Topology and its Applications 15 (1983) 159–172
North-Holland Publishing Company

APPROXIMATE FIBRATIONS AND BUNDLE MAPS ON HILBERT CUBE MANIFOLDS

C. Bruce HUGHES
Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

Received 30 November 1981
Revised 31 March 1982

This paper is concerned with parameterized families of approximate fibrations from a compact Hilbert cube manifold M to a compact polyhedron B. The main result shows how to straighten out certain of these families to be nearly like a product. As an application of this technique, it is shown that an approximate fibration $p:M\to B$ can be approximated arbitrarily closely by bundle maps if and only if p is homotopic via approximate fibrations to a bundle map. Another result is that the space of bundle maps from M to B is locally n-connected for each $n\geq 0$.

AMS Subj. Class. (1980): Primary 57N20, 55R65, 54C10; Secondary 55R10, 55R05, 54C35

Hilbert cube manifold   approximate fibration   bundle map   Hurewicz fibration

1. Introduction

Let $p:M\to B$ be an approximate fibration from a Hilbert cube manifold M onto an ANR B such that p is homotopic via approximate fibrations to a bundle map. The purpose of this paper is to prove that p can be approximated arbitrarily closely by bundle maps. The techniques used to prove this also provide local homotopy information about the space of bundle maps from M to B. In order to state our results more completely we need to make some definitions.

Except for the various function spaces which we consider, all spaces are locally compact, separable and metric. The Hilbert cube Q is the countable infinite product of closed intervals and a *Hilbert cube manifold* or *Q-manifold* is a manifold modeled on Q. A map $f:X\to Y$ (i.e., a continuous function) is *proper* provided $f^{-1}(C)$ is compact for all compact subsets C of Y. If α is an open cover of Y, then a proper map $f:X\to Y$ is said to be an *α-fibration* if for all maps $F:Z\times[0,1]\to Y$ and $g:Z\to X$ for which $fg=F_0$, there is a map $G:Z\times[0,1]\to X$ such that $G_0=g$ and fG is α-close to F (that is, given any (z,t) in $Z\times[0,1]$ there is a U in α containing both $fG(z,t)$ and $F(z,t)$). If $\varepsilon>0$, then we also use ε to denote the open cover of Y by balls of diameter ε. Thus, we speak of *ε-fibrations*.

A proper map $f:X \to Y$ between ANRs is an *approximate fibration* provided it is an α-fibration for each open cover α of Y. This notion was introduced in [7]. By a *bundle map* we mean a proper map which is the projection map of a locally trivial bundle.

Let I^n denote the n-cell $[0, 1]^n$ and let $\mathbf{0}$ denote the point $(0, 0, \ldots, 0)$ in I^n. A map $f:X \times I^n \to Y \times I^n$ is *fiber preserving* (f.p.) if $\pi f = \pi$ where π denotes projection onto I^n. If $f:X \times I^n \to Y \times I^n$ is a f.p. map and t is in I^n, then $f_t:X \to Y$ denotes the restriction $f|(X \times \{t\}):X \times \{t\} \to Y \times \{t\}$.

We can now state the main result.

Theorem. *Let M be a Q-manifold, let B be an* ANR, *and let $p:M \times I^n \to B \times I^n$ be a* f.p. *map such that $p_t:M \to B$ is an approximate fibration for each t in I^n and $p_0:M \to B$ is a bundle map. For every open cover α of B there exists a* f.p. *homeomorphism $H:M \times I^n \to M \times I^n$ such that $H_0 =$* id *and p_tH_t is α-close to p_0 for each t in I^n.*

This result has been obtained by Steve Ferry [10] for $B = S^1$ even when p_0 is not assumed to be a bundle map.

Our first application of the main theorem is concerned with the problem of approximating an approximate fibration arbitrarily closely by bundle maps. This problem has been considered by several authors. Let M be a Q-manifold and let $p:M \to B$ be an approximate fibration. Husch [12] proved that if $B = S^1$, then p can be approximated arbitrarily closely by bundle maps if and only if p is homotopic to a bundle map. On the other hand, Chapman and Ferry [6] constructed approximate fibrations $p:M \to S^2$ which are homotopic to bundle maps but cannot be approximated arbitrarily closely by bundle maps. A positive result due to Chapman [2] is when B is a polyhedron p can be approximated arbitrarily closely by block bundle maps provided that a certain π_1-condition on the homotopy fiber of p is satisfied (versions of this theorem when M is a topological manifold have been given by Quinn [13] and Chapman [3]). We have the following corollaries of the main theorem.

Corollary 1. *Let $p:M \to B$ be an approximate fibration from a Q-manifold M onto an* ANR *B. If p is homotopic via approximate fibrations to a bundle map, then p can be approximated arbitrarily closely by bundle maps.*

Corollary 2. *Let $p:M \to \mathbb{R}^m$ be an approximate fibration from a Q-manifold M onto euclidean m-space $\mathbb{R}^m$. Then p can be approximated arbitrarily closely by bundle maps if and only if p is boundedly homotopic to a bundle map.*

A homotopy is *bounded* if there is a constant $c > 0$ such that the diameter of the track of any point is less than c. Husch [12] has also obtained a proof of Corollary 2 for the case $m = 1$ only assuming that p is properly homotopic to a bundle map.

It follows from [11] that the converse of Corollary 1 is true when B is a polyhedron.

Our next results are concerned with local homotopy properties of certain spaces of maps. If X is a metric space (not necessarily locally compact) and $n \geq 0$ is an integer, then X is said to be *locally n-connected* (written LC^n) if for each x in X and each open subset U of X containing x, there exists an open subset V of X such that $x \in V \subset U$ and any map $f: \partial I^{n+1} \to V$ extends to a map $\tilde{f}: I^{n+1} \to U$. In [11] it was shown that the space of approximate fibrations from a compact Q-manifold M onto a compact polyhedron B is LC^n for each $n \geq 0$. In this paper spaces of maps are always given the compact-open topology.

Corollary 3. *Let M be a compact Q-manifold and let B be a compact polyhedron. The space of bundle maps from M onto B, the space of Hurewicz fibrations from M onto B, and the closure of the space of bundle maps from M onto B are* LC^n *for each* $n \geq 0$.

By the closure of the space of bundle maps from M onto B we mean its closure in the space of all maps from M onto B. Thus, we are considering all maps which are uniform limits of bundle maps. By a theorem of Chapman and Ferry [4] this is the same space as the closure of the space of Hurewicz fibrations from M onto B.

A key ingredient for the proofs in this paper is a parameterized engulfing theorem from [11] which we state below for the convenience of the reader. For notation B and Z will be fixed ANRs where Z is compact and $Z \times \mathbb{R}$ is an open subset of B. Let π_1 denote projection onto Z and π_2 projection onto $\mathbb{R}$. Let C be a closed (possibly empty) subset of ∂I^n. Let $\theta: \mathbb{R} \times I^n \to \mathbb{R} \times I^n$ be a f.p. homeomorphism such that $\theta | \mathbb{R} \times C$ is the identity, $x \leq \pi_2 \theta(x, t)$ for each x in $\mathbb{R}$ and t in I^n, and θ is supported on $[-1, 1] \times I^n$. Finally, let $\theta': B \times I^n \to B \times I^n$ denote the f.p. homeomorphism which extends $\mathrm{id}_Z \times \theta$ via the identity.

Theorem 1.1 ([11], Theorem 4.3). *For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if M is a Q-manifold and $p: M \times I^n \to B \times I^n$ is a* f.p. *map such that $p_t: M \to B$ is a δ-fibration over $Z \times [-2, 2]$ for each t in I^n, then there is a* f.p. *homeomorphism $\tilde{\theta}: M \times I^n \to M \times I^n$ such that*

(i) *$\tilde{\theta} | M \times C$ is the identity,*

(ii) *$p\tilde{\theta}$ is ε-close to $\theta' p$,*

(iii) *$\tilde{\theta}$ is supported on $p^{-1}(Z \times [-1, 1] \times I^n)$,*

(iv) *there is a* f.p. *homotopy $\tilde{\theta}_s: \mathrm{id} \simeq \tilde{\theta}$, $0 \leq s \leq 1$, which is a $(\pi_1 p)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ and is supported on $p^{-1}(Z \times [-1, 1] \times (I^n \backslash C))$.*

This paper is organized as follows. The proof of the main theorem is contained in Section 4. Section 2 contains the wrapping up construction needed for the proof of the handle theorem of Section 3. The main theorem is proved by using the handle theorem. The proofs of the corollaries are in Section 5.

2. Wrapping up

In this section we present a variation of the parameterized wrapping construction in [11] which in turn is a variation of Chapman's wrapping construction in [2]. Throughout this section n will denote a fixed non-negative integer and B and Z will denote fixed ANRs where Z is compact and $Z \times \mathbb{R}$ is an open subset of B. Let $e:\mathbb{R} \to S^1$ be the covering projection defined by $e(x) = \exp(\frac{1}{4}\pi i x)$ where S^1 is the set of complex numbers of absolute value 1 (thus e has period 8).

Theorem 2.1. *For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if M is a Q-manifold and $p: M \times I^n \to B \times I^n$ is a* f.p. *map such that $p_t: M \to B$ is an approximate fibration for each t in I^n, and $f: M \times I^n \to M \times I^n$ is a* f.p. *map such that $f_0 = \mathrm{id}$ and $(p_0 \times \mathrm{id})f$ is δ-close to p, then the following statement is true: for every $\mu > 0$ there is a compact Q-manifold $\bar{M}$ and f.p. maps $\bar{p}: \bar{M} \times I^n \to Z \times S^1 \times I^n$, $\bar{f}: \bar{M} \times I^n \to \bar{M} \times I^n$, and $\Psi: p^{-1}(Z \times (-1, 1) \times I^n) \to \bar{M} \times I^n$ such that $\bar{p}_t$ is a μ-fibration for each t in I^n, $\bar{f}_0 = \mathrm{id}$, $(\bar{p}_0 \times \mathrm{id})\bar{f}$ is ε-close to $\bar{p}$, Ψ is an open embedding and such that the following two diagrams commute:*

$$
\begin{array}{ccc}
\bar{M} \times I^n & \xrightarrow{\bar{p}} & Z \times S^1 \times I^n \\
\uparrow \Psi & & \uparrow \mathrm{id} \times e \times \mathrm{id} \\
p^{-1}(Z \times (-1, 1) \times I^n) & \xrightarrow{p|} & Z \times (-1, 1) \times I^n
\end{array}
$$

$$
\begin{array}{ccc}
\bar{M} \times I^n & \xrightarrow{\bar{f}} & \bar{M} \times I^n \\
\uparrow \Psi & & \uparrow \Psi_0 \times \mathrm{id} \\
p^{-1}(Z \times (-\frac{1}{2}, \frac{1}{2}) \times I^n) & \xrightarrow{f|} & (p_0 \times \mathrm{id})^{-1}(Z \times (-1, 1) \times I^n)
\end{array}
$$

Proof. In Section 5 of [11] the compact Q-manifold $\bar{M}$, the f.p. map $\bar{p}: \bar{M} \times I^n \to Z \times S^1 \times I^n$, and the open embedding $\Psi: p^{-1}(Z \times (-1, 1) \times I^n) \to \bar{M} \times I^n$ are constructed. We are only left with defining the f.p. map $\bar{f}: \bar{M} \times I^n \to \bar{M} \times I^n$. In order to do this we must recall some of the details of the construction of $\bar{M}$.

Let $\theta: \mathbb{R} \to \mathbb{R}$ be the PL homeomorphism which is supported on $[-3, 3]$ with the property that for $-2.2 \leq x \leq -1.8$ we have $\theta(x) = x + 4$. Let $\theta_t: \mathrm{id} \simeq \theta$, $0 \leq t \leq 1$, be the obvious isotopy. Let $\bar{\theta}_t: Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$, $0 \leq t \leq 1$, be the f.p. isotopy defined by $\bar{\theta}_t = \mathrm{id} \times \theta_t \times \mathrm{id}$. Then $\bar{\theta}_t$ extends via the identity to a f.p. isotopy defined on all of $B \times I^n$ which we continue to denote by $\bar{\theta}_t$. Using the parameterized engulfing result (Theorem 1.1) one produces a f.p. isotopy $\tilde{\theta}_t : M \times I^n \to M \times I^n$, $0 \leq t \leq 1$, such that $\tilde{\theta}_0 = \mathrm{id}$, $\tilde{\theta}_t$ is supported on $p^{-1}(Z \times [-3, 3] \times I^n)$ for each t in $[0, 1]$, and $p\tilde{\theta}_t$ is μ'-close to $\bar{\theta}_t p$ for each t in $[0, 1]$ where $\mu' > 0$ is as small as we choose. Perhaps we should indicate how Theorem 1.1 is used here to get the isotopy $\tilde{\theta}_t$ because the statement of that theorem only provides a homotopy $\mathrm{id} \simeq \tilde{\theta}_1$. One simply applies the theorem to the map

$$p \times \mathrm{id} : M \times I^n \times [0, 1] \to B \times I^n \times [0, 1]$$

to cover the f.p. homeomorphism $\theta' : B \times I^n \times [0, 1] \to B \times I^n \times [0, 1]$ defined by $\theta'(b, s, t) = (\bar{\theta}_t(b, s), t)$.

To define $\tilde{M}$ one first defines

$$Y = \tilde{\theta}_1 p^{-1}(Z \times (-\infty, -2] \times I^n) \backslash p^{-1}(Z \times (-\infty, -2) \times I^n).$$

Then define $\tilde{M} = Y/{\sim}$ where $\sim$ is the equivalence relation on Y generated by $y \sim \tilde{\theta}_1(y)$ for each y in $p^{-1}(Z \times \{-2\} \times I^n)$. As in [11, Section 5] we use the submersion theorem from [14] to identify $\tilde{M}$ with a product $\bar{M} \times I^n$ where $\bar{M}$ is the compact Q-manifold $Y_0/{\sim}$. Here Y_0 is the slice of Y over 0; that is,

$$Y_0 = \tilde{\theta}_1 p^{-1}(Z \times (-\infty, -2] \times \{0\}) \backslash p^{-1}(Z \times (-\infty, -2) \times \{0\}).$$

Recall that Ψ can be regarded as an inclusion map.

To define the map $\bar{f} : \bar{M} \times I^n \to \bar{M} \times I^n$ we first define an auxiliary map $f' : Y \to p^{-1}(Z \times \mathbb{R} \times \{0\}) \times I^n$. By a slight abuse of notation we identify $p^{-1}(Z \times \mathbb{R} \times \{0\}) \times I^n$ with $(p_0 \times \mathrm{id})^{-1}(Z \times \mathbb{R} \times I^n)$. Define subsets E_- and E_+ of Y by $E_- = p^{-1}(Z \times \{-2\} \times I^n)$ and $E_+ = \tilde{\theta}_1(E_-)$. The map f' is to have the following two properties:

(i) $f' \simeq f| \mathrm{rel}\, E_- \cup Y_0$ via a f.p. $(p_0 \times \mathrm{id})^{-1}(\mu'')$-homotopy where the size of $\mu'' > 0$ depends on the size of μ'.

(ii) $f'|E_+ = ((\tilde{\theta}_1 | M \times \{0\}) \times \mathrm{id}_{I^n}) \circ f \circ \tilde{\theta}_1^{-1} | E_+$.

Define a homotopy $g_t : E_- \cup E_+ \to p^{-1}(Z \times \mathbb{R} \times \{0\}) \times I^n$, $0 \leq t \leq 1$, by $g_t | E_- = f | E_-$ and

$$g_t | E_+ = ((\tilde{\theta}_t | M \times \{0\}) \times \mathrm{id}_{I^n}) \circ f \circ \tilde{\theta}_t^{-1} | E_+$$

for each t in $[0, 1]$. One sees that g_t, $0 \leq t \leq 1$, is a $(p_0 \times \mathrm{id})^{-1}(\mu'')$-homotopy and that $g_0 = f|$. By the estimated homotopy extension property [5] g_t extends to a homotopy $\tilde{g}_t$, $0 \leq t \leq 1$, defined on all of Y so that $f' = \tilde{g}_1$ satisfies conditions (i) and (ii) above.

Notice that $(p_0 \times \mathrm{id})f'$ is δ'-close to $p|$ where the size of $\delta' > 0$ depends on the size of δ and μ. This implies that we may assume that the image of f' lies in

$[\tilde{\theta}_1^{-1}(Y_0)\cup Y_0\cup\tilde{\theta}_1(Y_0)]\times I^n$. The quotient map $q: Y_0\times I^n\to\bar{M}\times I^n$ extends to

$$\tilde{q}:[\tilde{\theta}_1^{-1}(Y_0)\cup Y_0\cup\tilde{\theta}_1(Y_0)]\times I^n\to\bar{M}\times I^n$$

in the obvious way. Now $\tilde{q}f'$ factors through the appropriate equivalence classes to define the map $\bar{f}:\bar{M}\times I^n\to\bar{M}\times I^n$. □

3. The handle theorem

In this section we present a proof of the handle theorem which is needed for the proof of the main theorem in Section 4. As might be expected the handle theorem is proved by a torus trick. For notation let $\mathbb{R}^m$ denote euclidean m-space and let $T^m=S^1\times\cdots\times S^1$ (m times) denote the m-torus. For each $r>0$ let B_r^m denote the m-cell $[-r,r]^m$ in $\mathbb{R}^m$ and let $\mathring{B}_r^m$ denote its interior $(-r,r)^m$. Let $e^m=e\times\cdots\times e:\mathbb{R}^m\to T^m$ denote the product covering projection where $e:\mathbb{R}\to S^1$ is the covering projection of Section 2. Finally, let B be an ANR which contains $\mathbb{R}^m$ as an open subset and let $n\geqslant 0$ be an integer.

Theorem 3.1. *For every $\varepsilon>0$ there exists a $\delta>0$ so that if M is a Q-manifold, $p:M\times I^n\to B\times I^n$ is a* f.p. *map such that $p_t:M\to B$ is an approximate fibration for each t in I^n, and $f:M\times I^n\to M\times I^n$ is a* f.p. *map such that $f_0=\mathrm{id}$ and $(p_0\times\mathrm{id})f$ is δ-close to p, then there is a* f.p. *map $\tilde{f}:M\times I^n\to M\times I^n$ which is a homeomorphism over $p_0^{-1}(B_1^m)\times I^n$ and which is homotopic to f* rel $(M\times\{0\})\cup f^{-1}(p_0^{-1}(B\backslash\mathring{B}_2^m)\times I^n)$ *via a* f.p. *$(p_0\times\mathrm{id})^{-1}(\varepsilon)$-homotopy.*

Proof. As in [8, Section 8] we regard $T^{m-1}\times\mathbb{R}$ as an open subset of $\mathring{B}_3^m$ so that the composition

$$e^{m-1}\times\mathrm{id}:B_2^m=B_2^{m-1}\times[-2,2]\to T^{m-1}\times[-2,2]\subset T^{m-1}\times\mathbb{R}\subset\mathring{B}_3^m$$

is the inclusion. Given the hypothesis of the theorem we use Theorem 2.1 to find a compact Q-manifold $\bar{M}$ and a f.p. commutative diagram

$$\begin{array}{ccc}
\bar{M}\times I^n & \xrightarrow{\ \bar{p}\ } & T^m\times I^n\\
\Big\uparrow{\scriptstyle\Psi} & & \Big\uparrow{\scriptstyle \mathrm{id}\times e\times\mathrm{id}}\\
p^{-1}(T^{m-1}\times(-2,2)\times I^n) & \xrightarrow{\ p|\ } & T^{m-1}\times(-2,2)\times I^n
\end{array}$$

where Ψ is an open embedding and $\bar{p}_t:\bar{M}\to T^m$ is a μ-fibration for each t in I^n ($\mu>0$ is as small as we need). Moreover, there is a f.p. map $\bar{f}:\bar{M}\times I^n\to\bar{M}\times I^n$ such that $\bar{f}_0=\mathrm{id}$, $(\bar{p}_0\times\mathrm{id})\bar{f}$ is δ'-close to $\bar{p}$ (where the size of $\delta'>0$ depends on the size of δ), and such that the following diagram commutes:

$$\begin{array}{ccc} \bar{M}\times I^n & \xrightarrow{\bar{f}} & \bar{M}\times I^n \\ \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \Psi_0\times \mathrm{id}} \\ p^{-1}(B^m_{1.9}\times I^n) & \xrightarrow{f} & p_0^{-1}(\mathring{B}^m_2)\times I^n \end{array}$$

Note that we are again abusing notation in order to identify $p_0^{-1}(\mathring{B}^m_2)\times I^n$ with $(p_0\times \mathrm{id})^{-1}(\mathring{B}^m_2\times I^n)$. Also note that $fp^{-1}(B^m_{1.9}\times I^n)$ is contained in $(p_0\times \mathrm{id})^{-1}(\mathring{B}^m_2\times I^n)$ provided δ is small enough.

The next step is to form the pull-back diagram:

$$\begin{array}{ccc} M'' & \xrightarrow{p'} & \mathbb{R}^m\times I^n \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle e^m\times \mathrm{id}} \\ \bar{M}\times I^n & \xrightarrow{\bar{p}} & T^m\times I^n \end{array}$$

Since π is a bundle map we may identify M'' with $M'\times I^n$ where $M'=\pi^{-1}(\bar{M}\times\{0\})$. It follows that M' is a Q-manifold and $p'_t: M'\to\mathbb{R}^m$ is a μ'-fibration for each t in I^n (where the size of $\mu'>0$ depends on the size of μ).

Define $j:\bar{p}^{-1}(e^m(B^m_3)\times I^n)\to M''=M'\times I^n$ by

$$j(x,t)=((x,t),((e^m\,|\,B^m_3)\times \mathrm{id})^{-1}(\bar{p}(x,t)))$$

for $(x,t)\in\bar{p}^{-1}(e^m(B^m_3)\times I^n)\subset\bar{M}\times I^n$. In other words j is the lifting of the inclusion map $\bar{p}^{-1}(e^m(B^m_3)\times I^n)\subset\bar{M}\times I^n$ so that j is a homeomorphism onto $(p')^{-1}(B^m_3\times I^n)$. This map j induces a map

$$j_0\times \mathrm{id}:(\bar{p}_0\times \mathrm{id})^{-1}(e^m(B^m_3)\times I^n)\to M'\times I^n$$

which is a homeomorphism onto $(p'_0\times \mathrm{id})^{-1}(B^m_3\times I^n)=(p'_0)^{-1}(B^m_3)\times I^n$.

In the following assertion we show how the information of the map $\bar{f}$ is carried along by the pull-back construction.

Assertion 3.2. *There exists a* f.p. *map* $f':M'\times I^n\to M'\times I^n$ *such that* $f'_0=\mathrm{id}$, $(p'_0\times \mathrm{id})f'$ *is* δ''*-close to* p', *and such that the following diagram commutes*:

$$\begin{array}{ccc} M'\times I^n & \xrightarrow{f'} & M'\times I^n \\ \Big\uparrow{\scriptstyle j} & & \Big\uparrow{\scriptstyle j_0\times \mathrm{id}} \\ \bar{p}^{-1}(e^m(B^m_2)\times I^n) & \xrightarrow{\bar{f}} & (\bar{p}_0\times \mathrm{id})^{-1}(e^m(B^m_3)\times I^n) \end{array}$$

Proof. It will be useful to regard the copy of $M'\times I^n$ which is to be the image of f' as being obtained from the following pull-back diagram:

$$\begin{array}{ccc} M'\times I^n & \xrightarrow{p_0'\times \mathrm{id}} & \mathbb{R}^m\times I^n \\ {\scriptstyle \pi_0\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle e^m\times \mathrm{id}} \\ \bar{M}\times I^n & \xrightarrow{\bar{p}_0\times \mathrm{id}} & T^m\times I^n \end{array}$$

Then to define f' we only need to specify its two components, namely $(\pi_0\times \mathrm{id})f':M'\times I^n\to \bar{M}\times I^n$ and $(p_0'\times \mathrm{id})f':M'\times I^n\to \mathbb{R}^m\times I^n$. Of course, this needs to be done in such a way that

$$(\bar{p}_0\times \mathrm{id})(\pi_0\times \mathrm{id})f'=(e^m\times \mathrm{id})(p_0'\times \mathrm{id})f'.$$

The first component is defined by $(\pi_0\times \mathrm{id})f'=\bar{f}\pi$.

The second component will be defined by lifting a certain homotopy. Recall that $(\bar{p}_0\times \mathrm{id})\bar{f}$ is δ'-close to $\bar{p}$. Thus we may assume that there is a $\delta^{\#}$-homotopy $H:\bar{p}\pi\simeq(\bar{p}_0\times \mathrm{id})\bar{f}\pi$ in $T^m\times I^n$ where the size of $\delta^{\#}>0$ depends on the size of δ'. This homotopy is taken to be f.p. and since $\bar{f}_0=id$ we can also assume that H is rel $M'\times\{0\}$. Since $\bar{p}\pi=(e^m\times \mathrm{id})p'$ and e^m is a bundle map, we can lift H to get a f.p. homotopy $G_t:M'\times I^n\to\mathbb{R}^m\times I^n$, $0\leqslant t\leqslant 1$, $G_0=p'$, $(e^m\times \mathrm{id})G_t=H_t$, and $G_t|M'\times\{0\}=p_0'$ for each t in $[0,1]$. One now checks that by defining the second component by $(p_0'\times id)f'=G_1$, the map f' will satisfy the conditions of the assertion. The size of $\delta''>0$ depends on the size $\delta^{\#}$. $\square$

We now return to the proof of Theorem 3.1. The next step is to modify f'. Choose $K>3$ so large that $(p_0'f_t')^{-1}(B_3^m)\subset(p_0')^{-1}(B_K^m)$ for each t in I^n. Let $u:M'\to[0,1]$ be a map such that

$$u^{-1}(1)=(p_0')^{-1}(B_K^m)\quad\text{and}\quad u^{-1}(0)=(p_0')^{-1}(\mathbb{R}^m\backslash \mathring{B}_{K+1}^m).$$

Define $f'':M'\times I^n\to M'\times I^n$ by $f''(x,t)=(f'_{tu(x)}(x),t)$. Note that $f''=f'$ over $(p_0')^{-1}(B_3^m)\times I^n$ and if $L\geqslant K+1$ is chosen so large that

$$(p_0'f_t'')^{-1}(\mathbb{R}^m\backslash \mathring{B}_L^m)\cap(p_0'f_t')^{-1}(B_{K+1}^m)=\emptyset$$

for each t in I^n, then $f''=\mathrm{id}$ over $(p_0')^{-1}(\mathbb{R}^m\backslash \mathring{B}_L^m)\times I^n$. There is also a f.p. homotopy $f'\simeq f''$ which is rel $(M'\times\{0\})\cup(f')^{-1}((p_0')^{-1}(B_3^m)\times I^n)$.

It will be convenient to modify f'' slightly so that for some small neighborhood N of 0 in I^n it is the identity over $M'\times N$. This can be done by a small f.p. homotopy rel $M'\times\{0\}$, so we just assume that $f_t''=\mathrm{id}$ for each t in N.

Let $\gamma:\mathbb{R}^m\times I^n\to\mathbb{R}^m\times I^n$ be a radially defined f.p. homeomorphism such that γ is the identity on $B_2^m\times I^n$, γ_t takes B_L^m to B_3^m for t in $I^n\backslash N$, and for t in N, γ_t is phased out so that $\gamma_0=\mathrm{id}$. Notice that $\gamma_t p_t':M'\to\mathbb{R}^m$ is a μ''-fibration for each t in I^n where the size of $\mu''>0$ depends on the size of μ'.

The next step is to use engulfing to modify f'' so that it becomes a homeomorphism over $(p_0')^{-1}(B_2^m)\times I^n$. To this end let $\theta:\mathbb{R}^m\to\mathbb{R}^m$ be a homeomorphism supported on B_8^m such that θ preserves the last $(m-1)$-coordinates of any point and $\theta|B_2^m$ adds 5.1 units to the first coordinate of any point (in B_2^m). Use θ to define a f.p. homeomorphism $\bar\theta:\mathbb{R}^m\times I^n\to\mathbb{R}^m\times I^n$ such that $\bar\theta_t=\theta$ for each t in $I^n\backslash N$ and for t in N, $\bar\theta_t$ is phased out so that $\bar\theta_0=\mathrm{id}$. By the parameterized engulfing result (Theorem 1.1) there are f.p. homeomorphisms $\tilde\theta^1$, $\tilde\theta^2:M'\times I^n\to M'\times I^n$ such that $\gamma p'\tilde\theta^1$ is $\bar\mu$-close to $\theta\gamma p'$, $\gamma(p_0'\times\mathrm{id})\tilde\theta^2$ is $\bar\mu$-close to $\theta\gamma(p_0'\times\mathrm{id})$ where the size of $\bar\mu>0$ depends on the size of $\mu''>0$, and

$$\tilde\theta^1|(M'\times\{0\})=\mathrm{id}=\tilde\theta^2|(M'\times\{0\}).$$

Moreover, just as in Section 2, we may assume that there are f.p. isotopies $\tilde\theta_s^1$: $\mathrm{id}\simeq\tilde\theta^1$ and $\tilde\theta_s^2:\mathrm{id}\simeq\tilde\theta^2$, $0\leqslant s\leqslant 1$, rel $M'\times\{0\}$ such that if $\Phi_s:\mathrm{id}\simeq\bar\theta$, $0\leqslant s\leqslant 1$, is the obvious isotopy, then $\gamma p'\tilde\theta_s^1$ is $\bar\mu$-close to $\Phi_s\gamma p'$ and $\gamma(p_0'\times\mathrm{id})\tilde\theta_s^2$ is $\bar\mu$-close to $\Phi_s\gamma(p_0'\times\mathrm{id})$.

Now define $f^*:M'\times I^n\to M'\times I^n$ by $f^*=(\tilde\theta^2)^{-1}f''\tilde\theta^1$. It follows that $f_0^*=\mathrm{id}$, f^* is a homeomorphism over $(p_0')^{-1}(B_2^m)\times I^n$, and there is a f.p. homotopy $(\tilde\theta_s^2)^{-1}f''\tilde\theta_s^1$, $0\leqslant s\leqslant 1$, from f'' to f^* rel $M'\times\{0\}$ which is a small homotopy when projected to $\mathbb{R}^m\times I^n$ by $p_0'\times\mathrm{id}$. Putting all of this together we have a f.p. homotopy $H:f'\simeq f^*$ which is rel $M'\times\{0\}$ and which is a $(p_0'\times\mathrm{id})^{-1}(\varepsilon)$-homotopy over $(p_0'\times\mathrm{id})^{-1}(B_3^m\times I^n)$.

We now want to use the map f^* and the homotopy H to define the required map $\tilde f:M\times I^n\to M\times I^n$. To do this one must first trace through the constructions to see that there is a homeomorphism $g:p^{-1}(\mathring{B}_2^m\times I^n)\to(p')^{-1}(\mathring{B}_2^m\times I^n)$ which makes the following two diagrams commute:

$$\begin{array}{ccc}
p^{-1}(\mathring{B}_2^m\times I^n) & \xrightarrow{\;g\;} & (p')^{-1}(\mathring{B}_2^m\times I^n)\\
 & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\
 & \mathring{B}_2^m\times I^n &
\end{array}$$

$$\begin{array}{ccc}
p^{-1}(B_{1.5}^m\times I^n) & \xrightarrow{\;g\;} & (p')^{-1}(B_{1.5}^m\times I^n)\\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'}\\
(p_0\times\mathrm{id})^{-1}(\mathring{B}_2^m\times I^n) & \xrightarrow{\;g_0\times\mathrm{id}\;} & (p_0'\times\mathrm{id})^{-1}(\mathring{B}_2^m\times I^n)
\end{array}$$

With this homeomorphism in hand we identify $p^{-1}(B_{1.5}^m\times I^n)$ with $(p')^{-1}(B_{1.5}^m\times I^n)$. Then $\tilde f$ is defined so that $\tilde f=f^*$ over $(p_0'\times\mathrm{id})^{-1}(B_1^m\times I^n)$, $\tilde f=f$ over $(p_0\times$

$\mathrm{id})^{-1}((\mathbb{R}^m \backslash \mathring{B}^m_{1.5}) \times I^n)$, and over $(p_0 \times \mathrm{id})^{-1}((B^m_{1.5} \backslash B^m_1) \times I^n)$ $\tilde{f}$ is defined by the homotopy $H: f' \simeq f^*$. This completes the proof of the theorem. □

4. Proof of the main theorem

In this section we prove the main result of this paper which is stated as Theorem 4.3 below. This follows quickly from Proposition 4.2 which is the global version of Theorem 3.1 when p_0 is a bundle map. In the complement to Theorem 4.3 we consider the case when p_0 is a limit of bundle maps. We begin with a lemma needed for the proof of Proposition 4.2.

Lemma 4.1. *Let M be a Q-manifold, let X and Y be* ANRs, *$p: M \to X$ a bundle map, and $r: X \to Y$ a cell-like map. Then $rp: M \to Y$ can be approximated arbitrarily closely by bundle maps.*

Proof. Let $g: X \times Q \to Y \times Q$ be a homeomorphism close to $r \times \mathrm{id}_Q$ and let $u: M \times Q \to M$ be a homeomorphism close to projection (see [1]). If $\pi: Y \times Q \to Y$ denotes projection, then $\pi g(p \times \mathrm{id}_Q)u^{-1}: M \to Y$ is a bundle map close to rp. □

Proposition 4.2. *Let B be a polyhedron and let $n \geq 0$ be an integer. For every open cover α of B there exists an open cover β of B so that if M is a Q-manifold, $p: M \times I^n \to B \times I^n$ is a* f.p. *map such that $p_t: M \to B$ is an approximate fibration for each t in I^n and p_0 is a bundle map, and $f: M \times I^n \to M \times I^n$ is a* f.p. *map such that $f_0 = \mathrm{id}$ and $p_0 f_t$ is β-close to p_t for each t in I^n, then f is f.p.-homotopic* rel $M \times \{0\}$ *to a* f.p. *homeomorphism $H: M \times I^n \to M \times I^n$ via a $(p_0 \times \mathrm{id})^{-1}(\alpha \times I^n)$-homotopy (where $\alpha \times I^n$ denotes the open cover $\{U \times I^n \mid U \in \alpha\}$ of $B \times I^n$).*

Proof. We only treat the case where B is a compact polyhedron. The general case follows from the same argument. We proceed by induction on the dimension of B. The reader should consult [2, Section 9] for a similar induction argument. If $\dim B = 0$, the result is trivial. Assume that $\dim B = m > 0$ and that the result is true for $(m-1)$-dimensional polyhedra. For simplicity of notation we assume that B has only one principle simplex Δ^m. We identify $\mathring{\Delta}^m$ with $\mathbb{R}^m$ and let $B^{m-1} = B \backslash \mathbb{R}^m$ and $B_0 = B \backslash \mathring{B}^m_1$. Let $r: B_0 \to B^{m-1}$ denote the radially defined cell-like retraction.

By Theorem 3.1 we may assume that f is a homeomorphism over $p_0^{-1}(B^m_3) \times I^n$. By Lemma 4.1, $rp_0 | p_0^{-1}(B_0): p_0^{-1}(B_0) \to B^{m-1}$ is close to a bundle map p_0'. By the isotopy extension theorem (we have in mind the Q-manifold version of Corollary 1.2 of [8]; see [9] for the deformation theorem to make this work), there is a f.p. homeomorphism $h: p_0^{-1}(B_0) \times I^n \to f^{-1}(p_0^{-1}(B_0) \times I^n)$ such that $h_0 = \mathrm{id}$.

By [11, Theorem 7.2] the map $(p_0' \times \mathrm{id})fh$ is f.p. homotopic rel $p_0^{-1}(B_0) \times \{0\}$ via a small homotopy to a f.p. map $p': p_0^{-1}(B_0) \times I^n \to B^{m-1} \times I^n$ such that $p_t': p_0^{-1}(B_0) \to B^{m-1}$ is an approximate fibration for each t in I^n.

We are now in a position to apply the inductive hypothesis. There is a small homotopy of fh to a f.p. homeomorphism $g:p_0^{-1}(B_0)\times I^n \to p_0^{-1}(B_0)\times I^n$ such that $g_0=\mathrm{id}$ (the smallness of the homotopy is measured in B_0). By a sliced version of the strong Z-set unknotting theorem [2, Proposition 3.8] (a proof of this sliced version can be derived from [11, Proposition 3.10]), we may assume that this homotopy is rel $(fh)^{-1}(p_0^{-1}(\partial B_1^m)\times I^n)\cup(p_0^{-1}(B_0)\times\{0\})$. The homeomorphism gh^{-1} pieces together with f to give the desired homeomorphism H. To get the correct control on the homotopy from f to H, we must choose notation so that B_1^m is nearly all of Δ^m. $\square$

We are now ready for our main result.

Theorem 4.3. *Let M be a Q-manifold, let B be an* ANR, *and let $p:M\times I^n\to B\times I^n$ be a* f.p. *map such that $p_t:M\to B$ is an approximate fibration for each t in I^n and $p_0:M\to B$ is a bundle map. For every open cover α of B there exists a* f.p. *homeomorphism $H:M\times I^n\to M\times I^n$ such that $H_0=\mathrm{id}$ and p_tH_t is α-close to p_0 for each t in I^n.*

Proof. We first treat the case when B is a polyhedron. This will follow from Proposition 4.2 (replacing H_t by H_t^{-1}) once we produce a f.p. map $f:M\times I^n\to M\times I^n$ such that $f_0=\mathrm{id}$ and p_0f_t is close to p_t for each t in I^n. But such a map comes from taking a f.p. lift of a f.p. homotopy from $p_0\times\mathrm{id}$ to p, and since p_0 is a bundle map we actually get $p_0f_t=p_t$.

We next treat the case when B is a Q-manifold. By Chapman's theorem on the triangulation of Q-manifolds (see [1]) we can assume that $B=B_1\times Q$ where B_1 is a polyhedron and the fibers $\{b\}\times Q$, $b\in B_1$, are short with respect to the open cover α of B. Let $\pi:B\to B_1$ be projection and note that $\pi p_t:M\to B_1$ is an approximate fibration for each t in I^n and $\pi p_0:M\to B_1$ is a bundle map. By the case above for polyhedra, there is a f.p. homeomorphism $H:M\times I^n\to M\times I^n$ such that $H_0=\mathrm{id}$ and πp_tH_t is close to πp_0 for each t in I^n. If it is close enough, then p_tH_t is α-close to p_0 for each t in I^n.

Finally, we treat the general case when B is any ANR. Let $u:Q\times M\to M$ be a homeomorphism close to projection. Note that $(\mathrm{id}_Q\times p_t)u^{-1}:M\to Q\times B$ is an approximate fibration for each t in I^n and $(\mathrm{id}_Q\times p_0)u^{-1}$ is a bundle map. By Edwards' ANR theorem (see [1]) $Q\times B$ is a Q-manifold. By the Q-manifold case above it follows that there is a f.p. homeomorphism $H:M\times I^n\to M\times I^n$ such that $H_0=\mathrm{id}$ and $(\mathrm{id}_Q\times p_t)u^{-1}H_t$ is close to $(\mathrm{id}_Q\times p_0)u^{-1}$ for each t in I^n. If it is close enough, then p_tH_t is α-close to p_0 for each t in I^n. $\square$

It will be useful to know that Theorem 4.3 remains true when p_0 is not assumed to be a bundle map, but merely a map which can be approximated arbitrarily closely by bundle maps. As pointed out in the introduction such maps include proper Hurewicz fibrations from a Q-manifold to an ANR [4]. To see that such a variation on Theorem 4.3 is true, one merely traces through the same proofs of this section. We state this formally.

Complement 4.4. *Theorem* 4.3 *remains true when the map* p_0 *is merely assumed to be a map which can be approximated arbitrarily closely by bundle maps.*

5. Proof of corollaries

In this section we present the proofs of Corollaries 1, 2 and 3 which are stated in the introduction.

Proof of Corollary 1. If M is a Q-manifold, B is an ANR and $f:M\to B$ is an approximate fibration which is homotopic via approximate fibrations to a bundle map, then there is a f.p. map $p:M\times I\to B\times I$ such that $p_t:M\to B$ is an approximate fibration for each t in I, $p_1=f$, and p_0 is a bundle map. If α is a given open cover of B, then by Theorem 4.3 there is a f.p. homeomorphism $H:M\times I\to M\times I$ such that p_tH_t is α-close to p_0 for each t in I. It follows that $p_0H_1^{-1}$ is a bundle map α-close to f as required. □

Proof of Corollary 2. Let M be a Q-manifold and let $f:M\to\mathbb{R}^m$ be an approximate fibration. Suppose first that f is boundedly homotopic to a bundle map. Then there is a proper f.p. map $p:M\times I\to\mathbb{R}^m\times I$ such that p_0 is a bundle map and $p_1=f$, and for which there is some constant $c>0$ (possibly quite large) such that $p_t:M\to\mathbb{R}^m$ is a c-fibration for each t in I (we assume that $\mathbb{R}^m$ is endowed with the usual euclidean metric). Let $\varepsilon>0$ be given and note that p_t is an ε-fibration for t sufficiently close to 0 or 1. Choose $K>0$ large and let $\gamma:\mathbb{R}^m\times I\to\mathbb{R}^m\times I$ be the f.p. homeomorphism such that $\gamma_t(x)=x/K$ for t not close to 0 or 1 and such that $\gamma_0=\mathrm{id}=\gamma_1$. If this is done correctly, then $\gamma_tp_t:M\to\mathbb{R}^m$ is an ε-fibration for each t in I. By Remark 7.5 in [11] we see that there is a f.p. map $\tilde{p}:M\times I\to\mathbb{R}^m\times I$ such that $\tilde{p}_0=p_0$, $\tilde{p}_1=p_1$, and $\tilde{p}_t$ is an approximate fibration for each t in I. Now apply Corollary 1 to show that $p_1=f$ can be approximated arbitrarily closely by bundle maps.

Conversely, suppose that $f:M\to\mathbb{R}^m$ is an approximate fibration which can be approximated arbitrarily closely by bundle maps. Then f is boundedly homotopic to any bundle map which is sufficiently close to it. □

Let M be a compact Q-manifold and let B be a compact polyhedron. Corollary 3 is concerned with showing that certain spaces of maps from M onto B are LC^n. We now isolate the conditions that such a space must satisfy in order for our proof to show that it is LC^n. Let Γ be a space of maps from M onto B (endowed with the compact-open topology) such that

(1) every map in Γ is the uniform limit of bundle maps from M onto B, and

(2) if $p \in \Gamma$ and $h: M \to M$ is a homeomorphism which is isotopic to the identity, then $ph \in \Gamma$.
Corollay 3 follows immediately from the following theorem.

Theorem 5.1. *Γ is* LC^n *for each* $n \geq 0$.

Proof. Let $n \geq 0$ be given. It is clear that an inductive argument will prove the theorem once we show that: for every $\varepsilon > 0$ there exists a $\delta > 0$ so that given any $\mu > 0$ and any f.p. map $p: M \times \partial I^{n+1} \to B \times \partial I^{n+1}$ such that $p_t \in \Gamma$ and p_t is δ-close to p_0 for each t in ∂I^{n+1}, there is a f.p. map

$$G: M \times \partial I^{n+1} \times [0, 1] \to B \times \partial I^{n+1} \times [0, 1]$$

such that $G|M \times \partial I^{n+1} \times \{0\} = p$, $G|M \times \{0\} \times \{s\} = p_0$ for each s in $[0, 1]$, $G_{(t,s)}$ is ε-close to p_0 for each (t, s) in $\partial I^{n+1} \times [0, 1]$, $G_{(t,1)}$ is μ-close to p_0 for each t in ∂I^{n+1}, and $G_{(t,s)} \in \Gamma$ for each (t, s) in $\partial I^{n+1} \times [0, 1]$.

For G to be f.p. in this statement we mean that it preserves both the ∂I^{n+1} and $[0, 1]$ coordinates of any point. To prove this statement one first uses the fact that the space of approximate fibrations from M to B is LC^n [11] to extend p to a f.p. map $\tilde{p}: M \times I^n \to B \times I^n$ such that $\tilde{p}_t$ is $\frac{1}{3}\varepsilon$-close to $\tilde{p}_0 = p_0$ and $\tilde{p}_t$ is an approximate fibration for each t in I^n (this is possible if δ is small enough). Next use Theorem 4.3 to construct G as follows. Let $H: M \times I^n \to M \times I^n$ be a f.p. homeomorphism such that $H_0 = \mathrm{id}$ and $\tilde{p}_t H_t$ is μ-close to p_0 for each t in I^n. Let $r: I^n \times [0, 1] \to I^n$ be a homotopy such that $r(t, 0) = 0$, $r(t, 1) = t$ and $r(0, s) = 0$ for each (t, s) in $I^n \times [0, 1]$. Define G by $G(x, t, s) = \tilde{p}H(x, r(t, s))$ for each (x, t, s) in $M \times \partial I^{n+1} \times [0, 1]$. If $\mu < \frac{1}{3}\varepsilon$, then G satisfies our requirements. □

It should be clear that the proof of Theorem 5.1 actually shows that Γ is *uniformly* LC^n for each $n \geq 0$ (that is for every $n \geq 0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that every map $f: \partial I^{n+1} \to \Gamma$ with the diamter of $f(\partial I^{n+1})$ less than δ extends to a map $\tilde{f}: I^{n+1} \to \Gamma$ with the diameter of $f(I^{n+1})$ less than ε). We are of course giving Γ the usual sup metric. Whether the spaces of maps mentioned in Corollary 3 are locally contractible remains an open question.

Acknowledgement

The author expresses his gratitude to T.A. Chapman and Steve Ferry for their advice and suggestions concerning this work.

References

[1] T.A. Chapman, Lectures on Hilbert cube manifolds, C.B.M.S. Regional Conference Series in Math. 28 (1976).

[2] T.A. Chapman, Approximation results in Hilbert cube manifolds, Trans. Amer. Math. Soc. 262 (1980) 303–334.
[3] T.A. Chapman, Approximation results in topological manifolds, Mem. Amer. Math. Soc. 251 (1981).
[4] T.A. Chapman and Steve Ferry, Hurewicz fiber maps with ANR fibers, Topology 16 (1977) 131–143.
[5] T.A. Chapman and Steve Ferry, Approximating homotopy equivalences by homeomorphisms, Amer. J. Math. 101 (1979) 583–607.
[6] T.A. Chapman and Steve Ferry, Constructing approximate fibrations, preprint.
[7] D.S. Coram and P.F. Duvall, Jr., Approximate fibrations, Rocky Mtn. J. Math. 7 (1977) 275–288.
[8] R.D. Edwards and R.C. Kirby, Deformations of spaces of embeddings, Ann. of Math. 93 (1971) 63–88.
[9] A. Fathi and Y.M. Visetti, Deformation of open embeddings of Q-manifolds, Trans. Amer. Math. Soc. 224 (1976) 427–436.
[10] Steve Ferry, Approximate fibrations over S^1, in preparation.
[11] C. Bruce Hughes, Spaces of approximate fibrations on Hilbert cube manifolds, preprint.
[12] L.S. Husch, Approximating approximate fibrations by fibrations, Canad. J. Math. 29 (1977) 897–913.
[13] Frank Quinn, Ends of maps, I, Ann. of Math. 110 (1979) 275–331.
[14] L.C. Siebenmann, Deformation of homeomorphisms on stratified sets, Comment. Math. Helv. 47 (1972) 123–163.