : B \longrightarrow BTOP_i$. The following result is our main classification theorem.

Theorem (Hughes-Taylor-Williams). *The controlled homeomorphism classes of manifold approximate fibrations over B with fibre germ $q : V \longrightarrow \mathbb{R}^i$ are in one-to-one correspondence with vertical homotopy classes of liftings*

$$\begin{array}{ccc} & & BTOP^{\text{level}}(q) \\ & \nearrow & \downarrow \\ B & \xrightarrow{\tau} & BTOP_i \end{array}$$

This theorem was first proved in [7], but there the description of the space $BTOP^{\text{level}}(q)$ was more abstract. The more concrete description presented here appears in [8].

In some sense $BTOP^c(q)$ is the classifying space for manifold approximate fibrations with fibre germ q. However, it is twisted over $BTOP_i$ and the twisting appears because the tangent bundle of B might be twisted. When B has a trivial tangent bundle, then τ is homotopic to a constant and we get the following result.

Corollary. *If B is parallelizable, then controlled homeomorphism classes of manifold approximate fibrations over B with fibre germ q are in one-to-one correspondence with* $[B, BTOP^c(q)]$.

For example, if B is parallelizable, then there exists a manifold approximate fibration over B with any given fibre germ.

§7 Applications. The classification theorem for manifold approximate fibrations meshes well with the classification theorems for bundles and fibrations. Hence, the various theories can be compared. For example, if $p : E \longrightarrow B$ is a fibration, then we can identify the obstructions to finding a manifold approximate fibration over B which is controlled homotopy equivalent to p. Likewise, if $q : M \longrightarrow B$ is a manifold approximate fibration, then we can identify the obstructions to finding a fibre bundle over B (with closed manifold fibre) which is controlled homeomorphic to q. It follows that most of the original questions posed in the 1970s concerning the relationship between fibrations, bundles and approximate fibrations can now be answered. These applications appear in [7].

Another application concerns controlled surgery theory. If $p : E \longrightarrow B$ is a fibration and $f : M \longrightarrow E$ is a normal map, then in [9] we identify the obstructions to finding a normal cobordism from f to a controlled homotopy equivalence. The obstructions are reduced to bounded surgery over $\mathbb{R}^i$ and then to algebra.

One of the early motivations for the work in [5] was to study controlled homotopy topological structures. Specifically, let $p : E \longrightarrow B$ be a fibre bundle between closed manifolds. A *controlled structure* on p is a map $f : M \longrightarrow E$ where M is a closed manifold of the same dimension as E and f is a $p^{-1}(\epsilon)$-homotopy equivalence for every $\epsilon > 0$. An obstruction theory was developed in [6] which answers the question: when is f homotopic to a homeomorphism, with arbitrarily small metric control measured in B? The result is that one can suitably define a space of controlled structures on p and that space is homotopy equivalent to a space of sections of a bundle associated to the Whitney sum of p and the tangent bundle of B. The fibre of the associated bundle is the space of bounded structures on $F \times \mathbb{R}^i$ where F is the fibre of p and $i = \dim B$. The point is that this sectioning theorem is obtained as a corollary in [7] by comparing the classifications theorems of fibrations and manifold approximate fibrations over B. Moreover, the obstruction groups are identified in [3] in the infinite dimensional case and [9] in the finite dimensional case.

Our (Hughes, Taylor, Williams) current work involves using our theory of manifold approximate fibrations to study controlled topology over Riemannian manifolds of nonpositive sectional curvature. In addition, in a joint project with Shmuel Weinberger, we are applying our theory to study germs of neighborhoods of singular sets in singular manifolds. The main source of examples for this study comes from the theory of finite groups acting topologically on a manifold.

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