LOCAL HOMOTOPY PROPERTIES IN SPACES OF APPROXIMATE FIBRATIONS

DISSERTATION

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SECTION 1:
INTRODUCTION

This work is concerned with parameterized families of approximate fibrations from a Hilbert cube manifold (that is, a Q-manifold) M to a polyhedron B. We present a method for detecting those parameterized families of maps which are close to parameterized families of approximate fibrations. When M and B are compact this results in showing that the space of approximate fibrations from M to B is locally n-connected for each non-negative integer n.

Approximate fibrations were introduced by Coram and Duvall [8] as a generalization of both Hurewicz fibrations and cell-like maps. Since then approximate fibrations have been studied by several authors (see [5], [9], [13], [16]). Recently, Chapman proved the following important theorem:

THEOREM ([2, Theorem 1]). Let B be an ANR and let α be an open cover of B. There exists an open cover β of B so that if M is a Q-manifold and $f : M \to B$ is a β-fibration, then f is α-close to an approximate fibration.

Our main result is a parameterized version of Chapman's theorem.

THEOREM 1. (See Theorem 10.2). Let B be a polyhedron, let $n \geq 0$ be an integer, and let α be an open cover of B. There exists an open cover β of B so that if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a fiber preserving map such that $f_t : M \to B$ is a β-fibration for t in I^n and an approximate fibration for t in ∂I^n, then there is a fiber preserving map $\tilde{f} : M \times I^n \to B \times I^n$ such that $\tilde{f}_t$ is an approximate fibration

α-close to f_t for t in I^n and $\tilde{f}|M \times \partial I^n = f|M \times \partial I^n$.

The shell of the proof of Theorem 1 is the same as the proof of Chapman's theorem, however, there are several major technical differences. One of these differences is that in the course of the proof we encounter non-compact Q-manifolds parameterized by submersions to I^n. Hence, we are forced to recast some basic fibered Q-manifold theory in a new setting (see Section 3). Another point which requires delicacy at every step of the proof is achieving the condition $\tilde{f}|M \times \partial I^n = f|M \times \partial I^n$. This is also the reason that we restrict ourselves to polyhedral bases whereas Chapman's theorem allows for arbitrary ANRs. We remark that a relative version of Theorem 1 is also obtained.

Many authors have studied local properties of spaces of certain types of maps. Of particular relevance here are the theorems of Ferry [14] and Haver [15]. Ferry proved that the homeomorphism group of a compact Q-manifold is an ANR, while Haver proved a theorem which implies that the space of cell-like maps from a compact Q-manifold to itself is weakly locally contractible (and therefore, locally n-connected for each $n \geq 0$). Our main result implies the following:

COROLLARY. (See Section 10). Let M be a compact Q-manifold and let B be a compact polyhedron. Then the space of approximate fibrations from M to B endowed with the compact-open topology is locally n-connected for each $n \geq 0$. Moreover, the same is true of both the space of cell-like maps and the space of monotone approximate fibrations from M to B.

This paper is organized as follows. Section 2 consists of preliminary notations, definitions and facts. Section 3 contains the previously mentioned results on fibered Q-manifold theory. In Section 4 we introduce for technical reasons a variation on ε-fibrations which we call (ε,μ)-fibrations. Section 5 contains a key result about families of ε-fibrations parameterized by finite dimensional polyhedra. We show that such families have a certain sliced, or parameterized, lifting property. The restriction to finite dimensional parameter spaces is the main reason why we are unable to prove stronger results on spaces of approximate fibrations (for example, local contractibility). In Section 6 we prove that the various types of fibrations which we encounter have the appropriate stationary lifting properties.

Sections 7 through 9 contain the core of the proof of Theorem 1. These sections are modelled on the proof of Chapman's theorem [2] quoted above. Finally, in Section 10 we present the proofs and complete statements of our main results.

SECTION 2:
GENERAL PRELIMINARIES

Most of our notation and definitions are standard. Except for the various function spaces which we consider, all spaces are locally compact, separable and metric. We use $\mathbb{R}^n$ to denote euclidean n-space and B^n_r to denote the n-cell $[-r,r]^n \subset \mathbb{R}^n$. The circle is denoted by S^1 and the n-torus is $T^n = S^1 \times \cdots \times S^1$ (n times). The standard n-cell is $I^n = [0,1]^n$ and its (combinatorial) boundary is ∂I^n. If X is a space and $A \subset X$, then we use both $\mathring{A}$ and int(A) to denote the topological interior of A in X. The closure of A in X is denoted by cl(A). If X is a compact space, then c(X) denotes the cone over X. That is, c(X) = $X \times [0,+\infty]/\sim$, where $\sim$ is the equivalence relation generated by $(x,0) \sim (x',0)$ for all x, x' in X. Similarly, $\mathring{c}(X) = X \times [0,+\infty)/\sim$ denotes the open cone over X, and for any r in $[0,+\infty]$ let $c_r(X) = X \times [0,r]/\sim$ and $\mathring{c}_r(X) = X \times [0,r)/\sim$.

The Hilbert cube Q is represented by the countable infinite product of closed intervals [-1,1]. A space M is a *Hilbert cube manifold* or Q-*manifold* if it is locally homeomorphic to open subsets of Q. Our reference for Q-manifold theory is Chapman's book [1] which should be consulted by the reader unfamiliar with the basic machinery of Q-manifolds including the notion of Z-sets.

A map $f : X \to Y$ (i.e., a continuous function) is *proper* provided $f^{-1}(C)$ is compact for all compact subsets C of Y. If α is an open cover of Y, then a proper map $f : X \to Y$ is said to be an α-*fibration* if for all maps $F : Z \times [0,1] \to Y$ and $g : Z \to X$ for which $fg = F_0$, there is a map $G : Z \times [0,1] \to X$ such that $G_0 = g$ and fG is α-close to F (that is, given any $(z,t) \in Z \times [0,1]$ there is a $U \in \alpha$ containing both $fG(z,t)$ and $F(z,t)$). If $C \subset Y$ and α is an open cover of Y, then a proper map $f : X \to Y$ is said to be an *α-fibration over* C provided the condition above is satisfied when the maps $F : Z \times [0,1] \to Y$ are required to satisfy $F(Z \times [0,1]) \subset C$. If $\varepsilon > 0$, then we also use ε to denote the open cover of Y by balls of diameter ε. Thus, we speak of *ε-fibrations*.

A proper map $f : X \to Y$ is an *approximate fibration* provided it is an α-fibration for each open cover α of Y. We only consider approximate fibrations which are defined between ANRs. The abbreviation ANR is for *absolute neighborhood retract*. The following lemma is used repeatedly.

LEMMA 2.1. Let B be an ANR and let C be a compact subset of B with a compact neighborhood $\tilde{C}$. For every $\alpha > 0$ there exists a $\beta = \beta(\alpha, C, \tilde{C}, B) > 0$ such that if $\varepsilon > 0$ and $f : E \to B$ is an ε-fibration over $\tilde{C}$, then f has the following lifting property: given maps $F : Z \times [0,1] \to C \subset B$ and $g : Z \to E$ such that fg is β-close to F_0, there exists a map $G : Z \times [0,1] \to E$ such that $G_0 = g$ and fG is $(\alpha+\varepsilon)$-close to F.

For a proof of Lemma 2.1, see [8, Proposition 1.2] or [9, Lemma 1.1]. These two papers should also be consulted for other basic results on approximate fibrations.

A closed subset A of an ANR X is *cell-like* if it is contractible in any neighborhood of itself. A proper map $f : X \to Y$ between ANRs is *cell-like* provided $f^{-1}(y)$ is cell-like for each y in Y. A cell-like map is also an approximate fibration. A map $f : X \to Y$ is *monotone* provided $f^{-1}(y)$ is connected for each y in Y.

If α is an open cover of Y, then a homotopy $H : X \times [0,1] \to Y$ is an *α-homotopy* if for each x in X there exists a U in α containing

$H(\{x\} \times [0,1])$. For other similar conventions, the reader is referred to [2].

Finally, if X is a space (not necessarily locally compact) and $n \geq 0$ is an integer, then X is said to be *locally* n-*connected* (written LC^n) if for each x in X and each open subset U of X containing x, there exists an open subset V of X such that $x \in V \subset U$ and any map $f : \partial I^{n+1} \to V$ extends to a map $\tilde{f} : I^{n+1} \to U$.

SECTION 3:
BASIC Q-MANIFOLD THEORY IN A SUBMERSIVE SETTING

In this section parameterized versions of some basic results from Q-manifold theory are developed. The Q-manifolds are parameterized by a submersion over a polyhedron. The main results are a mapping replacement theorem (Proposition 3.7), a sliced Z-set unknotting theorem (Proposition 3.9), and a stability theorem (Proposition 3.15). There are certainly more general results than those presented here, but we restrict ourselves to proving only what is needed in the sequel. Related parameterized Q-manifold theories can be found in [6], [3, Section 2], and [14, Section 4]. In fact, we rely heavily on both the results and ideas of those papers (together with [1] of course) for the proofs of our theorems.

DEFINITION 3.1. A map $\pi : E \to B$ is a *submersion* if for each x in E, there is an open neighborhood F of x in $\pi^{-1}(\pi(x))$, an open neighborhood N of $\pi(x)$ in B, and an open embedding $\phi : F \times N \to E$ such that $\pi\phi$ is the projection $F \times N \to N \subset B$ and for each y in $F \subset E$, $\phi(y,\pi(x)) = y$. We call ϕ a *product chart about* F for π. If $C \subset E$ and for each x in C, the neighborhood N of $\pi(x)$ can be chosen to be all of B, then we say π has *nice cross sections on* C.

If $\pi : E \to B$ is a proper submersion for which the fibers $\pi^{-1}(b)$, $b \in B$, are Q-manifolds, then π is actually a bundle projection. This is proved in [17] with [12] supplying the necessary deformation theorem (see also [7]). Unfortunately, we will encounter in the sequel submersions whose fibers are non-compact Q-manifolds, and it is that fact which makes this section necessary. However, we will be working on compact pieces of the submersion and the following theorem due to Siebenmann is the main technical tool which allows us to deal with this situation (again, see [12] for a major ingredient).

PROPOSITION 3.2. ([17, Corollary 6.15]). Let $\pi : E \to B$ be a submersion such that $\pi^{-1}(b)$ is a Q-manifold for each b in B and let $C \subset E$ be a compactum such that $\pi(C)$ is a point. Then there exist an open neighborhood F of C in $\pi^{-1}(\pi(C))$, an open neighborhood N of $\pi(C)$ in B, and a product chart $\phi : F \times N \to E$ about F for π.

The following definition is a slight generalization of that given in [14] for sliced Z-sets in products.

DEFINITION 3.3. Let $\pi : E \to B$ be a submersion and let K be a closed subset of E. Then K is said to be a *sliced Z-set* if for every open cover $\mathcal{U}$ of E there is a map $f : E \to E \setminus K$ such that f is $\mathcal{U}$-close to id and $\pi f = \pi$.

The following theorem characterizes sliced Z-sets in certain products.

PROPOSITION 3.4. ([6, Theorem 3.1]). Let $\pi : M \times B \to B$ be projection where M is a Q-manifold and B is a polyhedron and let $K \subset M \times B$ be closed. Then K is a sliced Z-set if and only if $K \cap \pi^{-1}(b)$ is a Z-set in $\pi^{-1}(b) = M \times \{b\}$ for each $b \in B$.

REMARKS ON PROOF. Since one will not find Proposition 3.4 worded exactly like this in [6], we indicate here how it may be derived from the results of [6]. The non-trivial part of this proposition is to start with a closed set $K \subset M \times B$ such that $K \cap \pi^{-1}(b)$ is a Z-set in $\pi^{-1}(b)$ for each b in B and then show that K is a sliced Z-set. Under these conditions we can use [6, Theorem 3.1] to get a homeomorphism $h : M \times B \to M \times Q \times B$ such that $h(K) \subset M \times \{0\} \times B$ and $ph = \pi$ where $p : M \times Q \times B \to B$ is projection (recall $Q = [-1,1]^{\infty}$). Given an open cover $\mathcal{U}$ of $M \times Q \times B$, it will suffice to produce a map $f : M \times Q \times B \to (M \times Q \times B) \setminus (M \times \{0\} \times B)$ such that f is $\mathcal{U}$-close to id and $pf = p$. To this end, write $M \times B = \cup_{i=1}^{\infty} C_i$ where each C_i is compact and $C_i \subset \operatorname{int} C_{i+1}$. Choose integers $n_1 < n_2 < n_3 < \cdots$ so that if $f : M \times Q \times B \to M \times Q \times B$ is a map with the property that $f(m,q,b) = (m_1(q_1,q_2,\ldots,q_{n_1},q'_{n_1+1},q'_{n_1+2},\ldots),b)$ for each $(m,b) \in C_i \setminus \operatorname{int} C_{i-1}$ and $q \in Q$, then f is $\mathcal{U}$-close to id. Let $\phi : M \times B \to [1,+\infty)$ be a map such that $\phi^{-1}([1,n_i]) = C_i$ for each $i = 1,2,\ldots$. Construct a map $\alpha : Q \times [1,+\infty) \to Q$ so that if $n_i \le r \le n_{i+1}$ and $q \in Q$, then

$$\alpha(q,r) = (q_1,q_2,\ldots,q_{n_{i+1}},q'_{n_{i+1}+1},q'_{n_{i+1}+2},\ldots,q'_{n_{i+2}},1,1,1,\ldots) .$$

Finally, define $f : M \times Q \times B \to (M \times Q \times B) \setminus (M \times \{0\} \times B)$ by $f(m,q,b) = (m,\alpha(q,\phi(m,b)),b)$ for each (m,q,b) in $M \times Q \times B$. □

Using Propositions 3.2 and 3.4 we now characterize compact sliced Z-sets in certain submersions.

PROPOSITION 3.5. Let $\pi : M \to B$ be a submersion where B is a polyhedron and $\pi^{-1}(b)$ is a Q-manifold for each b in B and let $K \subset M$ be compact. Then K is a sliced Z-set if and only if $K \cap \pi^{-1}(b)$ is a Z-set in $\pi^{-1}(b)$ for each b in B.

PROOF. We only need to prove the "if" part of this proposition. Using Proposition 3.2 and the compactness of K, we find Q-manifolds F_i, open subsets N_i of B, compact polyhedra $B_i \subset N_i$, and product charts

$\phi_i : F_i \times N_i \to M$ for $i = 1,2,\ldots,k$ such that $K \subset \cup_{i=1}^{k} \phi_i(F_i \times B_i)$ and $K \cap \pi^{-1}(B_i) \subset \phi_i(F_i \times B_i)$. Using Proposition 3.4 one sees that $K \cap \phi_i(F_i \times B_i)$ is a sliced Z-set in $\phi_i(F_i \times N_i)$ for each $i = 1,2,\ldots,k$. From this it follows by a standard argument that each $K \cap \pi^{-1}(B_i) = K \cap \phi_i(F_i \times B_i)$ is a sliced Z-set in M (see the proof of Theorem 3.1(4) in [1]). By another standard argument it follows that K is a sliced Z-set in M (see the proof of Theorem 3.1(3) in [1]). □

We now state a mapping replacement theorem from [6] from which we will derive an analogous result. A *sliced Z-embedding* is an embedding onto a sliced Z-set.

PROPOSITION 3.6. ([6, Theorem 4.1]). Let $\pi : M \times B \to B$ be projection where M is a Q-manifold and B is a polyhedron. Let A_0 be a closed subset of the space A and let $f : A \to M \times B$ be a proper map such that $f|A_0 : A_0 \to M \times B$ is a sliced Z-embedding. Then for any open cover $\mathcal{U}$ of $M \times B$ there is a sliced Z-embedding $\tilde{f} : A \to M \times B$ such that $\tilde{f}$ is $\mathcal{U}$-close to f, $\tilde{f}|A_0 = f|A_0$, and $\pi\tilde{f} = \pi f$.

PROPOSITION 3.7. Let $\pi : M \to B$ be a submersion where B is a polyhedron and $\pi^{-1}(b)$ is a Q-manifold for each b in B. Let $A_0 \subset A$ be compacta and let $f : A \to M$ be a map such that $f|A_0 : A_0 \to M$ is a sliced Z-embedding. Then for every $\varepsilon > 0$ there is a sliced Z-embedding $\tilde{f} : A \to M$ such that $\tilde{f}$ is ε-close to f, $\tilde{f}|A_0 = f|A_0$, and $\pi\tilde{f} = \pi f$.

PROOF. As in the proof of Proposition 3.5, we find Q-manifolds F_i, open subsets N_i of B, compact polyhedra $B_i \subset N_i$, and product charts $\phi_i : F_i \times N_i \to M$ for $i = 1,2,\ldots,k$ such that $f(A) \subset \cup_{i=1}^{k} \phi_i(F_i \times B_i)$ and $f(A) \cap \pi^{-1}(B_i) \subset \phi_i(F_i \times B_i)$. Define $A_i = f^{-1}(f(A) \cap \phi_i(F_i \times B_i))$

for $i = 1,\ldots,k$. Note that $A = \cup_{i=1}^{k} A_i$ and that each A_i is compact. Inductively define maps $f = \tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_k = \tilde{f}$ from A into M such that for $i = 1,\ldots,k$ we have:

i) $\tilde{f}_i$ is ε/k-close to $\tilde{f}_{i-1}$;

ii) $\tilde{f}_i | A_i$ is a sliced Z-embedding;

iii) $\tilde{f}_i | A_0 \cup A_1 \cup \cdots \cup A_{i-1} = \tilde{f}_{i-1} | A_0 \cup A_1 \cup \cdots \cup A_{i-1}$;

iv) $\pi \tilde{f}_i = \pi f$.

It is then clear that $\tilde{f} = \tilde{f}_k$ satisfies the conclusions of the proposition. To get the maps $\tilde{f}_i$, first use Proposition 3.6 to get $f_i : f^{-1}(f(A) \cap \phi_i(F_i \times N_i)) \to \phi_i(F_i \times N_i)$ such that f_i is close to $\tilde{f}_{i-1} | f^{-1}(f(A) \cap \phi_i(F_i \times N_i))$, f_i is a sliced Z-embedding into $\phi_i(F_i \times N_i)$, $f_i | [(A_0 \cup A_1 \cup \cdots \cup A_{i-1}) \cap f^{-1}(f(A) \cap \phi_i(F_i \times N_i))] = \tilde{f}_{i-1}|$, and $\pi f_i = \pi f|$. If f_i is close enough to $\tilde{f}_{i-1}|$, then f_i extends via $\tilde{f}_{i-1}$ to our desired map $\tilde{f}_i : A \to M$. □

The next step is a result on the separation of sliced Z-sets. Again the proof will be to argue locally using the appropriate result from [6].

PROPOSITION 3.8. Let $\pi : M \to B$ be a submersion where B is a polyhedron and $\pi^{-1}(b)$ is a Q-manifold for each b in B. Let K and L be compact sliced Z-sets in M, let U be an open subset of M containing K, and let $\mathcal{U}$ be an open cover of U. Then there is an isotopy $H : M \times [0,1] \to M$ such that

i) $H_0 = \mathrm{id}$;

ii) $H_t | M \setminus U = \mathrm{id}$ for each t in $[0,1]$;

iii) $H|U \times [0,1]$ is a $\mathcal{U}$-isotopy;

iv) $\pi H(m,t) = \pi(m)$ for each (m,t) in $M \times [0,1]$;

v) $H_1(K) \cap L = \emptyset$.

PROOF. As usual choose Q-manifolds F_i, open subsets N_i of B, compact polyhedra $B_i \subset N_i$, and product charts $\phi_i : F_i \times N_i \to U \subset M$ for $i = 1,2,\ldots,k$ such that $K \subset \cup_{i=1}^{k} \phi_i(F_i \times B_i)$ and $K \cap \pi^{-1}(B_i) \subset \phi_i(F_i \times B_i)$. For each $i = 1,2,\ldots,k$ let $K_i = K \cap \pi^{-1}(B_i)$. Using [6, Theorem 2.1] we find for each $i = 1,2,\ldots,k$ an isotopy $H^i : M \times [0,1] \to M$ such that

i) $H_0^i = id$;

ii) $H_t^i|M \setminus \phi_i(F_i \times N_i) = id$ for each t in $[0,1]$;

iii) $H^i|\phi_i(F_i \times N_i) \times [0,1]$ is a small isotopy;

iv) $\pi H^i(m,t) = \pi(m)$ for each (m,t) in $M \times [0,1]$;

v) $H_1^i(K_i) \cap L = \emptyset$.

Our isotopy $H : M \times [0,1] \to M$ is defined by $H(m,t) = H_t^k \circ H_t^{k-1} \circ \cdots \circ H_t^1(m)$ for each (m,t) in $M \times [0,1]$. If each H^i is a small enough isotopy, then $H|U \times [0,1]$ will be a $\mathcal{U}$-isotopy and $H_1(K) \cap L = \emptyset$. The other requirements on H are trivial to check. □

We are now in a position to prove the following sliced Z-set unknotting theorem.

PROPOSITION 3.9. Let $\pi : M \to B$ be a submersion where B is a polyhedron and $\pi^{-1}(b)$ is a Q-manifold for each b in B. Let A be a compactum and let $F : A \times [0,1] \to M$ be a map such that F_0 and F_1 are sliced Z-embeddings and $\pi F(a,t) = \pi F(a,0)$ for each (a,t) in $A \times [0,1]$. Let U be an open subset of M containing $F(A \times [0,1])$ and let $\mathcal{U}$ be an open cover of U by which F is limited. Then there is an isotopy $H : M \times [0,1] \to M$ such that

i) $H_0 = \mathrm{id}$;

ii) $H_t | M \setminus U = \mathrm{id}$ for each t in $[0,1]$;

iii) $H | U \times [0,1]$ is a $\mathcal{U}$-isotopy;

iv) $\pi H(m,t) = \pi(m)$ for each (m,t) in $M \times [0,1]$;

v) $H_1 F_0 = F_1$.

PROOF. By Propositions 3.7 and 3.8 we may assume that F is a sliced Z-embedding. Choose Q-manifolds F_i, open subsets N_i of B, compact polyhedra $B_i \subset N_i$, and product charts $\phi_i : F_i \times N_i \to U \subset M$ for $i = 1,2,\ldots,k$ such that $F(A \times [0,1]) \subset \cup_{i=1}^{k} \phi_i(F_i \times B_i)$ and $F(A \times [0,1]) \cap \pi^{-1}(B_i) \subset \phi_i(F_i \times B_i)$. Since F is an embedding we can find an open cover $\mathcal{V}$ of U by which F is limited such that $st^k(\mathcal{V})$ refines $\mathcal{U}$. Using [6, Theorem 5.1] we can find for each $i = 1,2,\ldots,k$ a homeomorphism $h_i : F_i \times N_i \to F_i \times N_i \times [0,1]$ which takes $F(\{a\} \times [0,1])$ linearly onto $\phi^{-1}F(a,0) \times [1/3,2/3] \subset F_i \times N_i \times [0,1]$ for each a in $F^{-1}(F_0(A) \cap \pi^{-1}(N_i))$. We also require that $ph_i = p$ where p denotes projection onto N_i. Define an isotopy $G_t^1 : F_1 \times N_1 \times [0,1] \to F_1 \times N_1 \times [0,1]$, $0 \le t \le 1$, so that $G_0^1 = \mathrm{id}$, G_t^1 affects only the $[0,1]$-coordinate of any point, and G_t^1 slides $h_1[F(A \times \{0\}) \cap \pi^{-1}(B_1)]$ up to $h_1[F(A \times \{1\}) \cap \pi^{-1}(B_1)]$. If G_t^1 is limited by an appropriate open cover, then the isotopy $\phi h_1^{-1} G_t^1 h_1 \phi^{-1}$, $0 \le t \le 1$, extends via the identity to a $\mathcal{V}$-isotopy $H^1 : M \times [0,1] \to M$. Define $\rho_1 : A \to [0,1]$ so that

$H_1^1(F(a,0)) = F(a,\rho_1(t))$ for each a in A. Then $\rho_1 | F_0^{-1}(F_0(A) \cap \pi^{-1}(B_1)) = 1$ and $\rho_1 | F_0^{-1}(F_0(A) \setminus \pi^{-1}(N_1)) = 0$.

Using this same procedure again, we construct a $\mathcal{V}$-isotopy $H^2 : M \times [0,1] \to M$ and a map $\rho_2 : A \to [0,1]$ such that $H_1^2 H_1^1(F(a,0)) = F(a,\rho_2(t))$ for each a in A and $\rho_2 | F_0^{-1}(F_0(A) \cap \pi^{-1}(B_1 \cup B_2)) = 1$. Continuing this process will give us isotopies $H^1, H^2, \ldots, H^k$ such that the isotopy $H : M \times [0,1] \to M$ defined by $H_t = H_t^k \circ \cdots \circ H_t^2 \circ H_t^1$, $0 \le t \le 1$, satisfies the conclusions of the proposition. □

We will now deduce a strong relative version of sliced Z-set unknotting.

PROPOSITION 3.10. Let $\pi : M \to B$ be a submersion where B is a polyhedron and $\pi^{-1}(b)$ is a Q-manifold for each b in B. Let $A_0 \subset A$ be compacta and let $F : A \times [0,1] \to M$ be a map such that F_0 and F_1 are sliced Z-embeddings and $\pi F(a,t) = \pi F(a,0)$ for each (a,t) in $A \times [0,1]$. Suppose further that $F_t | A_0 : A_0 \to M$ is a sliced Z-embedding for each t in $[0,1]$. Let U be an open subset of M containing $F(A \times [0,1])$ and let $\mathcal{U}$ be an open cover of U by which F is limited. Then there is an isotopy $H : M \times [0,1] \to M$ such that

i) $H_0 = \mathrm{id}$;

ii) $H_t | M \setminus U = \mathrm{id}$ for each t in $[0,1]$;

iii) $H | U \times [0,1]$ is an $\mathrm{st}(\mathcal{U})$-isotopy;

iv) $\pi H(m,t) = \pi(m)$ for each (m,t) in $M \times [0,1]$;

v) $H_1 F_0 = F_1$;

vi) $H_t F_0 | A_0 = F_t | A_0$ for each t in $[0,1]$.

PROOF. Define $f : A \times [0,1] \to M \times [0,1]$ by $f(a,t) = (F(a,t),t)$ for each (a,t) in $A \times [0,1]$. Using Proposition 3.5 it is easy to see that $f|(A_0 \times [0,1]) \cup (A \times \{0,1\}) : (A_0 \times [0,1]) \cup (A \times \{0,1\}) \to M \times [0,1]$ is a sliced Z-embedding with respect to the submersion $\pi \times id : M \times [0,1] \to B \times [0,1]$. Moreover, $f(A \times [0,1]) \subset U \times [0,1]$ and for each a in A there is a U_a in $\mathcal{U}$ such that $f(\{a\} \times [0,1]) \subset U_a \times [0,1]$ (we fix such a U_a for each a in A). Using our mapping replacement theorem (Proposition 3.7) we find a map $\tilde{f} : A \times [0,1] \to M \times [0,1]$ close to f such that $\tilde{f}|(A_0 \times [0,1]) \cup (A \times \{0,1\}) = f|$, $p\tilde{f} = pf$ where $p = \pi \times id : M \times [0,1] \to B \times [0,1]$, and such that $\tilde{f}$ is a sliced Z-embedding with respect to p. If $\tilde{f}$ is close enough to f, then $\tilde{f}(\{a\} \times [0,1]) \subset U_a \times [0,1]$ for each a in A.

Define a homotopy $G_s : A \times [0,1] \to M \times [0,1]$, $0 \leq s \leq 1$, by $G_s(a,t) = (p_M\tilde{f}(a,st),t)$ for each (a,t) in $A \times [0,1]$ where $p_M : M \times [0,1] \to M$ is projection. Note that $G_0(a,t) = (F(a,0),t)$ and $G_1(a,t) = \tilde{f}(a,t)$. Also, $pG_s(a,t) = (\pi F(a,0),t)$ and $\{G_s\}$ is limited by $\{U_a \times [0,1] | a \in A\}$.

Using sliced Z-set unknotting (Proposition 3.9) we get a homeomorphism $g : M \times [0,1] \to M \times [0,1]$ such that $g|(M \setminus U) \times [0,1] = id$, $g|U \times [0,1]$ is $\{U' \times [0,1] | U' \in \mathcal{U}\}$-close to id, $pg = p$, and $g \circ G_1 = G_0$. Our isotopy $H : M \times [0,1] \to M$ can now be defined by

$$H(m,t) = p_M g^{-1}(g_0(m),t) \quad . \square$$

The next result is a collaring theorem for submanifolds which are sliced Z-sets.

PROPOSITION 3.11. Let $\pi : M \to B$ be a submersion where B is a polyhedron and $\pi^{-1}(b)$ is a Q-manifold for each b in B. Let $N \subset M$ be a sliced Z-set such that $\pi|N : N \to B$ is a submersion and $\pi^{-1}(b) \cap N$ is a Q-manifold for each b in B. Then there is an open embedding $\psi : N \times [0,1) \to M$ such that $\psi(n,0) = n$ for each n in N and $\pi\psi(n,t) = \pi(n)$ for each (n,t) in $N \times [0,1)$.

PROOF. Using small product charts for π and $\pi|N$ and the fact that Z-set Q-submanifolds are collared [1, Theorem 16.2] it is easy to see that N is locally collared in M by embeddings which respect π. Then the usual proofs that locally collared implies collared show the existence of our ψ (the reader should consult [3, Proposition 2.5] and [14, Corollary 4.10] for similar invocations of the proof of Brown's collaring theorem). □

We now begin proving a sequence of lemmas which will be used to establish our stability result (Proposition 3.15). The strategy is somewhat different here in that we will work in nice horizontal neighborhoods rather than in vertical neighborhoods as in the preceding propositions.

LEMMA 3.12. Let G be an open subset of $Q \times I^n$ and let $\mathcal{G}$ be an open cover of G. Then there exists a homeomorphism $h : G \times Q \to G$ $\mathcal{G}$-close to projection such that $ph = p$ where p denotes projection to I^n.

PROOF. The proof is only a slight modification of the proof of stability for open subsets of Q as presented in [1, Section 13]. We only sketch the argument here; for more details see [1].

Construct a map $\alpha : Q \times I^n \times Q \times [1,+\infty) \to Q \times I^n$ such that if $\alpha_r : Q \times I^n \times Q \to Q \times I^n$ is defined by $\alpha_r(q,t,q') = \alpha(q,t,q',r)$ for $r \geq 1$, then

i) each α_r is a homeomorphism;

ii) if $m \leq r$, where m is an integer, then

$$\alpha_r((q_i),t,q') = ((q_1,q_2,\dots,q_m,q''_{m+1},q''_{m+2},\dots),t) .$$

Write $G = \cup_{m=1}^{\infty} A_m$ where each A_m is compact, $A_m \subset \text{int } A_{m+1}$, and for integers $k_1 < k_2 < \cdots$, we have $((q_1,q_2,\dots,q_{k_m},q'_{k_m+1},q'_{k_m+2},\dots),t)$ in A_m whenever $((q_i),t)$ is in A_m.

Choose integers $\ell_1 < \ell_2 < \cdots$ so that each $k_m < \ell_m$ and if $f : G \times Q \to G$ is a map which preserves the first ℓ_m-coordinates and the I^n-coordinate of any point in $(A_m \setminus \text{int } A_{m-1}) \times Q$, then f is $\mathcal{G}$-close to projection (we are requiring f to send (q,t,q') to $((q_1,q_2,\dots,q_{\ell_m},q''_{\ell_m+1},q''_{\ell_m+2},\dots),t)$ whenever (q,t,q') is in $(A_m \setminus \text{int } A_{m-1}) \times Q)$.

Define a director map $\phi : G \to [1,+\infty)$ so that if (q,t) is in G and $j = \min\{\ell_m | (q,t) \in A_m\}$, then

i) $\phi((q,t)) \geq j$;

ii) $\phi((q,t)) = \phi((q',t))$ if $q_i = q'_i$ for $i \leq j$.

Finally, define a homeomorphism $h : G \times Q \to G$ by $h((q,t,q')) = \alpha(q,t,q',\phi(q,t))$ and check that ϕ satisfies the conclusions of the lemma. □

The next lemma is similar to [1, Lemma 14.1]. For a definition of variable products see [1, Section 14].

LEMMA 3.13. Let U be an open subset of $Q \times I^n$ and let $\mathcal{U}$ be an open cover of U. If $(Q \times I^n) \times_r Q$ is a variable product, then there exists a homeomorphism $g : U \times_r Q \to U$ such that

i) $pg = p$ where p denotes projection to I^n;

ii) g is $\mathcal{U}$-close to projection onto U.

PROOF. Let $G = \{x \in U \mid r(x) \neq 0\}$. Then G is open in $Q \times I^n$ and we can use Lemma 3.12 to find a homeomorphism $h : G \times Q \to G$ which induces the desired homeomorphism $g : U \times_r Q \to U$. □

LEMMA 3.14. Let $W \subset I^n$ be open and let $\mathcal{U}$ be an open cover of $U = Q \times [0,1] \times W \subset Q \times [0,1] \times I^n$. If $(Q \times [0,1] \times I^n) \times_r Q$ and $(Q \times [0,1] \times I^n) \times_s Q$ are variable products such that $r|Q \times \{1\} \times I^n = s|Q \times \{1\} \times I^n$, then there is a homeomorphism $g : U \times_r Q \to U \times_s Q$ such that

i) $g|(Q \times \{1\} \times W) \times_r Q = \text{id}$,

ii) $pg = p$ where p denotes projection to I^n,

iii) qg is $\mathcal{U}$-close to q where q denotes projection to $U \subset Q \times [0,1] \times I^n$.

PROOF. By Lemma 3.13 there are homeomorphisms $h : U \times_r Q \to U$ and $f : U \times_s Q \to U$ close to projection such that $ph = p = pf$. Note that the inclusions $(Q \times \{1\} \times W) \times_r Q) \subset U \times_r Q$ and $(Q \times \{1\} \times W) \times_s Q) \subset U \times_s Q$ are sliced Z-embeddings with respect to projection to I^n. It follows that $h|(Q \times \{1\} \times W) \times_r Q$ and $f|(Q \times \{1\} \times W) \times_s Q$ are close sliced Z-embeddings. By sliced Z-set unknotting there is a homeomorphism $\tilde{g} : Q \times [0,1] \times W \to Q \times [0,1] \times W$ such that $p\tilde{g} = p$, $\tilde{g}$ is close to id and $\tilde{g} \circ h|(Q \times \{1\} \times W) \times_r Q = f|$ (we are using [6, Theorem 5.1] here which works in a non-compact setting). Then $g = f^{-1}\tilde{g}h$ is the desired homeomorphism. □

We are now ready for our stability theorem.

PROPOSITION 3.15. Let $\pi : M \to I^n$ be a submersion such that $\pi^{-1}(t)$ is a Q-manifold for each t in I^n. Let $W \subset I^n$ be an open set and let $C \subset M$ be a compactum such that π has nice cross sections on some compact neighborhood $\tilde{C}$ of C in M. For every open cover $\mathcal{U}$ of M there is a map $f : M \times Q \to M$ with the following properties:

i) f is $\mathcal{U}$-close to projection;

ii) $f|(M \setminus \pi^{-1}(W)) \times Q : (M \setminus \pi^{-1}(W)) \times Q \to M \setminus \pi^{-1}(W)$ is projection;

iii) f is a homeomorphism over $C \cap \pi^{-1}(W)$;

iv) $\pi f(m,q) = \pi(m)$ for each (m,q) in $M \times Q$.

PROOF. Using the facts that π has nice cross sections on $\tilde{C}$ and that Q-manifolds are locally homeomorphic to $Q \times [0,1)$, we can find open embeddings $\phi_i : Q \times [0,2) \times I^n \to M$, $i = 1,2,\ldots,k$, such that $\tilde{C} \subset \cup_{i=1}^k \phi_i(Q \times [0,1/2] \times I^n)$ and $\pi\phi_i = p$ where p denotes projection to I^n.

We inductively define maps $s_i : M \to [0,1]$ for $i = 0,1,\ldots,k$, as follows. First set $s_0 = 1$. Assuming $1 \le i \le k$ and that s_{i-1} has been defined let s_i be a map such that $s_i = 0$ on $\cup_{j=1}^i \phi_j(Q \times [0,1/2] \times I^n)$ and $s_i = s_{i-1}$ on $M \setminus \phi_i(Q \times [0,1) \times I^n)$.

Using Lemma 3.14 for each $i = 1,2,\ldots,k$ choose a homeomorphism $f_i : \phi_i(Q \times [0,1] \times W) \times_{s_{i-1}} Q \to \phi_i(Q \times [0,1] \times I^n) \times_{s_i} Q$ such that $f_i|\phi_i(Q \times \{1\} \times W) \times_{s_{i-1}} Q$ is the identity, $pf_i = p$, and $p_M f_i$ is close to p_M where p_M denotes projection to M. Each f_i extends via the identity to a homeomorphism $\tilde{f}_i : \pi^{-1}(W) \times_{s_{i-1}} Q \to \pi^{-1}(W) \times_{s_i} Q$. If each $p_M f_i$ is close enough to p_M, then the composition $\tilde{f}_k \circ \cdots \circ \tilde{f}_1 : \pi^{-1}(W) \times Q \to \pi^{-1}(W) \times_{s_k} Q$ followed by projection $\pi^{-1}(W) \times_{s_k} Q \to \pi^{-1}(W)$ extends via projection to the desired map $f : M \times Q \to M$. □

REMARK 3.16. A theorem similar to Proposition 3.15 can be proved whose conclusion is the existence of a map $f : M \times [0,1] \to M$ with the same properties as the map f of Proposition 3.15. This is accomplished by first proving a lemma analogous to Lemma 3.12 where now we want a homeomorphism $h : G \times [0,1] \to G$ (such a lemma in fact follows from Lemma 3.12). Then notice that Proposition 3.15 followed rather mechanically from Lemma 3.12 so that we may replace Q by [0,1] in the appropriate places.

SECTION 4:
(ε,μ)-FIBRATIONS

In this section we study a technical variation of the definition of an ε-fibration which we call an (ε,μ)-fibration. Our main result is Theorem 4.8 which roughly says that a map which is an ε-fibration globally and a μ-fibration over a piece of the range has a certain lifting property which takes into account both the ε and μ control. This result will be used in Sections 7 and 9. The proof of Theorem 4.8 is modelled on the proof of Proposition 2.2 in [2]. In fact, Theorem 4.8 is implicitly assumed in the proofs of Theorems 3.3 and 5.2 in [2].

Preliminaries for this section include a generalization of the fact that close maps into an ANR are homotopic by a small homotopy (Proposition 4.2) and a variation on the estimated homotopy extension property (Proposition 4.3).

DEFINITION 4.1. Let $A \subset B$, $\varepsilon > 0$, and $\mu > 0$. Let $f : X \to B$ be a map.

i) A map $g : X \to B$ is (ε,μ)-*close to* f *with respect to* A if f is ε-close to g and $f|f^{-1}(A)$ is μ-close to $g|f^{-1}(A)$.

ii) A map $g : X \to B$ is (ε,μ)-*homotopic to* f *with respect to* A if there is an ε-homotopy $H : X \times [0,1] \to B$ from f to g such that $H|f^{-1}(A) \times [0,1]$ is a μ-homotopy. We call H an (ε,μ)-*homotopy from* f *to* g *with respect to* A.

(Beware: These relations are *not* symmetric).

PROPOSITION 4.2. Let B be an ANR and let A and C be compact subsets of B with $A \subset C$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

> if $f,g : X \to C \subset B$ are maps such that g is (δ,ν)-close to f with respect to A, then g is (ε,μ)-homotopic to f with respect to A (as maps into B).

Moreover, the homotopy H from f to g can be chosen so that it is stationary wherever these maps agree (that is, if x is in X and $f(x) = g(x)$, then $H(x,t) = f(x)$ for each t in $[0,1]$).

PROOF. Let $\varepsilon > 0$ be given and let $\tilde{C}$ be a compact neighborhood of C in B. Choose $\delta > 0$ so that any two 2δ-close maps into $\tilde{C}$ are $\varepsilon/2$-homotopic in B (with the homotopy stationary wherever the two maps agree).

Let $\mu > 0$ and set $\mu' = \min(\mu,\delta,\varepsilon/2)$. Choose $\nu > 0$ so that any two ν-close maps into $\tilde{C}$ are μ'-homotopic in B (with the homotopy stationary wherever the two maps agree). Also assume that the μ'-neighborhood of C in B is contained in $\tilde{C}$.

Let $f,g : X \to C \subset B$ be two maps with g (δ,ν)-close to f with respect to A. Since $f|f^{-1}(A)$ is ν-close to $g|f^{-1}(A)$, there is a μ'-homotopy $F : f^{-1}(A) \times [0,1] \to B$ such that $F_0 = f|f^{-1}(A)$ and $F_1 = g|f^{-1}(A)$. By the estimated homotopy extension property (see [4, Proposition 2.1]) there is a μ'-homotopy $\tilde{F} : X \times [0,1] \to B$ which extends F and such that $\tilde{F}_0 = f$. Note that the image of $\tilde{F}$ is contained in $\tilde{C}$.

Now $\tilde{F}_1$ is 2δ-close to g and $\tilde{F}_1 = g$ on $f^{-1}(A)$, so there is an $\varepsilon/2$-homotopy $G : X \times [0,1] \to B$ such that $G_0 = \tilde{F}_1$, $G_1 = g$, and $G_t|f^{-1}(A) = g|f^{-1}(A)$ for each t in $[0,1]$. We define $H : X \times [0,1] \to B$ by

$$H(x,t) = \begin{cases} \tilde{F}(x,2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ G(x,2t-1) & \text{if } \frac{1}{2} \le t \le 1 . \end{cases}$$

Then H is an ε-homotopy and $H|f^{-1}(A) \times [0,1]$ is a μ-homotopy.

We now modify the proof so that H will be stationary wherever f and g agree. Let $Z \subset X$ be the agreement set of f and g. In the proof above let F be defined on $(f^{-1}(A) \cup Z) \times [0,1]$ and assume that it is stationary on Z. Then $\tilde{F}_1 = g$ on $f^{-1}(A) \cup Z$ so we can assume that $G_t|f^{-1}(A) \cup Z = g|f^{-1}(A) \cup Z$ for each t in [0,1]. □

PROPOSITION 4.3. Let A be a closed subset of the ANR B and let X be a closed subset of the space Y. Let $\varepsilon > 0$ and $\mu > 0$. If $f : Y \to B$ is a map and $F : X \times [0,1] \to B$ is an (ε,μ)-homotopy from $f|X$ to F_1 with respect to A, then F extends to an (ε,μ)-homotopy $\tilde{F} : Y \times [0,1] \to B$ from f to $\tilde{F}_1$ with respect to A.

PROOF. We assume that $\mu \le \varepsilon$. Since $F|[f^{-1}(A) \cap X] \times [0,1]$ is a μ-homotopy, the estimated homotopy extension property implies that $F|[f^{-1}(A) \cap X] \times [0,1]$ extends to a μ-homotopy $\hat{F} : f^{-1}(A) \times [0,1] \to B$ with $\hat{F}_0 = f|f^{-1}(A)$. Since F and $\hat{F}$ agree on $[f^{-1}(A) \cap X] \times [0,1]$ we can define $F^* : [f^{-1}(A) \cup X] \times [0,1] \to B$ by $F^*|f^{-1}(A) \times [0,1] = \hat{F}|$ and $F^*|X \times [0,1] = F|$.

Now F^* is an (ε,μ)-homotopy from $f|$ to F_1^* with respect to A. In particular F^* is an ε-homotopy so by another application of the estimated homotopy extension property, F^* extends to an ε-homotopy $\tilde{F} : Y \times [0,1] \to B$ with $\tilde{F}_0 = f$. Then $\tilde{F}$ is clearly an (ε,μ)-homotopy from f to $\tilde{F}_1$ with respect to A. □

DEFINITION 4.4. Let D and C be subsets of the ANR B with $D \subset C$.

A map $p : E \to B$ is said to be an (ε,μ)-*fibration over* (C,D) where $\varepsilon > 0$ and $\mu > 0$, if given maps $F : Z \times [0,1] \to C \subset B$ and $f : Z \to E$ with $pf = F_0$, then there is a map $G : Z \times [0,1] \to E$ such that $G_0 = f$ and pG is (ε,μ)-close to F with respect to D. The map G is called an (ε,μ)-*lift of* F *with respect to* D.

Similarly, we say that $p : E \to B$ is an (ε,μ)-*fibration over* (C,D) *for compacta* when the condition above is required to hold only when Z is compact.

LEMMA 4.5. Let B be an ANR and let C and D be compact subsets of B with compact neighborhoods $\tilde{C}$ and $\tilde{D}$, respectively, in B such that $C \subset \tilde{C} \subset D \subset \tilde{D}$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that if $p : E \to B$ is a (δ,ν)-fibration over $(\tilde{C},\tilde{D})$ for compacta, then the following statement is true:

Let $Z = Z_1 \cup Z_2$, $X_1 \subset Z_1 \setminus Z_2$, and $X_2 \subset Z_2 \setminus Z_1$ be given where Z, Z_1, Z_2, X_1, and X_2 are compact. Let there be given maps $F : Z \times [0,1] \to C$ and $f : Z \to E$ with $pf = F_0$, $G^i : Z_i \times [0,1] \to E$ with $G^i_0 = f|Z_i$ and pG^i (δ,ν)-close to $F|Z_i \times [0,1]$ with respect to D for $i = 1,2$. Then there is a map $G : Z \times [0,1] \to E$ such that pG is (ε,μ)-close to F with respect to D, $G_0 = f$, and $G|X_i \times [0,1] = G^i|X_i \times [0,1]$ for $i = 1,2$.

PROOF. Given $\varepsilon > 0$ choose $\delta' > 0$ such that $\delta' < \varepsilon/2$ and the δ'-neighborhood of C in B is contained in $\tilde{C}$. Choose $\delta < 0$ such that $\delta < \delta'$ and such that $\delta < \delta(\delta')$ where $\delta(\delta')$ comes from Proposition 4.2 (with $A = D$ and $C = \tilde{C}$).

Given $\mu > 0$ choose $\nu' > 0$ such that $\nu' < \mu/2$ and the ν'-neighborhood of D in B is contained in $\tilde{D}$. Choose $\nu > 0$ such that $\nu < \nu'$ and $\nu < \nu(\nu')$ where $\nu(\nu')$ comes from Proposition 4.2 (with $A = D$ and $C = \tilde{C}$).

Assume that we are given the situation of Lemma 4.5. Choose a compact neighborhood $\tilde{X}_i$ of X_i in Z_i such that $\tilde{X}_i \cap Z_1 \cap Z_2 = \emptyset$ for $i = 1,2$. Define $\bar{G} : ((\tilde{X}_1 \cup \tilde{X}_1) \times [0,1]) \cup (Z \times \{0\}) \to E$ by $\bar{G}|\tilde{X}_i \times [0,1] = G^i|$ for $i = 1,2$ and $\bar{G}|Z \times \{0\} = f$. Then $p\bar{G}$ is (δ,ν)-close to $F|$ with respect to D. It follows from Proposition 4.2 and the choices of δ and ν that there is a (δ',ν')-homotopy from $F|$ to $p\bar{G}$ with respect to D.

By Proposition 4.3 the map $p\bar{G}$ extends to a map $F' : Z \times [0,1] \to B$ which is (δ',ν')-close to F with respect to D. Note that the image of F' is contained in $\tilde{C}$.

Define a homotopy $R_t : Z \times [0,1] \to Z \times [0,1]$, $0 \le t \le 1$, such that

i) $R_1 = id$,

ii) $R_t|[(X_1 \cup X_2) \times [0,1]) \cup (Z \times \{0\})] = id$ for each t in $[0,1]$,

iii) $R_0(Z \times [0,1]) \subset ((\tilde{X}_1 \cup \tilde{X}_2) \times [0,1]) \cup (Z \times \{0\})$.

Note that $p\bar{G}R_0 = F'R_0$. Since p is a (δ,ν)-fibration over $(\tilde{C},\tilde{D})$ for compacta, there is a homotopy $H_t : Z \times [0,1] \to E$, $0 \le t \le 1$, such that

i) $H_0 = \bar{G}R_0$,

ii) pH_t is (δ,ν)-close to $F'R_t$ with respect to $\tilde{D}$ for each t in $[0,1]$,

iii) $H_t|[((X_1 \cup X_2) \times [0,1]) \cup (Z \times \{0\})] = H_0|$ for each t in $[0,1]$.

(To achieve condition iii) we are using the stationary lifting property. See Section 6.)

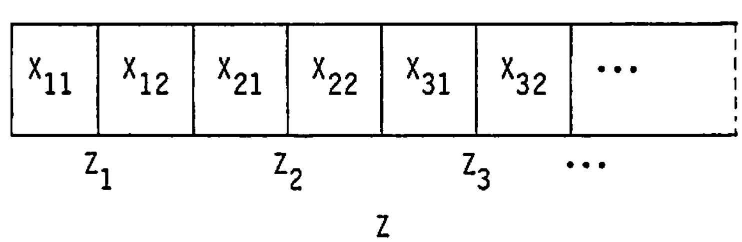
It can be checked that $G = H_1 : Z \times [0,1] \to E$ is the desired (ε,μ)-lift of F with respect to D. □

LEMMA 4.6. Let B be an ANR and let C and D be compact subsets of B with compact neighborhoods $\tilde{C}$ and $\tilde{D}$, respectively, in B such that $C \subset \tilde{C} \subset D \subset \tilde{D}$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if $p : E \to B$ is a (δ,ν)-fibration over $(\tilde{C},\tilde{D})$ for compacta,

then p is an (ε,μ)-fibration over (C,D).

PROOF. Our δ and ν are chosen according to Lemma 4.5. We also assume that $\delta \le \varepsilon$ and $\nu \le \mu$. Let $p : E \to B$ be given as in the hypothesis. We want to show that p is an (ε,μ)-fibration over (C,D). To this end let Z be a space (a local compactum of course) and suppose we are given maps $F : Z \times [0,1] \to C \subset B$ and $f : Z \to E$ such that $pf = F_0$. Write $Z = \cup_{i=1}^{\infty} Z_i$ where each Z_i is compact and $Z_i \cap Z_j \neq \emptyset$ implies $|i-j| \le 1$. For each i write $Z_i = X_{i1} \cup X_{i2}$ where X_{ij} is compact for $j = 1,2$, $X_{i2} \cap Z_{i-1} = \emptyset$, and $Z_i \cap Z_{i+1} \subset X_{i2} \cap X_{(i+1)1}$.



Since p is a (δ,ν)-fibration over $(\tilde{C},\tilde{D})$ for compacta, we can find for each i, a map $G^i : Z_i \times [0,1] \to E$ such that $G_0^i = f|Z_i$ and pG^i is (δ,ν)-close to $F|Z_i \times [0,1]$ with respect to D. For each i we will inductively define a map $\hat{G}^i : (\cup_{j=1}^{i} Z_j) \times [0,1] \to E$ with the following properties:

i) $\hat{G}^i_0 = f|$;

ii) $\hat{G}^i|[(\cup_{j=1}^{i-2} Z_j) \cup X_{(i-1)1}] \times [0,1] = \hat{G}^{i-1}|$;

iii) $\hat{G}^i|X_{i2} \times [0,1] = G^i|$;

iv) $p\hat{G}^i$ is (ε,μ)-close to F with respect to D.

To start off the induction set $\hat{G}^1 = G^1$. Assuming that $\hat{G}^i$ has been defined, we proceed to define $\hat{G}^{i+1}$. Since $\hat{G}^{i+1}$ is to extend $\hat{G}^i|[(\cup_{j=1}^{i-1} Z_j) \cup X_{i1}] \times [0,1]$, we will just define $\hat{G}^{i+1}$ on $X_{i2} \cup Z_{i+1}$ (in such a way that $\hat{G}^{i+1}|(X_{i1} \cap X_{i2}) \times [0,1] = \hat{G}^i|$). To do this we simply use Lemma 4.5 to piece together $\hat{G}^i|X_{i2} \times [0,1] = G^i|$ and G^{i+1} to get an (ε,μ)-lift $\hat{G}^{i+1}$ such that $\hat{G}^{i+1}|(X_{i1} \cap X_{i2}) \times [0,1] = \hat{G}^i|$ and such that $\hat{G}^{i+1}|X_{(i+1)2} \times [0,1] = G^{i+1}|$.

Finally, the map $G : Z \times [0,1] \to E$, defined so that $G|Z_i \times [0,1] = \hat{G}^{i+1}$, is the required (ε,μ)-lift of F. □

LEMMA 4.7. Let B be an ANR and let $K \subset \tilde{K} \subset V \subset C \subset U \subset B$ where K and C are compact, V and U are open, and $\tilde{K}$ is a compact neighborhood of K. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that if $p : E \to B$ is a δ-fibration over U and a ν-fibration over V, then the following statement is true:

> whenever $F : Z \times [0,1] \to C$ and $f : Z \to E$ are maps such that $pf = F_0$ and for each z in Z, $F(\{z\} \times [0,1])$ lies either in int $\tilde{K}$ or in $C \setminus K$, then there is a map $G : Z \times [0,1] \to E$ such that $G_0 = f$ and pG is (ε,μ)-close to F with respect to K.

PROOF. Given $\varepsilon > 0$ simply choose $\delta > 0$ so that $\delta < \varepsilon$. Given $\mu > 0$ choose $\nu' > 0$ so that $\delta + \nu' < \varepsilon$, $\nu' < \mu$, and the ν'-neighborhood of C in

B is contained in U. Choose $\nu > 0$ so that any two ν-close maps into $\tilde{K}$ are ν'-homotopic in V.

Let p, F and f be given as in the hypothesis. Define $Z_1 = \{z \in Z | F(\{z\} \times [0,1]) \subset \text{int } \tilde{K}\}$ and $\hat{Z}_1 = \{z \in Z | F(\{z\} \times [0,1])$ meets $K\}$. Note that $\hat{Z}_1 \subset Z_1$, $\hat{Z}_1$ is closed in Z, and Z_1 is open in Z. Since p is a ν-fibration over V, there is a map $G^1 : Z_1 \times [0,1] \to E$ such that $G^1_0 = f|Z_1$ and such that pG^1 is ν-close to $F|(Z_1 \times [0,1])$. Thus pG^1 and $F|Z_1 \times [0,1]$ are ν'-homotopic in V. Using the estimated homotopy extension property we can extend pG^1 to a map $F' : Z \times [0,1] \to B$ such that $F'_0 = f$ and such that F' is ν'-close to F. Note that the image of F' is contained in U.

Since $\hat{Z}_1$ is closed in Z and Z_1 is open in Z, we can use a Uryshon map to define a homotopy $R_t : Z \times [0,1] \to Z \times [0,1]$, $0 \le t \le 1$, with the following properties:

i) $R_1 = \text{id}$;

ii) $R_t|(\hat{Z}_1 \times [0,1]) \cup (Z \times \{0\}) = \text{id}$ for each t in [0,1];

iii) $R_0(Z \times [0,1]) \subset (Z_1 \times [0,1]) \cup (Z \times \{0\})$.

Define $\bar{G}^1 : (Z_1 \times [0,1]) \cup (Z \times \{0\}) \to E$ by $\bar{G}^1|(Z_1 \times [0,1]) = G^1$ and $\bar{G}^1|(Z \times \{0\}) = f$. Note that $p\bar{G}^1 R_0 = F'R_0$. Since p is a δ-fibration over U, there is a homotopy $H_t : Z \times [0,1] \to E$, $0 \le t \le 1$, such that

i) $H_0 = \bar{G}^1 R_0$,

ii) $H_t|(\hat{Z}_1 \times [0,1]) \cup (Z \times \{0\}) = \bar{G}^1|$ for each t in [0,1],

iii) pH_t is δ-close to $F'R_t$ for each t in [0,1].

(To achieve condition ii) we are again using the stationary lifting property. See Section 6.)

Finally, the map $G = H_1 : Z \times [0,1] \to E$ is seen to be the desired (ε,μ)-lift of F over (C,K). □

We are now ready for our main result of this section.

THEOREM 4.8. Let B be an ANR and let $K \subset V \subset C \subset U \subset B$ where C and K are compact and U and V are open in B. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if $p : E \to B$ is a δ-fibration over U and a ν-fibration over V, then p is an (ε,μ)-fibration over (C,K).

PROOF. Let $K = K_0 \subset K_1 \subset K_2 \subset K_3 \subset V$ where each K_i is a compact neighborhood of K_{i-1} in B and let $C = C_0 \subset C_1 \subset C_2 \subset U$ where each C_i is a compact neighborhood of C_{i-1} in B. Let $\varepsilon > 0$ be given and choose $\delta' > 0$ such that $\delta' < \varepsilon/2$, the δ'-neighborhood of $C \setminus K_1$ in B is contained in $C_1 \setminus K$, and the δ'-neighborhood of K_2 in B is contained in int K_3. Choose $\delta_1 > 0$ such that $\delta_1 \le \delta(\delta')$ of Proposition 4.2 with $A = K$. Finally, choose $\delta > 0$ such that $\delta \le \delta(\min(\varepsilon,\delta_1))$ of Lemma 4.7 with $\tilde{K} = K_3$, $K = K_1$, and $C = C_1$.

Let $\mu > 0$ be given and choose $\nu' > 0$ such that $\nu' < \mu/2$ and the ν'-neighborhood of K in B is contained in K_1. Choose $\nu_1 > 0$ such that $\nu_1 \le \nu(\nu')$ of Proposition 4.2 with $A = K$. Finally choose $\nu > 0$ such that $\nu \le \nu(\min(\mu/2,\nu_1))$ of Lemma 4.7 with the choices described above.

With these choices we will show that $p : E \to B$ is an (ε,μ)-fibration over (C,K) for compacta. Of course, to be technically correct we should replace C, K by $\tilde{C}$, $\tilde{K}$ and use Lemma 4.6 to conclude that p is an (ε',μ')-fibration over (C,K), but we will omit these further epsilonics.

Let $F : Z \times [0,1] \to C \subset B$ and $f : Z \to E$ be maps such that $pf = F_0$ and Z is compact. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$ so that each $F(\{z\} \times [t_{i-1},t_i])$ lies in either int K_2 or $C \setminus K_1$. We will inductively define maps $G^i : Z \times [0,t_i] \to E$ for $i = 0,1,\ldots,n$ with the following properties:

i) G^i extends G^{i-1} if $i > 0$ and $G^0 = f$;

ii) pG^i is (ε,μ)-close to $F|Z \times [0,t_i]$ with respect to K;

iii) $pG^i|Z \times \{t_i\}$ is (δ_1,ν_1)-close to $F|Z \times \{t_i\}$ with respect to K for $i < n$.

Of course, G^n will be the desired (ε,μ)-lift of F with respect to K.

Assuming that G^i has been defined, we proceed to define G^{i+1} $(0 \le i < n)$. Since $pG^i|Z \times \{t_i\}$ is (δ_1,ν_1)-close to $F|Z \times \{t_i\}$ with respect to K, there is a (δ',ν')-homotopy H from $F|Z \times \{t_i\}$ to $pG^i|Z\times\{t_i\}$ with respect to K. Using the homotopy H we can "pull $F|Z \times [t_i,1]$ around the corner created by H" to get a new map $F' : Z \times [0,1] \to B$ satisfying:

i) $F'|Z \times [0,t_i] = pG^i$;

ii) $F'|Z \times [t_{i+1},1] = F|$;

iii) $F'|Z \times [t_i,1]$ is (δ',ν')-close to $F|$ with respect to K;

iv) each $F'(\{z\} \times [t_i,t_{i+1}])$ lies in either int K_3 or $C_1 \setminus K$.

Since G^{i+1} is to extend G^i we only need to define G^{i+1} on $Z \times [t_i,t_{i+1}]$. Using Lemma 4.7 we can find a map $G^{i+1} : Z\times[t_i,t_{i+1}] \to E$ so that

i) $G^{i+1}|Z \times \{t_i\} = G^i|Z \times \{t_i\}$,

ii) pG^{i+1} is $(\min(\varepsilon/2,\delta_1),\min(\mu/2,\nu_1))$-close to $F'|Z\times[t_i,t_{i+1}]$ with respect to K_1.

This completes the inductive step and also the proof of the theorem. □

SECTION 5:
SLICED ε-FIBRATIONS

In this section we present a key lemma for our proof that certain spaces of approximate fibrations are locally n-connected. The main result here is Theorem 5.3 which says that a family of δ-fibrations parameterized by a finite dimensional polyhedron has a certain sliced ε-lifting property. We begin with some definitions.

DEFINITION 5.1. Let E, B, and X be spaces and let $C \subset B$. A map $f : E \times X \to B \times X$ is said to be *fiber preserving* (f.p.) if $p_X f = p_X$ where p_X denotes projection to X. If $\varepsilon > 0$ and $f : E \times X \to B \times X$ is a proper f.p. map, then we say f is a *sliced* ε*-fibration over* $C \times X$ if f satisfies the following *sliced* ε*-lifting property over* $C \times X$:

> if $F : Z \times [0,1] \times X \to C \times X \subset B \times X$ and $g : Z \times X \to E \times X$ are f.p. maps such that $fg = F_0$, then there exists a f.p. map $G : Z \times [0,1] \times X \to E \times X$ such that $G_0 = g$ and fG is ε-close to F.

If $C = B$, then f is called a *sliced* ε*-fibration*.

We will need the following lemma for the proof of Theorem 5.3.

LEMMA 5.2. Let $f : E \times X \to B \times X$ be a proper f.p. map where E and B are ANRs. Let C be a compact subset of B and let $\tilde{C}$ be a compact neighborhood of C in B. Let $\varepsilon > 0$ and suppose for each x in X $f_x = f|E \times \{x\} : E = E \times \{x\} \to B \times \{x\} = B$ is an ε-fibration over $\tilde{C}$. For every $\alpha > 0$ there is an open cover $\mathcal{U}$ of X such that if X_0 is any subset of X contained in some member of $\mathcal{U}$, then $f|E \times X_0 : E \times X_0 \to B \times X_0$ is an $(\alpha+\varepsilon)$-fibration over $C \times X_0$.

PROOF. Given the hypothesis above, choose $\beta = \beta(\alpha/3,C,\tilde{C},B)$ by Lemma 2.1. Choose the open cover $\mathcal{U}$ of X so that the diameter of any member of $\mathcal{U}$ is less than $\alpha/3$ and so that f_x is β-close to f_y over $\tilde{C}$ whenever x and y are elements of a common member of $\mathcal{U}$.

Let X_0 be a subset of X contained in some member of $\mathcal{U}$. Let maps $F : Z \times [0,1] \to C \times X_0$ and $g : Z \to E \times X_0$ be given such that $fg = F_0$. Since the diagram

$$\begin{array}{ccc} E \times X_0 & \xrightarrow{p_E = \text{proj}} & E \\ f \downarrow & & \downarrow f_x \\ B \times X_0 & \xrightarrow{p_B = \text{proj}} & B \end{array}$$

β-commutes over $\tilde{C}$ where $x \in X_0$ is fixed, the choice of β implies the existence of a map $G : Z \times [0,1] \to E$ such that $G_0 = p_E g$ and $f_x G$ is $((\alpha/3)+\varepsilon)$-close to $p_B F$. Define $H : Z \times [0,1] \to E \times X_0$ by $H(z,t) = (G(z,t), p_X g(z))$. Then H is seen to be an $(\alpha+\varepsilon)$-lift of F. □

THEOREM 5.3. Let C be a compact subset of the ANR B and let $\tilde{C}$ be a compact neighborhood of C in B. Let n be an integer. For every $\varepsilon > 0$ there exists a $\delta > 0$, $\delta = \delta(\varepsilon,n,C,\tilde{C},B)$, such that if E is an ANR, X is an n-dimensional polyhedron, and $f : E \times X \to B \times X$ is a proper f.p. map such that f_x is a δ-fibration over $\tilde{C}$ for each x in X, then f is a sliced ε-fibration over $C \times X$.

PROOF. The proof is by induction on n. The theorem is clearly true for $n = 0$ by taking $\delta(\varepsilon,0,C,\tilde{C},B) = \varepsilon$. Assume $n > 0$ and that the theorem is true for $n - 1$. Let $\varepsilon > 0$ be given and choose $\beta = \beta(\varepsilon/4,C,\tilde{C},B)$

by Lemma 2.1. Let $\delta = \delta(\varepsilon,n,C,\tilde{C},B) = \min\{\varepsilon/8, \delta(\beta,n-1,C,\tilde{C},B)\}$. Let $f : E \times X \to B \times X$ be given as in the hypothesis. By Lemma 5.2 we can consider X to have such a fine triangulation that if σ is any simplex of X, then $f|E \times \sigma : E \times \sigma \to B \times \sigma$ is an $(\varepsilon/4)$-fibration over $C \times \sigma$. Note that $p_B f|E \times \sigma : E \times \sigma \to B$ is also an $(\varepsilon/4)$-fibration over C. We also assume that f_x is $(\varepsilon/2)$-close to f_y over $\tilde{C}$ whenever x and y are in σ.

Given f.p. maps $F : Z \times [0,1] \times X \to C \times X \subset B \times X$ and $g : Z \times X \to E \times X$ such that $fg = F_0$, the inductive assumption implies the existence of a f.p. map $G : Z \times [0,1] \times X^{n-1} \to E \times X^{n-1}$ such that $G_0 = g|Z \times X^{n-1}$ and fG is β-close to $F|Z \times [0,1] \times X^{n-1}$ (here X^{n-1} denotes the $(n-1)$-skeleton of X). Define $\tilde{g} : [Z \times [0,1] \times X^{n-1}] \cup [Z \times \{0\} \times X] \to E \times X$ by $\tilde{g}|Z \times [0,1] \times X^{n-1} = G$ and $\tilde{g}(z,0,x) = g(z,x)$. Let σ be an n-simplex of X and note that the pair $(Z \times [0,1] \times \sigma, [Z \times [0,1] \times \partial\sigma] \cup [Z \times \{0\} \times \sigma])$ is homeomorphic to the pair $(Z \times [0,1] \times I^n, Z \times \{0\} \times I^n)$. Hence, there exists a map $\tilde{F}_\sigma : Z \times [0,1] \times \sigma \to E \times \sigma$ such that $\tilde{F}_\sigma|[Z \times [0,1] \times \partial\sigma] \cup [Z \times \{0\} \times \sigma] = \tilde{g}|$ and $p_B f\, \tilde{F}_\sigma$ is $(\varepsilon/2)$-close to $p_B F|$. ($\tilde{F}_\sigma$ is not assumed to be f.p. .) Define $\tilde{F} : Z \times [0,1] \times X \to E \times X$ by $\tilde{F}|Z \times [0,1] \times \sigma = \tilde{F}_\sigma$ whenever σ is an n-simplex of X and $\tilde{F}|Z \times [0,1] \times X^{n-1} = G$. Finally, define $H : Z \times [0,1] \times X \to E \times X$ by $H(z,t,x) = (p_E\tilde{F}(z,t,x),x)$. Then H is seen to be the desired sliced ε-lift of F. □

The proof of Theorem 5.3 lends itself to a technical generalization which we state without proof below. First we need another definition.

DEFINITION 5.4. Let E, B, and X be spaces and let $K \subset C \subset B$. If $\varepsilon,\mu > 0$ and $f : E \times X \to B \times X$ is a proper f.p. map, then we say f is a *sliced* (ε,μ)-*fibration over* $(C \times X, K \times X)$ if f satisfies the following lifting property:

if $F : Z \times [0,1] \times X \to C \times X \subset B \times X$ and $g : Z \times X \to E \times X$ are f.p. maps such that $fg = F_0$, then there exists a f.p. map $G : Z \times [0,1] \times X \to E \times X$ such that $G_0 = g$ and fG is (ε,μ)-close to F with respect to $K \times X$.

PROPOSITION 5.5. Let $K \subset V \subset C \subset U \subset B$ where C and K are compact and U and V are open subsets of the ANR B. For every integer n and $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if E is an ANR, X is an n-dimensional polyhedron, and $f : E \times X \to B \times X$ is a proper f.p. map such that each f_x is a (δ,ν)-fibration over (U,V), then f is a sliced (ε,μ)-fibration over $(C \times X, K \times X)$.

SECTION 6:
STATIONARY LIFTING PROPERTIES

In this section we present a proof that ε-fibrations have the stationary ε-lifting property. The same result for approximate fibrations appears in [8]. Their proof is based on the usual proof of the analogous result for Hurewicz fibrations as found in [10, Chapter XX]. Our proof is somewhat different from those proofs in that we do not use lifting functions. A disadvantage of this is that we do not recover the stationary lifting property for abstract spaces (of course, this is not important to us). An advantage is that our simple proof readily generalizes to other types of fibrations that we encounter, for example sliced ε-fibrations and (ε,μ)-fibrations. We begin with a definition.

DEFINITION 6.1. Let $p : E \to B$ be a map and let $C \subset B$. If $\varepsilon > 0$, then p is said to have the *stationary ε-lifting property over* C if given maps $F : Z \times [0,1] \to C \subset B$ and $g : Z \to E$ such that $pg = F_0$, there exists a map $G : Z \times [0,1] \to E$ such that $G_0 = g$, pG is ε-close to F, and G is *stationary with* F, that is, whenever $z \in Z$ such that $F(z,t) = F(z,0)$ for each t in $[0,1]$, then $G(z,t) = G(z,0)$ for each t in $[0,1]$.

THEOREM 6.2. If $\varepsilon > 0$ and $p : E \to B$ is an ε-fibration over $C \subset B$, then p has the stationary ε-lifting property over C.

PROOF. Let $F : Z \times [0,1] \to C \subset B$ and $g : Z \to E$ be maps such that $pg = F_0$. Let $A = \{z \in Z | F(z,t) = F(z,0) \text{ for each } t \text{ in } [0,1]\}$. Then A is a closed subset of Z. Let $\rho : Z \to [0,1]$ be a map such that $\rho^{-1}(0) = A$. For each $z \in Z \setminus A$, let $\theta(z,\cdot) : [0,\rho(z)] \to [0,1]$ be the unique linear map which takes 0 to 0 and $\rho(z)$ to 1. Define $\psi : Z\times[0,1]\to Z\times[0,1]$ by

$$\psi(z,t) = \begin{cases} (z,0) & \text{if } z \in A, \\ (z,\theta^{-1}(z,t)) & \text{if } z \in Z \setminus A. \end{cases}$$

Thus, ψ is the map which is the identity on $Z \times \{0\}$, squeezes $Z \times \{1\}$ down to the graph of ρ, and is linear on each interval $\{z\} \times [0,1]$.

Define $F^* : Z \times [0,1] \to C$ by

$$F^*(z,t) = \begin{cases} F(z,\theta(z,t)) & \text{if } z \in Z \setminus A \text{ and } 0 \le t \le \rho(z), \\ F_1(z) & \text{if } \rho(z) \le t \le 1. \end{cases}$$

Since p is an ε-fibration over C, there exists a map $G^* : Z \times [0,1] \to E$ such that $G_0^* = g$ and pG^* is ε-close to F^*. Define $G : Z \times [0,1] \to E$ by

$$G(z,t) = \begin{cases} G^*(z,\theta^{-1}(z,t)) & \text{if } z \in Z \setminus A, \\ g(z) & \text{if } z \in A. \end{cases}$$

Then $G_0 = g$ and G is stationary with F. Also pG is ε-close to F. To see that G is continuous, just notice that $G = G^* \circ \psi$. □

DEFINITION 6.3. Let $p : E \times X \to B \times X$ be a f.p. map, let $\varepsilon > 0$, and let $C \subset B$. Then p is said to have the *sliced stationary* ε-*lifting property over* $C \times X$ if given f.p. maps $F : Z \times [0,1] \times X \to C \times X$ and $g : Z \times X \to E \times X$ such that $pg = F_0$, there exists a f.p. map $G : Z \times [0,1] \times X \to E \times X$ such that $G_0 = g$, pG is ε-close to F, and G is stationary with F.

PROPOSITION 6.4. If $\varepsilon > 0$ and $p : E \times X \to B \times X$ is a f.p. map such that p is a sliced ε-fibration over $C \times X$, $C \subset B$, then p has the sliced stationary ε-lifting property over $C \times X$.

REMARKS ON PROOF. The proof of this proposition is almost word-for-word like the proof of Theorem 6.2. The map F^* will be f.p. and thus G^* can be assume to be f.p. . It then follows that G will be f.p. . □

We could also define the *stationary* (ε,μ)*-lifting property over* (C,D).

PROPOSITION 6.5. If $\varepsilon,\mu > 0$ and $p : E \to B$ is an (ε,μ)-fibration over (C,D), $D \subset C \subset B$, then p has the stationary (ε,μ)-lifting property over (C,D).

REMARKS ON PROOF. The proof of this proposition is almost word-for-word like the proof of Theorem 6.2. Just require G^* to be an (ε,μ)-lift of F^* with respect to D. □

Of course, there are also analogous results for sliced (ε,μ)-fibrations and for those fibrations which have the lifting property for a certain class of spaces (for example, compacta).

SECTION 7:
PARAMETERIZED ENGULFING

In this section we establish the key engulfing results used in the sequel. These results are stated as Theorems 7.3, 7.8, and 7.9. Lemma 7.1 contains the basic geometric engulfing move used in the proof of Theorem 7.3. Generalizations of this result are contained in Lemmas 7.4 and 7.5. These lemmas are established using sliced Z-set unknotting.

DATA FOR LEMMA 7.1. Let B and Z denote ANRs where Z is compact and $Z \times \mathbb{R}$ is an open subset of B. Given an integer $n \geq 0$, let $\alpha_+ : I^n \to [0,1]$, $\alpha_- : I^n \to [-1,0]$, and $\rho : I^n \to [-1;0]$ be maps satisfying the following conditions:

i) $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0) \subset \partial I^n$,

ii) $\rho = 0$ on a neighborhood of $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0)$,

iii) $\alpha_-(t) < \rho(t)$ for each $t \in I^n \setminus \alpha_-^{-1}(0)$.

We now define four subsets of $Z \times \mathbb{R} \times I^n \subset B \times I^n$. Let $E = \{(z,x,t) | \alpha_-(t) \leq x \leq \alpha_+(t)\}$, $E_- = \{(z,x,t) | \alpha_-(t) = x\}$, $E_+ = \{(z,x,t) | \alpha_+(t) = x\}$, and $X = \{(z,x,t) | \alpha_-(t) \leq x \leq \rho(t)\}$. Note that $X \subset E$ and $\mathrm{Bd}E = E_- \cup E_+$.

Finally, let Y be a compact subset of E which misses E_+ and $Z \times \mathbb{R} \times [\alpha_-^{-1}(0) \cup \alpha_+^{-1}(0)]$. Denote projection to Z by p_1 and projection to $\mathbb{R}$ by p_2.

LEMMA 7.1. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if M is a Q-manifold and $f : M \times [0,1] \times I^n \to B \times I^n$ is a f.p. map which is a sliced δ-fibration over $Z \times [-2,2] \times I^n$, then there is a f.p.

homeomorphism $u : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that

i) $f^{-1}(Y) \cap (M \times \{0\} \times I^n) \subset uf^{-1}(X)$,

ii) u is supported on $f^{-1}(\text{int } E)$,

iii) there is a f.p. homotopy $u_s : \text{id} \simeq u$, $0 \le s \le 1$, which is supported on $f^{-1}(\text{int } E)$ and which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$.

PROOF. Let N_1 be a compact neighborhood of $\alpha_-^{-1}(0) \cup \alpha_+^{-1}(0)$ in I^n such that Y misses $Z \times \mathbb{R} \times N_1$ and such that $\rho = 0$ on N_1. Choose a compact neighborhood N_2 of $\alpha_-^{-1}(0) \cup \alpha_+^{-1}(0)$ in I^n such that $N_2 \subset \text{int } N_1$. Choose maps $\beta_+ : I^n \to [0,1]$ and $\beta_- : I^n \to [-1,0]$ with the following properties:

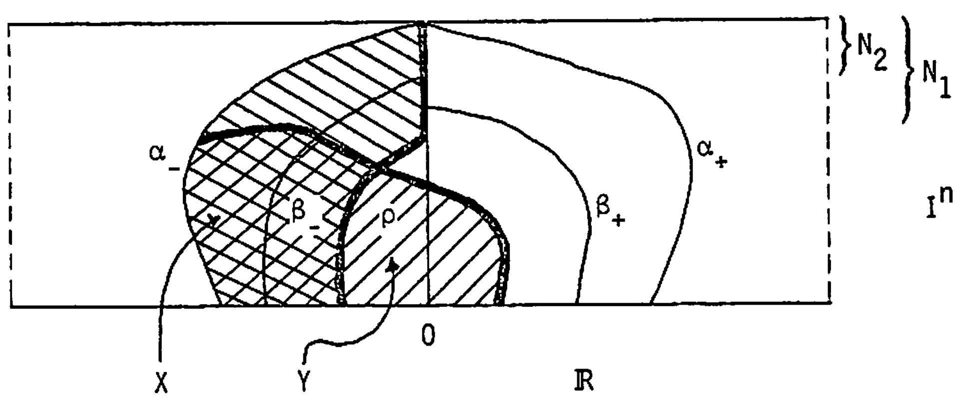
i) $\beta_+^{-1}(0) = N_1$, $\beta_-^{-1}(0) = N_2$;

ii) $\beta_+(t) < \alpha_+(t)$ for $t \in I^n \setminus \alpha_+^{-1}(0)$, $\alpha_-(t) < \beta_-(t)$ for $t \in I^n \setminus \alpha_-^{-1}(0)$;

iii) $Y \subset \{(z,x,t) \in Z \times \mathbb{R} \times I^n | x < \beta_+(t)\}$;

iv) $\beta_-(t) < \rho(t)$ for $t \in I^n \setminus N_2$.

Here is a picture of the situation when $Z = \{\text{point}\}$ and $n = 1$:



Choose a f.p. isotopy $g_s : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$, $0 \le s \le 1$, which slides the graph of β_+ over to the graph of β_-. More specifically, we require g_s to satisfy the following properties:

i) $g_0 = \text{id}$;

ii) g_s affects only the $\mathbb{R}$-coordinate of any point;

iii) $g_s | Z \times \mathbb{R} \times N_2 = \text{id}$;

iv) g_s is supported on a compact subset K of int E;

v) $g_1(z,\beta_+(t),t) = (z,\beta_-(t),t)$ for each $(z,t) \in Z \times I^n$.

Now given a Q-manifold M and a f.p. map $f : M \times [0,1] \times I^n \to B \times I^n$ which is a sliced δ-fibration over $Z \times [-2,2] \times I^n$, define a f.p. homotopy $\tilde{g}_s : M \times [0,1] \times I^n \to B \times I^n$, $0 \le s \le 1$, so that $\tilde{g}_s = g_s \circ f$ on $f^{-1}(Z \times \mathbb{R} \times I^n)$ and $g_s = f$ on $(M \times [0,1] \times I^n) \setminus f^{-1}(Z \times \mathbb{R} \times I^n)$. Since f is a sliced δ-fibration over $Z \times [-2,2] \times I^n$, there is a f.p. homotopy $G_s : f^{-1}(Z \times [-2,2] \times I^n) \to M \times [0,1] \times I^n$ such that $G_0 = \text{id}$, fG_s is δ-close to $\tilde{g}_s|$ for each s, and G_s is stationary with respect to $\tilde{g}_s | f^{-1}(Z \times [-2,2] \times I^n)$ (Proposition 6.4).

Observe that $G_s | [f^{-1}(Z \times [-2,2] \times I^n) \cap (M \times \{0\} \times I^n)]$ extends via the identity to a map $G_s' : M \times \{0\} \times I^n \to M \times [0,1] \times I^n$. Using Proposition 3.6 we can approximate G_1' by a f.p. embedding $\tilde{G}_1 : M \times \{0\} \times I^n \to M \times [0,1] \times I^n$ possessing the following properties:

i) $\tilde{G}_1(M \times \{0\} \times I^n)$ is a sliced Z-set;

ii) $\tilde{G}_1$ is supported on $f^{-1}(K) \cap (M \times \{0\} \times I^n)$;

iii) $\tilde{G}_1(f^{-1}(\text{int } E) \cap (M \times \{0\} \times I^n)) \subset f^{-1}(\text{int } E)$.

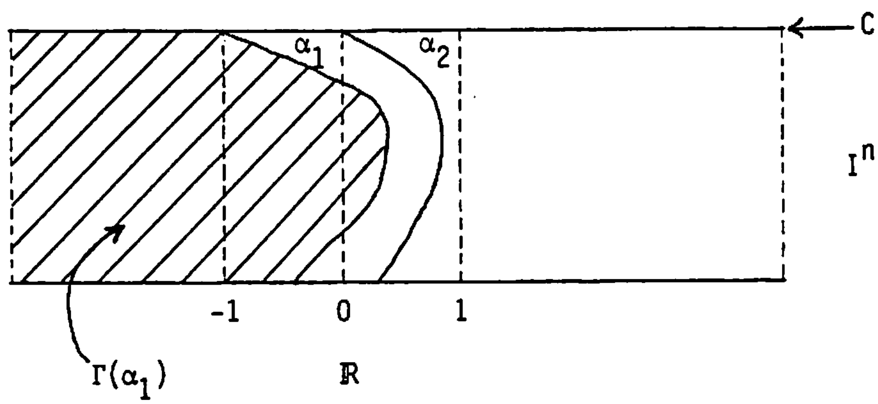
We can further assume that $\tilde{G}_1$ is so close to $G_1|$ that $\tilde{G}_1$ is f.p. homotopic to $G_1|$ via maps satisfying conditions ii) and iii).

Using sliced Z-set unknotting [6, Theorem 5.1] we can find a f.p. isotopy $u_s : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$, $0 \le s \le 1$, so that $u_0 = \mathrm{id}$, $u_1 | \tilde{G}_1(M \times \{0\} \times I^n) = \tilde{G}_1^{-1}$, and $u_s | (M \times [0,1] \times I^n) \setminus f^{-1}(\mathrm{int}\, E) = \mathrm{id}$. It is now easy to see that u_1 satisfies the conclusion of the lemma. It only remains to observe that we can assume u_t is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$. This is because of the control on the f.p. isotopy u_s given by [6, Theorem 5.1]. □

ADDENDUM TO LEMMA 7.1. Let F be a compact Q-manifold such that $F \times B \subset M$ and $f | F \times B \times \{0\} \times I^n : F \times B \times \{0\} \times I^n \to B \times I^n$ is projection. Extend $g_s : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$ via the identity to a f.p. homeomorphism $\hat{g}_s : B \times I^n \to B \times I^n$, $0 \le s \le 1$. Then the f.p. homotopy $u_s : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$, $0 \le s \le 1$, can be chosen to additionally satisfy $u_s | F \times B \times \{0\} \times I^n = \mathrm{id}_F \times \hat{g}_s^{-1}$, $0 \le s \le 1$.

PROOF. We indicate here how to modify the proof of Lemma 7.1 in order to attain the added condition on u_s. Since $\tilde{g}_s | F \times B \times \{0\} \times I^n = f \circ (\mathrm{id}_F \times \hat{g}_s)$ for $0 \le s \le 1$ and $F \times B \times \{0\}$ is collared in $M \times [0,1]$, it can be assumed that $G'_s | F \times B \times \{0\} \times I^n = (\mathrm{id}_F \times \hat{g}_s)$ for $0 \le s \le 1$. Using the full strength of Proposition 3.6, the f.p. embedding $\tilde{G}_1 : M \times \{0\} \times I^n \to M \times [0,1] \times I^n$ can be chosen so that $\tilde{G}_1 | F \times B \times \{0\} \times I^n = (\mathrm{id}_F \times \hat{g}_1)$ and the homotopy from G'_1 to $\tilde{G}_1$ is rel $F \times B \times \{0\} \times I^n$. After a reparameterization of the homotopy from G'_0 to $\tilde{G}_1$, one simply uses the strong relative version of sliced Z-set unknotting (Proposition 3.10) to produce $u_s : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$, $0 \le s \le 1$, with the desired properties instead of [6, Theorem 5.1]. □

DATA FOR LEMMA 7.2. Let B and Z denote ANRs where Z is compact and $Z \times \mathbb{R}$ is an open subset of B. Given an integer $n \geq 0$, let $\alpha_1 : I^n \to [-1,1]$ and $\alpha_2 : I^n \to [0,1]$ denote maps such that $\alpha_1(t) < \alpha_2(t)$ for each $t \in I^n$ and $\alpha_1^{-1}(-1) = \alpha_2^{-1}(0) = C \subset \partial I^n$. Let $\Gamma(\alpha_1) = \{(z,x,t) \in Z \times \mathbb{R} \times I^n | x \leq \alpha_1(t)\}$.



LEMMA 7.2. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a f.p. map which is a sliced δ-fibration over $Z \times [-2,2] \times I^n$, then there is a f.p. homeomorphism $h : M \times I^n \to M \times I^n$ such that

i) $h|M \times C$ is the identity,

ii) $f^{-1}(\Gamma(\alpha_1)) \subset hf^{-1}(Z \times (-\infty,0] \times I^n)$,

iii) there is a f.p. homotopy $h_s : id \simeq h$, $0 \leq s \leq 1$, which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ where p_1 denotes projection to Z,

iv) h_s is supported on $f^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | -.9 \leq x \leq \alpha_2(t), t \in I^n \setminus C\}$ for each $0 \leq s \leq 1$.

PROOF. Given $\varepsilon > 0$, $\delta > 0$ is chosen by Lemma 7.1 so that the two basic engulfing moves described below can be made. Given a f.p. map

$f : M \times I^n \to B \times I^n$ as in the hypothesis, choose a f.p. map $k : M \times [0,1] \times I^n \to M \times I^n$ close to projection such that $k|M \times [0,1] \times C$ is projection and $k|M \times [0,1] \times (I^n \setminus C) : M \times [0,1] \times (I^n \setminus C) \to M \times (I^n \setminus C)$ is a homeomorphism (see Remark 3.16 or [14, Theorem 4.6]). It then follows that $fk : M \times [0,1] \times I^n \to B \times I^n$ is also a sliced δ-fibration over $Z \times [-2,2] \times I^n$. The homeomorphism h will be defined as a composition $h = k \circ v \circ w \circ u \circ k^{-1}$ where v, w, and u are constructed below.

Use Lemma 7.1 to produce a f.p. homeomorphism $u : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that $(fk)^{-1}(\Gamma(\alpha_1)) \cap (M \times \{0\} \times I^n) \subset u(fk)^{-1}(Z \times (-\infty,0) \times I^n)$ and u is supported on $(fk)^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | -.5 \le x < \alpha_2(t),\ t \in I^n \setminus C\}$.

Let $S_1 = (fk)^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | -.6 \le x \le \alpha_1(t)\}$ and let $S_2 = (fk)^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \max(0,\alpha_1(t)) < x < \alpha_2(t)\}$. Use Lemma 7.1 again to produce a f.p. homeomorphism $v : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that $S_1 \cap (M \times \{1\} \times I^n) \subset v^{-1}(S_2)$ and v is supported on $(fk)^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | -.7 \le x < \alpha_2(t),\ t \in I^n \setminus C\}$.

Let $U = (fkv)^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \max(0,\alpha_1(t)) < x\}$ and observe that if k is close enough to projection, then $S_1 \subset [\pi(U \cap (M \times \{1\} \times I^n))] \times [0,1]$ where $\pi : M \times [0,1] \times I^n \to M \times I^n$ is projection. Then $w : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ is a f.p. homeomorphism affecting only the [0,1]-coordinate of any point such that $w[S_1 \setminus u(fk)^{-1}(Z \times (-\infty,0) \times I^n)] \subset U$. The support of w is on $(fk)^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | -.7 \le x < \alpha_2(t)\}$.

It is easily verified that $h = k \circ v \circ w \circ u \circ k^{-1}$ satisfies the conclusions of the lemma. The homotopy of the identity to h comes from composing three homotopies of u, v, and w to the identity. These homotopies for u and v are provided by Lemma 7.1; the homotopy needed for w comes from pushing along the [0,1]-factor in $M \times [0,1] \times I^n$. □

ADDENDUM TO LEMMA 7.2. Let N_1 and N_2 be compact neighborhoods of C in I^n such that $N_2 \subset \text{int } N_1$ and $\Gamma(\alpha_1)$ misses $Z \times [-.5,+\infty) \times N_1$. Choose maps $\beta_+ : I^n \to [0,1]$ and $\beta_- : I^n \to (-.5,0]$ with the following properties:

i) $\beta_+^{-1}(0) = N_1$, $\beta_-^{-1}(0) = N_2$;

ii) $\alpha_1(t) < \beta_+(t)$ for each $t \in I^n$;

iii) $\beta_+(t) < \alpha_2(t)$ for each $t \in I^n \setminus C$.

Suppose we are given a f.p. isotopy $g_s : B \times I^n \to B \times I^n$, $0 \leq s \leq 1$, such that $g_s | Z \times \mathbb{R} \times I^n$ satisfies the properties listed for g_s in the proof of Lemma 7.1 where now $E = \{(z,x,t) \in Z \times \mathbb{R} \times I^n | -.5 \leq x \leq \alpha_2(t)\}$. Let F be a compact Q-manifold such that $F \times B$ is a Z-set in M and $f | F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection. Then the f.p. homeomorphism $h : M \times I^n \to M \times I^n$ can be chosen to additionally satisfy $h | F \times B \times I^n = id_F \times g_1^{-1}$ and the f.p. homotopy $h_s : id \simeq h$, $0 \leq s \leq 1$, can be chosen to additionally satisfy $h_s | F \times B \times I^n = id_F \times g_s^{-1}$, $0 \leq s \leq 1$.

PROOF. Just three modifications need to be made in the proof of Lemma 7.2. First, choose the f.p. map $k : M \times [0,1] \times I^n \to M \times I^n$ to additionally satisfy $k | F \times B \times \{0\} \times I^n : F \times B \times \{0\} \times I^n \to F \times B \times I^n$ is the identity. This is possible by Proposition 3.6. Secondly, choose the f.p. homeomorphism $u : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ and the f.p. homotopy $u_s : id \simeq u$, $0 \leq s \leq 1$, so that $u_s | F \times B \times \{0\} \times I^n = id \times g_s^{-1}$. This is possible by the Addendum to Lemma 7.1. Finally, choose the f.p. homeomorphism $v : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ and the f.p. homotopy

$v_s : id \simeq v$, $0 \leq s \leq 1$, so that $v_s|M \times \{0\} \times I^n$ is the identity. To see that this is possible, recall that v is provided by Lemma 7.1 and reexamine its proof.

With these modifications it is now easy to see that $h = k \circ v \circ w \circ u \circ k^{-1}$ satisfies the conclusion of the addendum. □

DATA FOR THEOREM 7.3. Let B and Z denote ANRs where Z is compact and $Z \times \mathbb{R}$ is an open subset of B. Let p_1 denote projection onto Z and p_2 projection onto $\mathbb{R}$. Let $n \geq 0$ be an integer and let C be a closed subset of ∂I^n. Let $\theta : \mathbb{R} \times I^n \to \mathbb{R} \times I^n$ be a f.p. homeomorphism with the following properties:

i) $\theta|\mathbb{R} \times C$ is the identity;

ii) $x \leq p_2\theta(x,t)$ for each $x \in \mathbb{R}$ and $t \in I^n$;

iii) θ is supported on $[-1,1] \times I^n$.

Let $\theta' : B \times I^n \to B \times I^n$ denote the f.p. homeomorphism which extends $id_Z \times \theta$ via the identity.

For each $\bar{x} \in \mathbb{R}$, let $\Gamma(\theta,\bar{x}) = \{(z,x,t) \in Z \times \mathbb{R} \times I^n | x \leq p_2\theta(\bar{x},t)\}$.

THEOREM 7.3. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a f.p. map which is a sliced δ-fibration over $Z \times [-2,2] \times I^n$, then there is a f.p. homeomorphism $\tilde{\theta} : M \times I^n \to M \times I^n$ such that

i) $\tilde{\theta}|M \times C$ is the identity,

ii) $f\tilde{\theta}$ is ε-close to $\theta' f$,

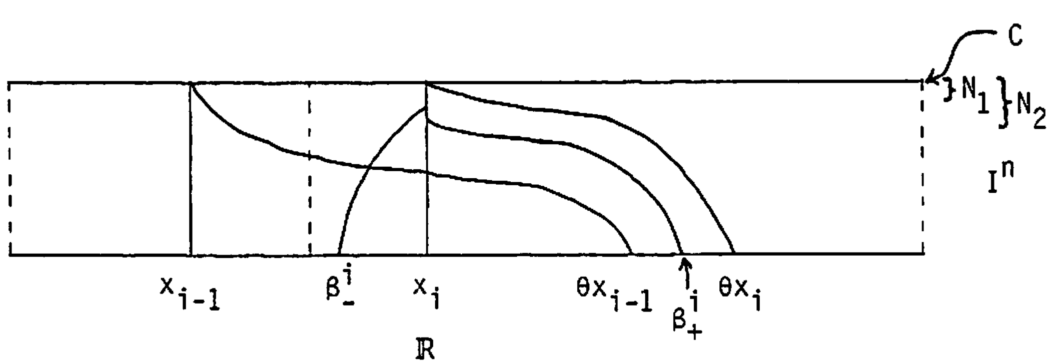
iii) $\tilde{\theta}$ is supported on $f^{-1}(Z \times [-1,1] \times I^n)$,

iv) there is a f.p. homotopy $\tilde{\theta}_s : id \simeq \tilde{\theta}$, $0 \leq s \leq 1$, which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ and is supported on $f^{-1}(Z \times [-1,1] \times (I^n \setminus C))$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then $\tilde{\theta}$ can be chosen so that $\tilde{\theta}|F \times B \times I^n = id_F \times \theta'$ and the homotopy $\tilde{\theta}_s$, $0 \le s \le 1$, can be chosen so that $p_1\tilde{\theta}_s|F \times Z \times \mathbb{R} \times I^n = p_1$ for $0 \le s \le 1$.

PROOF. Let $\varepsilon > 0$ be given. Choose a partition $-1 = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = 1$ of $[-1,1]$ so fine that the interval $[p_2\theta(x_{i-2},t), p_2\theta(x_i,t)]$ has diameter less than $\varepsilon/2$ for each $i = 2,3,\ldots,m$ and $t \in I^n$. Then $\delta > 0$ is chosen according to Lemma 7.2 so that each of the m-1 engulfing moves described below can be performed.

Given a Q-manifold M and a f.p. map $f : M \times I^n \to B \times I^n$ which is a sliced δ-fibration over $Z \times [-2,2] \times I^n$, we proceed to define f.p. homeomorphisms $\tilde{\theta}^i : M \times I^n \to M \times I^n$ and $\theta^i : \mathbb{R} \times I^n \to \mathbb{R} \times I^n$. Choose compact neighborhoods N_1 and N_2 of C in I^n such that $N_2 \subset \text{int } N_1$ and such that $\Gamma(\theta, x_{i-1})$ misses $Z \times [1/2(x_{i-1} + x_i), +\infty) \times N_1$ for $i = 1,\ldots,m-1$. For $i = 1,\ldots,m-1$ choose maps $\beta_+^i : I^n \to [x_i, +\infty)$ and $\beta_-^i : I^n \to (1/2(x_{i-1} + x_i), x_i]$ such that $(\beta_+^i)^{-1}(x_i) = N_1$, $(\beta_-^i)^{-1}(x_i) = N_2$, and $p_2\theta(x_{i-1},t) < \beta_+^i(t) < p_2\theta(x_i,t)$ for each $t \in I^n \setminus C$.



Now $\theta^i : \mathbb{R} \times I^n \to \mathbb{R} \times I^n$ is defined to be the f.p. homeomorphism which is supported on $\{(x,t) \in \mathbb{R} \times I^n | 1/2(x_{i-1} + x_i) < x < p_2\theta(x_i,t),\ t \in I^n \setminus N_2\}$ and which slides the graph of β^i_- over to the graph of β^i_+; that is, $\theta^i(\beta^i_-(t),t) = (\beta^i_+(t),t)$ for each $t \in I^n$. Let $\bar{\theta}^i : B \times I^n \to B \times I^n$ denote the f.p. homeomorphism which extends $id_Z \times \theta^i$ via the identity. There is an obvious f.p. isotopy $\theta^i_s : id \simeq \theta^i$, $0 \le s \le 1$, and $id_Z \times \theta^i_s$ extends via the identity to $\bar{\theta}^i_s : id \simeq \bar{\theta}^i$, $0 \le s \le 1$.

According to Lemma 7.2 for each $i = 1,\dots,m-1$ there exists a f.p. homeomorphism $\tilde{\theta}^i : M \times I^n \to M \times I^n$ such that

i) $\tilde{\theta}^i | M \times C$ is the identity,

ii) $f^{-1}(\Gamma(\theta,x_{i-1})) \subset \tilde{\theta}^i f^{-1}(Z \times (-\infty,x_i] \times I^n)$,

iii) $\tilde{\theta}^i$ is supported on $f^{-1}(\Gamma(\theta,x_i)) \setminus f^{-1}(Z \times (-\infty,x_{i-1}] \times I^n)$,

iv) there is a f.p. homotopy $\tilde{\theta}^i_s : id \simeq \tilde{\theta}^i$, $0 \le s \le 1$, which is a $(p_1 f)^{-1}(\varepsilon/2m)$-homotopy over $Z \times \mathbb{R} \times I^n$ and which is supported on

$$[f^{-1}(\Gamma(\theta,x_i)) \setminus f^{-1}(Z \times (-\infty,x_{i-1}] \times I^n)] \setminus (M \times C).$$

Moreover, if F is a compact Q-manifold such that $F \times B$ is a Z-set in M and $f | F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then $\tilde{\theta}^i$ can be chosen so that $\tilde{\theta}^i | F \times B \times I^n = id_F \times \bar{\theta}^i$ and $\tilde{\theta}^i_s$ can be chosen so that $\tilde{\theta}^i_s | F \times B \times I^n = id_F \times \bar{\theta}^i_s$, $0 \le s \le 1$.

Consider the following compositions:

$$\bar{\theta} = \bar{\theta}^1 \circ \bar{\theta}^2 \circ \dots \circ \bar{\theta}^{m-1}, \qquad \bar{\theta}_s = \bar{\theta}^1_s \circ \bar{\theta}^2_s \circ \dots \circ \bar{\theta}^{m-1}_s,$$

$$\tilde{\tilde{\theta}} = \tilde{\theta}^1 \circ \tilde{\theta}^2 \circ \dots \circ \tilde{\theta}^{m-1}, \text{ and } \tilde{\tilde{\theta}}_s = \tilde{\theta}^1_s \circ \tilde{\theta}^2_s \circ \dots \circ \tilde{\theta}^{m-1}_s .$$

It follows from the construction that $\Gamma(\theta,x_{i-1}) \subset \bar{\theta}(Z \times (-\infty,x_i] \times I^n) \subset \Gamma(\theta,x_i)$

and $f^{-1}(\Gamma(\theta,x_{i-1})) \subset \tilde{\bar{\theta}} f^{-1}(Z \times (-\infty,x_i] \times I^n) \subset f^{-1}(\Gamma(\theta,x_i))$ for $i = 1,2,\ldots,m$. From this it follows that $\bar{\theta}$ is ε-close to θ and $f\tilde{\tilde{\theta}}$ is ε-close to $\theta' f$. Also $\tilde{\tilde{\theta}}_s : id \simeq \tilde{\tilde{\theta}}$, $0 \le s \le 1$, is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$. Since $\tilde{\tilde{\theta}}|F \times B \times I^n = id_F \times \bar{\theta}$, we must modify $\tilde{\tilde{\theta}}$ to get the required $\tilde{\theta}$.

Since $F \times B$ is a Z-set in M, there is a collar about $F \times B$ in M. Thus, we can consider $F \times B \times [0.2)$ as an open subset of M with $F \times B$ and $F \times B \times \{0\}$ identified. Let $\psi : M \to M$ be an embedding which is supported on $F \times B \times [0,1.5]$ and just pushes M in along the collar so that $\psi(f,b,0) = (f,b,1)$ for $(f,b) \in F \times B$. There is a f.p. isotopy $H_s : \theta \simeq \theta^1 \circ \theta^2 \circ \cdots \circ \theta^{m-1}$, $0 \le s \le 1$, which is supported on $[-1,1] \times (I^n \setminus C)$. If the partition $-1 = x_0 < x_1 < \cdots < x_m = 1$ is fine, then H_s, $0 \le s \le 1$, is a small isotopy. Let $\bar{H}_s : B \times I^n \to B \times I^n$, $0 \le s \le 1$, denote the f.p. isotopy which extends $id_Z \times H_s$ via the identity. Define the f.p. homeomorphism $\tilde{\theta} : M \times I^n \to M \times I^n$ as follows. First, let $\tilde{\theta}|[M \setminus (F \times B \times [0,1))] \times I^n = (\psi \times id_{I^n}) \circ \tilde{\tilde{\theta}} \circ (\psi^{-1} \times id_{I^n})$. Then, for $(f,b,u,t) \in F \times B \times [0,1] \times I^n$, let $\tilde{\theta}(f,b,u,t) = (f,p_B\bar{H}_u(b,t),u,t)$ where p_B denotes projection onto B. By making the collar on $F \times B$ short in M, it can be seen that $\tilde{\theta}$ is close to $\tilde{\tilde{\theta}}$ and satisfies the conclusions of the theorem. The appropriate f.p. isotopy $\tilde{\theta}_s : id \simeq \tilde{\theta}$, $0 \le s \le 1$, comes by first using the isotopy $\tilde{\tilde{\theta}}_s : id \simeq \tilde{\tilde{\theta}}$, $0 \le s \le 1$, and then using the collar coordinate and the definition of $\tilde{\theta}$ to get an isotopy from $\tilde{\tilde{\theta}}$ to $\tilde{\theta}$. □

We are now ready to begin a generalization of Theorem 7.3. This generalization is Theorem 7.8 below and is the main result of this section. Four lemmas are needed for the proof of Theorem 7.8. The first two, Lemmas 7.4 and 7.5, are similar to Lemma 7.1 and their proofs are omitted. Next, two lemmas similar to Lemma 7.2 are stated. Of these two, only the proof of Lemma 7.7 is presented. We now fix some notation which will be used for the rest of this section.

DATA FOR THE REMAINDER OF SECTION 7. Let B and Z denote ANRs where $Z \times \mathbb{R}$ is an open subset of B. Let $\phi : Z \to [0,+\infty)$ be a proper map, and for each $r \in [0,+\infty)$ let $Z_r = \phi^{-1}([0,r])$ and $Z^r = \phi^{-1}([r,+\infty))$. Let p_1 denote projection onto Z and p_2 projection onto $\mathbb{R}$. Let an integer $n \geq 0$ be given. For any map $\alpha : Z \times I^n \to \mathbb{R}$ define $\Gamma(\alpha) = \{(z,x,t) \in Z \times \mathbb{R} \times I^n | x \leq \alpha(x,t)\}$, and for each $r \in [0,+\infty)$ define $\Gamma_r(\alpha) = \Gamma(\alpha) \cap (Z_r \times \mathbb{R} \times I^n)$ and $\Gamma^r(\alpha) = \Gamma(\alpha) \cap (Z^r \times \mathbb{R} \times I^n)$.

DATA FOR LEMMA 7.4. Let $\alpha_+ : Z \times I^n \to [0,1]$, $\alpha_- : Z \times I^n \to [-1,0]$, and $\rho : Z \times I^n \to [-1,0]$ be maps satisfying the following conditions:

i) $Z^1 \times I^n \subset \alpha_+^{-1}(0) \cap \alpha_-^{-1}(0)$;

ii) $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0) \subset (Z^1 \times I^n) \cup (Z \times \partial I^n)$;

iii) $\rho = 0$ on a neighborhood of $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0)$;

iv) $\alpha_-(z,t) < \rho(z,t)$ for each $(z,t) \in (Z \times I^n) \setminus \alpha_-^{-1}(0)$.

Let Y be a compact subset of $\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \alpha_-(z,t) \leq x < \alpha_+(z,t), \phi(z) > 5/12$, and $\alpha_-(z,t) \neq 0 \neq \alpha_+(z,t)\}$.

LEMMA 7.4. For every $\mu > 0$ there exists a $\nu > 0$ such that if M is a Q-manifold and $f : M \times [0,1] \times I^n \to B \times I^n$ is a f.p. map which is a sliced ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n$, then there is a f.p. homeomorphism $u : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that

i) $f^{-1}(Y) \cap (M \times \{0\} \times I^n) \subset uf^{-1}(\Gamma^{5/12}(\rho))$,

ii) u is supported on $f^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n |$ $\alpha_-(z,t) < x < \alpha_+(z,t), \phi(z) > 5/12\}$,

iii) there is a f.p. homotopy $u_s : id \simeq u$, $0 \le s \le 1$, which is supported on $f^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \alpha_-(z,t) < x < \alpha_+(z,t)$, $\phi(z) > 5/12\}$ and which is a $(p_1 f)^{-1}(\mu)$-homotopy over $Z \times \mathbb{R} \times I^n$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times \{0\} \times I^n : F \times B \times \{0\} \times I^n \to B \times I^n$ is projection, then u and u_s can be chosen so that $u|F \times Z \times \mathbb{R} \times \{0\} \times I^n = id_F \times \hat{u}$ and $u_s|F \times Z \times \mathbb{R} \times \{0\} \times I^n = id_F \times \hat{u}_s$, $0 \le s \le 1$, where $\hat{u} : Z \times \mathbb{R} \times \{0\} \times I^n \to Z \times \mathbb{R} \times \{0\} \times I^n$ is a homeomorphism and $\hat{u}_s : Z \times \mathbb{R} \times \{0\} \times I^n \to Z \times \mathbb{R} \times \{0\} \times I^n$, $0 \le s \le 1$, is an isotopy so that $\hat{u}$ and $\hat{u}_s$ are f.p. over Z and I^n.

REMARKS ON PROOF. The proof is almost word-for-word like the proof of Lemma 7.1 and its Addendum. The only significant change is that extra care must be taken in defining maps and homotopies in order to allow for the extra degree of freedom in the Z-direction. □

DATA FOR LEMMA 7.5. Let $\alpha_+ : Z \times I^n \to [0,1]$, $\alpha_- : Z \times I^n \to [-1,0]$, and $\rho_1 : Z \times I^n \to [-1,0]$ be maps satisfying the following conditions:

i) $Z^1 \times I^n \subset \alpha_+^{-1}(0) \cap \alpha_-^{-1}(0)$;

ii) $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0) \subset (Z^1 \times I^n) \cup (Z \times \partial I^n)$;

iii) $\rho_1 = 0$ on a neighborhood of $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0)$;

iv) $\alpha_-(z,t) < \rho_1(z,t)$ for each $(z,t) \in (Z \times I^n) \setminus \alpha_-^{-1}(0)$.

Let $5/12 = r_4 < r_3 < r_2 < r_1 < r_0 = 1/2$ be a partition of the interval $[5/12,1/2]$. Let Y_1 be a compact subset of $\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \alpha_-(z,t) \le x < \alpha_+(z,t), \phi(z) \le r_1$, and $\alpha_-(z,t) \ne 0 \ne \alpha_+(z,t)\}$.

LEMMA 7.5. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if we are given maps $\omega_+ : Z \times I^n \to [0,1]$, $\omega_- : Z \times I^n \to [-1,0]$ and $\rho_2 : Z \times I^n \to [-1,0]$, and a compact subset Y_2 of $\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \omega_-(z,t) \le x < \omega_+(z,t), \phi(z) \ge r_1$, and $\omega_-(z,t) \ne 0 \ne \omega_+(z,t)\}$ satisfying the following conditions:

i) $Z^1 \times I^n \subset \omega_+^{-1}(0) \cap \omega_-^{-1}(0)$;

ii) $\omega_+^{-1}(0) \cup \omega_-^{-1}(0) \subset (Z^1 \times I^n) \cup (Z \times \partial I^n)$;

iii) $\rho_2 = 0$ on a neighborhood of $\omega_+^{-1}(0) \cup \omega_-^{-1}(0)$;

iv) $\omega_-(z,t) < \rho_2(z,t)$ for each $(z,t) \in (Z \times I^n) \setminus \omega_-^{-1}(0)$;

v) $\omega_+ = \alpha_+$ and $\omega_- = \alpha_-$ on $Z_{r_2} \times I^n$;

vi) $\rho_2 = \rho_1$ on $Z_{r_3} \times I^n$,

then for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

> if M is a Q-manifold and $f : M \times [0,1] \times I^n \to B \times I^n$ is a f.p. map which is a sliced δ-fibration over $Z_3 \times [-3,3] \times I^n$ and a sliced ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n$, then there is a f.p. homeomorphism $u : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that
>
> i) $f^{-1}(Y_1 \cup Y_2) \cap (M \times \{0\} \times I^n) \subset uf^{-1}(\Gamma(\rho_2))$,
>
> ii) u is supported on $f^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \omega_-(z,t) < x < \omega_+(z,t), \phi(z) \le 2\}$,

iii) there is a f.p. homotopy $u_s : \mathrm{id} \simeq u$, $0 \le s \le 1$, which is supported on $f^{-1}\{(z,x,t) \in Z \times \mathbb{R} \times I^n | \omega_-(z,t) < x < \omega_+(z,t), \phi(z) \le 2\}$ and which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ and a $(p_1 f)^{-1}(\mu)$-homotopy over $Z^{5/12} \times \mathbb{R} \times I^n$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times \{0\} \times I^n : F \times B \times \{0\} \times I^n \to B \times I^n$ is projection, then u and u_s can be chosen so that $u|F \times Z \times \mathbb{R} \times \{0\} \times I^n = \mathrm{id}_F \times \hat{u}$ and $u_s|F \times Z \times \mathbb{R} \times \{0\} \times I^n = \mathrm{id}_F \times \hat{u}_s$, $0 \le s \le 1$, where $\hat{u} : Z \times \mathbb{R} \times \{0\} \times I^n \to Z \times \mathbb{R} \times \{0\} \times I^n$ is a homeomorphism and $\hat{u}_s : Z \times \mathbb{R} \times \{0\} \times I^n \to Z \times \mathbb{R} \times \{0\} \times I^n$, $0 \le s \le 1$, is an isotopy so that $\hat{u}$ and $\hat{u}_s$ are f.p. over Z and I^n.

REMARKS ON PROOF. The proof of Lemma 7.5, just like that of Lemma 7.4, is almost word-for-word like the proof of Lemma 7.1 and its Addendum. The only significant change now is that Proposition 5.5 (see also Section 6) must be invoked so that we can assume that f has the sliced stationary (δ,ν)-lifting property over $(Z_3 \times [-3,3] \times I^n, (Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n)$. This is needed when constructing the homotopies that are used for sliced Z-set unknotting. □

DATA FOR LEMMA 7.6 AND LEMMA 7.7. Let C be a closed subset of ∂I^n and let $\alpha_1 : Z \times I^n \to [-1,1]$ and $\alpha_2 : Z \times I^n \to [0,1]$ denote maps such that $\alpha_1(z,t) < \alpha_2(z,t)$ for each $(z,t) \in Z \times I^n$ and $\alpha_1^{-1}(-1) = \alpha_2^{-1}(0) = (Z^1 \times I^n) \cup (Z \times C)$.

LEMMA 7.6. For every $\mu > 0$ there exists a $\nu > 0$ such that if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a f.p. map which is a sliced

ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n$, then there is a f.p. homeomorphism $h : M \times I^n \to M \times I^n$ such that

i) $h|M \times C$ is the identity,

ii) $f^{-1}(\Gamma^{1/2}(\alpha_1)) \subset hf^{-1}(Z \times (-\infty,0] \times I^n)$,

iii) there is a f.p. homotopy $h_s : id \simeq h$, $0 \leq s \leq 1$, which is a $(p_1 f)^{-1}(\mu)$-homotopy over $Z \times \mathbb{R} \times I^n$,

iv) h_s is supported on $f^{-1}(\Gamma_2(\alpha_2)) \setminus [f^{-1}(Z \times (-\infty,-.9] \times I^n) \cup f^{-1}(Z_{5/12} \times \mathbb{R} \times I^n) \cup (M \times C)]$ for each $0 \leq s \leq 1$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then h and h_s can be chosen so that $h|F \times Z \times \mathbb{R} \times I^n = id_F \times \hat{h}$ and $h_s|F \times Z \times \mathbb{R} \times I^n = id_F \times \hat{h}_s$, $0 \leq s \leq 1$, where $\hat{h} : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$ is a homeomorphism and $\hat{h}_s : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$, $0 \leq s \leq 1$, is an isotopy so that $\hat{h}$ and $\hat{h}_s$ are f.p. over Z and I^n.

REMARKS ON PROOF. Lemma 7.6 follows from Lemma 7.4 in almost the exact way that Lemma 7.2 follows from Lemma 7.1. □

LEMMA 7.7. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if we are given $-.5 < \omega < 0$ and a map $\alpha_3 : Z \times I^n \to [-1,1]$ such that $\alpha_1(z,t) \leq \alpha_3(z,t) < \alpha_2(z,t)$ for each $(z,t) \in Z \times I^n$ and $\alpha_3^{-1}(\omega) = (Z^1 \times I^n) \cup (Z \times C)$, then for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a f.p. map which is a sliced δ-fibration over $Z_3 \times [-3,3] \times I^n$ and a sliced ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n$, then there is a f.p. homeomorphism $h : M \times I^n \to M \times I^n$ such that

i) $h|M \times C$ is the identity,

ii) $f^{-1}(\Gamma(\alpha_1) \cup \Gamma^{1/2}(\alpha_3)) \subset hf^{-1}(Z \times (-\infty,0] \times I^n)$,

iii) there is a f.p. homotopy $h_s : id \simeq h$, $0 \le s \le 1$, which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ and a $(p_1 f)^{-1}(\mu)$-homotopy over $Z^{5/12} \times \mathbb{R} \times I^n$,

iv) h_s is supported on $f^{-1}(\Gamma_2(\alpha_2)) \setminus [f^{-1}(Z \times (-\infty,-.9] \times I^n) \cup f^{-1}(Z^{5/12} \times (-\infty,\omega] \times I^n) \cup (M \times C)]$ for each $0 \le s \le 1$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then h and h_s can be chosen so that $h|F \times Z \times \mathbb{R} \times I^n = id_F \times \hat{h}$ and $h_s|F \times Z \times \mathbb{R} \times I^n = id_F \times \hat{h}_s$, $0 \le s \le 1$, where $\hat{h} : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$ is a homeomorphism and $\hat{h}_s : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$, $0 \le s \le 1$, is an isotopy so that $\hat{h}$ and $\hat{h}_s$ are f.p. over Z and I^n.

PROOF. The proof is very similar to the proof of Lemma 7.2. The choices for $\delta > 0$ and $\nu > 0$ are made using Lemma 7.5 so that the two engulfing moves (u and v) described below can be performed. Given a map $f : M \times I^n \to B \times I^n$ as in the hypothesis, choose a f.p. map $k : M \times [0,1] \times I^n \to M \times I^n$ close to projection such that $k|M \times [0,1] \times C$ is projection and $k|M \times [0,1] \times (I^n \setminus C) : M \times [0,1] \times (I^n \setminus C) \to M \times (I^n \setminus C)$ is a homeomorphism (see Remark 3.16 or [14, Theorem 4.6]).

Choose a partition $5/12 = r_5 < r_4 < r_3 < r_2 < r_1 < r_0 = 1/2$ of the interval $[5/12,1/2]$. Use Lemma 7.5 to produce a f.p. homeomorphism $u : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that $(fk)^{-1}(\Gamma(\alpha_1) \cup \Gamma^{r_1}(\alpha_3)) \cap (M \times \{0\} \times I^n) \subset u(fk)^{-1}(Z \times (-\infty,0) \times I^n)$ and u is supported on $(fk)^{-1}\{(z,x,t) | .5\omega \le x \le \alpha_2(z,t), r_2 \le \phi(z) \le 2, t \notin C \text{ or } -.5 \le x \le \alpha_2(z,t), \phi(z) \le r_2, t \notin C\}$.

Let $S_1 = (fk)^{-1}\{(z,x,t) | -.6 \le x \le \alpha_1(z,t), \phi(z) \le r_4$ or $.6\omega \le x \le \alpha_1(z,t), \phi(z) \ge r_4$ or $.6\omega \le x \le \alpha_3(z,t), \phi(z) \ge r_2\}$. Let $S_2 = (fk)^{-1}\{(z,x,t) | \max(0,\alpha_3(z,t)) < x < \alpha_2(z,t)$ or $\max(0,\alpha_1(z,t)) < x < \alpha_2(z,t), \phi(z) < r_2\}$. Use Lemma 7.5 again to produce a f.p. homeomorphism $v : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ such that $S_1 \cap (M \times \{1\} \times I^n) \subset v^{-1}(S_2)$ and v is supported on $(fk)^{-1}\{(z,x,t) | .7\omega \le x \le \alpha_2(z,t), \phi(z) \le 2, t \notin C$ or $-.7 \le x \le \alpha_2(z,t), \phi(z) \le r_3, t \notin C\}$.

Let $U = (fkv)^{-1}\{(z,x,t) | \max(0,\alpha_3(z,t)) < x$ or $\max(0,\alpha_1(z,t)) < x, \phi(z) < r_2\}$ and observe that if k is close enough to projection, then $S_1 \subset [\pi(U \cap (M \times \{1\} \times I^n)] \times [0,1]$ where $\pi : M \times [0,1] \times I^n \to M \times I^n$ is projection. Then $w : M \times [0,1] \times I^n \to M \times [0,1] \times I^n$ is defined to be a f.p. homeomorphism affecting only the [0,1]-coordinate of any point such that $w[S_1 \setminus u(fk)^{-1}(Z \times (-\infty,0) \times I^n)] \subset U$. The support of w is on $(fk)^{-1}\{(z,x,t) | .8\omega \le x \le \alpha_2(z,t), r_2 \le \phi(z) \le 2$ or $-.8 \le x \le \alpha_2(z,t), \phi(z) \le r_2\}$.

It is easily verified that $h = k \circ v \circ w \circ u \circ k^{-1}$ satisfies the conclusions of the lemma. As in the proof of Lemma 7.2, the homotopy of the identity to h comes from the construction. The modification of the proof above to treat the relative case when we are given a compact Q-manifold F as in the hypothesis is exactly like the proof of the Addendum to Lemma 7.2. □

We are now ready for the main result of this section. This key engulfing theorem is a refinement of Theorem 7.3.

DATA FOR THEOREM 7.8. Let C be a closed subset of ∂I^n and let $\tilde{\theta}_r : \mathbb{R} \times I^n \to \mathbb{R} \times I^n$, $0 \le r < +\infty$, be a f.p. isotopy with the following properties:

i) $\bar{\theta}_r$ is the identity for $r \geq 1$;

ii) $\bar{\theta}_r | \mathbb{R} \times C$ is the identity for $r \geq 0$;

iii) $x \leq p_2\bar{\theta}_r(x,t)$ for each $(r,x,t) \in [0,+\infty) \times \mathbb{R} \times I^n$;

iv) $\bar{\theta}_r$ is supported on $[-1,1] \times I^n$ for $r \geq 0$.

Define $\theta : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$ by $\theta(z,x,t) = (z,\theta_{\phi(z)}(x,t))$ for each $(z,x,t) \in Z \times \mathbb{R} \times I^n$. Then θ is a f.p. homeomorphism supported on $Z_1 \times [-1,1] \times (I^n \setminus C)$ which extends via the identity to a f.p. homeomorphism $\theta' : B \times I^n \to B \times I^n$.

For any $\bar{x} \in \mathbb{R}$ and $r \in [0,+\infty)$, define $\Gamma(\theta,\bar{x}) = \{(z,x,t) \in Z \times \mathbb{R} \times I^n | x \leq p_2\bar{\theta}_{\phi(z)}(x,t)\}$, $\Gamma^r(\theta,\bar{x}) = \Gamma(\theta,\bar{x}) \cap (Z^r \times \mathbb{R} \times I^n)$, and $\Gamma_r(\theta,\bar{x}) = \Gamma(\theta,\bar{x}) \cap (Z_r \times \mathbb{R} \times I^n)$.

THEOREM 7.8. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a f.p. map which is a sliced δ-fibration over $Z_3 \times [-3,3] \times I^n$ and a sliced ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n$, then there is a f.p. homeomorphism $\tilde{\theta} : M \times I^n \to M \times I^n$ such that

i) $\tilde{\theta} | M \times C$ is the identity,

ii) $f\tilde{\theta}$ is ε-close to $\theta' f$,

iii) $f\tilde{\theta}$ is μ-close to $\theta' f$ over $Z^{2/3} \times \mathbb{R} \times I^n$,

iv) $\tilde{\theta}$ is supported on $f^{-1}(Z_1 \times [-1,1] \times I^n)$,

v) there is a f.p. homotopy $\tilde{\theta}_s : id \simeq \tilde{\theta}$, $0 \leq s \leq 1$, which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ and a $(p_1 f)^{-1}(\mu)$-homotopy over $Z^{2/3} \times \mathbb{R} \times I^n$, and which is supported on $f^{-1}(Z_1 \times [-1,1] \times (I^n \setminus C))$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then $\tilde{\theta}$ can be chosen so that $\tilde{\theta}|F \times B \times I^n = id_F \times \theta'$ and the homotopy $\tilde{\theta}_s$, $0 \le s \le 1$, can be chosen so that $p_1\tilde{\theta}_s|F \times Z \times \mathbb{R} \times I^n = p_1$ for $0 \le s \le 1$.

PROOF. Given $\varepsilon > 0$, choose a fine partition $-1 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ of the interval $[-1,1]$. Then $\delta > 0$ is chosen according to Lemma 7.7 so that certain engulfing moves described below can be performed. Given $\mu > 0$, choose a fine refinement $-1 = y_0 < y_1 < y_2 < \cdots < y_\ell = 1$ of the partition $-1 = x_0 < x_1 < x_2 < \cdots < x_m = 1$. Then $\nu > 0$ is chosen according to Lemma 7.6 and Lemma 7.7 so that each of the ℓ-1 engulfing moves described below can be performed. Let $J = \{i = 0,1,\ldots,\ell | y_i = x_j \text{ for some } j = 0,1,\ldots,m\}$. For $i \in J$, let i_J denote the unique integer in $\{0,1,\ldots,m\}$ such that $y_i = x_{i_J}$. Let $J' = J \setminus \{0,\ell\}$.

For each $i = 1,\ldots,\ell-1$, let $\gamma^i : Z \times I^n \to [y_i,+\infty)$ be the map defined by $\gamma^i(z,t) = p_2\theta(z,y_i,t)$ for $\phi(z) \ge 1/2$, $\gamma^i(z,t) = p_2\theta(z,y_\alpha,t)$ for $\phi(z) \le 5/12$ where α is the greatest integer $\le i$ such that $\alpha \in J$, and γ^i is defined "linearly" on $\phi^{-1}([5/12,1/2]) \times I^n$. (These maps are given before ν is calculated.)

Now given $f : M \times I^n \to B \times I^n$ as in the hypothesis, we will define for each $i = 1,\ldots,\ell-1$ a f.p homeomorphism $\tilde{\theta}^i : M \times I^n \to M \times I^n$ and a f.p. homotopy $\tilde{\theta}^i_s : id \simeq \tilde{\theta}^i$, $0 \le s \le 1$. For $i \notin J$ use Lemma 7.6 to construct a f.p. homeomorphism $\tilde{\theta}^i : M \times I^n \to M \times I^n$ such that

i) $\tilde{\theta}^i|M \times C$ is the identity,

ii) $f^{-1}(\Gamma^{1/2}(\theta,y_{i-1})) \subset \tilde{\theta}^i f^{-1}(Z \times (-\infty,y_i] \times I^n)$,

iii) $\tilde{\theta}^i$ is supported on $f^{-1}(\Gamma_2(\gamma^i)) \setminus f^{-1}(Z \times (-\infty, y_{i-1}] \times I^n)$,

iv) there is a f.p. homotopy $\tilde{\theta}^i_s : \mathrm{id} \simeq \tilde{\theta}^i$, $0 \le s \le 1$, which is a $(p_1 f)^{-1}(\mu/2\ell)$-homotopy over $Z \times \mathbb{R} \times I^n$ and is supported on $f^{-1}(\Gamma_2(\gamma^i)) \setminus [f^{-1}(Z \times (-\infty, y_{i-1}] \times I^n) \cup (M \times C)]$.

For $i \in J'$ use Lemma 7.7 to construct a f.p. homeomorphism $\tilde{\theta}^i : M \times I^n \to M \times I^n$ such that

i) $\tilde{\theta}^i | M \times C$ is the identity,

ii) $f^{-1}(\Gamma(\gamma^{i-1})) \subset \tilde{\theta}^i f^{-1}(Z \times (-\infty, y_i] \times I^n)$,

iii) $\tilde{\theta}^i$ is supported on $f^{-1}(\Gamma_2(\theta, y_i)) \setminus [f^{-1}(Z \times (-\infty, x_{i_J - 1}] \times I^n) \cup f^{-1}(Z^{5/12} \times (-\infty, y_{i-1}] \times I^n)]$,

iv) there is a f.p. homotopy $\tilde{\theta}^i_s : \mathrm{id} \simeq \tilde{\theta}^i$, $0 \le s \le 1$, which is a $(p_1 f)^{-1}(\varepsilon/3m)$-homotopy over $Z \times \mathbb{R} \times I^n$ and a $(p_1 f)^{-1}(\mu/2\ell)$-homotopy over $Z^{5/12} \times \mathbb{R} \times I^n$, and which is supported on $f^{-1}(\Gamma_2(\theta, y_i)) \setminus [f^{-1}(Z \times (-\infty, x_{i_J - 1}] \times I^n) \cup f^{-1}(Z^{5/12} \times (-\infty, y_{i-1}] \times I^n) \cup (M \times C)]$.

Define $\tilde{\theta} = \tilde{\theta}^1 \circ \tilde{\theta}^2 \circ \cdots \circ \tilde{\theta}^{\ell-1}$ and $\tilde{\theta}_s = \tilde{\theta}^1_s \circ \tilde{\theta}^2_s \circ \cdots \circ \tilde{\theta}^{\ell-1}_s$, $0 \le s \le 1$. One can verify the following three facts:

i) $f^{-1}(\Gamma(\theta, x_{i_J - 1})) \subset \tilde{\theta} f^{-1}(Z \times (-\infty, y_i] \times I^n) \subset f^{-1}(\Gamma(\theta, y_i))$ for $i \in J$;

ii) $f^{-1}(\Gamma^{7/12}(\theta, y_{i-1})) \subset \tilde{\theta} f^{-1}(Z \times (-\infty, y_i] \times I^n)$ for $i = 1, 2, \ldots, \ell-1$;

iii) $\tilde{\theta} f^{-1}(Z^{2/3} \times (-\infty, y_i] \times I^n) \subset f^{-1}(\Gamma^{7/12}(\theta, y_i))$ for $i = 1, 2, \ldots, \ell-1$.

It follows that $\tilde{\theta}$ satisfies the conclusions of the theorem if the partitions $-1 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ and $-1 = y_0 < y_1 < y_2 < \cdots < y_\ell = 1$ are sufficiently fine. The treatment of the embellishment of the theorem when we are given a compact Q-manifold F as in the hypothesis is adequately illustrated in the proof of Theorem 7.2. □

It should be remarked here that the embellishment of Theorem 7.8 works equally well when we are given a sliced Z-embedding $g : F \times B \times I^n \to M \times I^n$ such that fg is projection.

Finally, we will need the following variation of Theorem 7.8 in Section 9 when we wrap up and encounter non-proper submersions. The proof of Theorem 7.9 is exactly like the proof of Theorem 7.8 only now we must rely on the Q-manifold results of Section 3. We adopt the data of Theorem 7.8.

THEOREM 7.9. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

> if M is a Q-manifold, $\pi : M \to I^n$ is a submersion with Q-manifold fibers, $f : M \to B \times I^n$ is a proper f.p. map (i.e., $\text{proj} \circ f = \pi$) such that $f_t : \pi^{-1}(t) \to B$ is a δ-fibration over $Z_3 \times [-3,3]$ and a ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3]$ for each t in I^n, and π has nice cross sections on $f^{-1}(Z_3 \times [-3,3] \times I^n)$, then there is a f.p. homeomorphism $\tilde{\theta} : M \to M$ such that
>
> i) $\tilde{\theta}|\pi^{-1}(C)$ is the identity,
>
> ii) $f\tilde{\theta}$ is ε-close to $\theta' f$,
>
> iii) $f\tilde{\theta}$ is μ-close to $\theta' f$ over $Z^{2/3} \times \mathbb{R} \times I^n$,

iv) $\tilde{\theta}$ is supported on $f^{-1}(Z_1 \times [-1,1] \times I^n)$,

v) there is a f.p. isotopy $\tilde{\theta}_s : id \simeq \tilde{\theta}$, $0 \le s \le 1$, which is a $(p_1 f)^{-1}(\varepsilon)$-homotopy over $Z \times \mathbb{R} \times I^n$ and a $(p_1 f)^{-1}(\mu)$-homotopy over $Z^{2/3} \times \mathbb{R} \times I^n$ and which is supported on $f^{-1}(Z_1 \times [-1,1] \times (I^n \setminus C))$.

Moreover, if we are additionally given a compact Q-manifold F and a sliced Z-embedding $g : F \times B \times I^n \to M$ such that πg is projection and fg is projection, then $\tilde{\theta}$ can be chosen so that $\tilde{\theta} g = id_F \times \theta'$ and the homotopy $\tilde{\theta}_s$, $0 \le s \le 1$, can be chosen so that $p_1 \tilde{\theta}_s g = p_1$ for $0 \le s \le 1$.

Perhaps the only thing that has not been mentioned above about the proof of Theorem 7.9 is the establishment of the corresponding results of Section 5 which are needed in this situation. However, the required results follow immediately from Proposition 3.2 and Theorem 5.3.

SECTION 8:
PARAMETERIZED WRAPPING

In this section we present the details of a parameterized version of Chapman's construction for wrapping up δ-fibrations around S^1. Before stating the main result of this section, Theorem 8.2, we first state the more palatable Proposition 8.1 which is actually a corollary of Theorem 8.2. For notation let B and Z denote ANRs where $Z \times \mathbb{R}$ is an open subset of B. Let $\phi : Z \to [0,+\infty)$ be a proper map and for r in $[0,+\infty)$ define $Z_r = \phi^{-1}([0,r])$ and $Z^r = \phi^{-1}([r,+\infty))$. Let $n \geq 0$ be an integer. The map p_1 denotes projection onto Z, p_2 projection onto $\mathbb{R}$, and p_3 projection onto I^n. Finally, let $e : \mathbb{R} \to S^1$ be the covering projection defined by $e(x) = \exp(\pi i x/4)$ (thus e has period 8). This notation will be used throughout this section.

PROPOSITION 8.1. Suppose Z is compact. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map which is a sliced δ-fibration over $Z \times [-3,3] \times I^n$, then there is a compact Q-manifold $\tilde{M}$, a f.p. map $f : \tilde{M} \times I^n \to Z \times S^1 \times I^n$ which is a sliced ε-fibration, and a f.p. open embedding $\psi : f^{-1}(Z \times (-1,1) \times I^n) \to \tilde{M} \times I^n$ for which the following diagram commutes:

$$\begin{array}{ccc}
\tilde{M} \times I^n & \xrightarrow{\tilde{f}} & Z \times S^1 \times I^n \\
\uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \mathrm{id} \times e \times \mathrm{id}} \\
f^{-1}(Z \times (-1,1) \times I^n) & \xrightarrow{f|} & Z \times (-1,1) \times I^n
\end{array}$$

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that there is a sliced Z-embedding $g : F \times Z \times S^1 \times I^n \to \tilde{M} \times I^n$ for which $\tilde{f}g : F \times Z \times S^1 \times I^n \to Z \times S^1 \times I^n$ is projection and for which the following diagram commutes:

$$\begin{array}{ccc} F \times Z \times S^1 \times I^n & \xrightarrow{\quad g \quad} & \tilde{M} \times I^n \\ \uparrow{\scriptstyle id \times e \times id} & & \uparrow{\scriptstyle \psi} \\ F \times Z \times (-1,1) \times I^n & \subset & f^{-1}(Z \times (-1,1) \times I^n) \end{array}$$

We now state the main result. Notice that Z is no longer required to be compact.

THEOREM 8.2. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map which is a sliced δ-fibration over $Z_3 \times [-3,3] \times I^n$ and a sliced ν-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3,3] \times I^n$, then there is a Q-manifold $\tilde{M}$, a submersion $\pi : \tilde{M} \to I^n$ with Q-manifold fibers, a f.p. map $\tilde{f} : \tilde{M} \to \mathring{Z}_{2.5} \times S^1 \times I^n$ such that $\tilde{f}_t : \pi^{-1}(t) \to \mathring{Z}_{2.5} \times S^1$ is an ε-fibration over $Z_2 \times S^1$ and a μ-fibration over $(Z_2 \setminus \mathring{Z}_{2/3}) \times S^1$ for each t in I^n, and a f.p. open embedding $\psi : f^{-1}(\mathring{Z}_1 \times (-1,1) \times I^n) \to \tilde{M}$ for which the following diagram commutes:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\quad \tilde{f} \quad} & \mathring{Z}_{2.5} \times S^1 \times I^n \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle id \times e \times id} \\ f^{-1}(\mathring{Z}_1 \times (-1,1) \times I^n) & \xrightarrow{\quad f| \quad} & \mathring{Z}_1 \times (-1,1) \times I^n \end{array}$$

Also, the submersion $\pi : \tilde{M} \to I^n$ has nice cross sections on $\tilde{f}^{-1}(Z_{2.1} \times S^1 \times I^n)$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that there is a sliced Z-embedding $g : F \times \mathring{Z}_{2.5} \times S^1 \times I^n \to \tilde{M}$ for which $\tilde{f}g : F \times \mathring{Z}_{2.5} \times S^1 \times I^n \to \mathring{Z}_{2.5} \times S^1 \times I^n$ is projection and for which the following diagram commutes:

$$\begin{array}{ccc} F \times \mathring{Z}_{2.5} \times S^1 \times I^n & \xrightarrow{\quad g \quad} & \tilde{M} \\ \uparrow \scriptstyle{id \times e \times id} & & \uparrow \scriptstyle{\psi} \\ F \times \mathring{Z}_1 \times (-1,1) \times I^n & \subset & f^{-1}(\mathring{Z}_1 \times (-1,1) \times I^n) \end{array}$$

REMARKS ON THE PROOF OF PROPOSITION 8.1. A direct proof may be given along the lines of the proof of Theorem 8.2 below. The procedure is simplified because Z is compact. Alternatively, one may simply notice that Proposition 8.1 follows from Theorem 8.2. The only additional information needed is that a proper submersion with Q-manifold fibers is a bundle projection (see Section 3). □

PROOF OF THEOREM 8.2. Let $\bar{\theta}_r : \mathbb{R} \times I^n \to \mathbb{R} \times I^n$, $0 \le r < +\infty$, be the f.p. isotopy such that for $0 \le r \le 2.7$ $\bar{\theta}_r$ is the f.p. PL homeomorphism supported on $[-2.4,2.4] \times I^n$ with the property that $\bar{\theta}_r(x,t) = (x+4,t)$ for $-2.2 \le x \le -1.8$ and $t \in I^n$. For $2.7 \le r \le 2.8$, $\bar{\theta}_r$ is phased out to the identity so that $\bar{\theta}_r = id$ for $r \ge 2.8$. Define $\theta : Z \times \mathbb{R} \times I^n \to Z \times \mathbb{R} \times I^n$ by $\theta(z,x,t) = (z,\bar{\theta}_{\phi(z)}(x,t))$. By engulfing (Theorem 7.8) there is a f.p. homotopy $h_s : id \simeq h_1$, $0 \le s \le 1$, on $M \times I^n$ where

$h_1 : M \times I^n \to M \times I^n$ is a f.p. homeomorphism such that fh_1 is δ'-close to θf over $Z \times \mathbb{R} \times I^n$, and fh_1 is ν'-close to θf over $Z^{1/2} \times \mathbb{R} \times I^n$, and the homotopy is supported on $f^{-1}(Z_3 \times [-3,3] \times I^n)$ (Theorem 7.8 also gives some control on the size of the homotopy which we will need.) Here, δ' and ν' are small if δ and ν are small, respectively. Moreover, if we are given a compact Q-manifold F as in the hypothesis, then we may assume that $h_1|F \times Z \times \mathbb{R} \times I^n = id_F \times \theta$ and $p_1 h_s|F \times Z \times \mathbb{R} \times I^n = p_1$ for $0 \leq s \leq 1$.

Let $Y = h_1 f^{-1}(Z_{2.7} \times (-\infty,-2] \times I^n) \setminus f^{-1}(Z \times (-\infty,-2) \times I^n)$, $E_- = Y \cap f^{-1}(Z \times \{-2\} \times I^n)$, and $E_+ = h_1 f^{-1}(Z_{2.7} \times \{-2\} \times I^n)$. Let $\sim$ be the equivalence relation on Y generated by the rule: if y is in $Y \cap f^{-1}(Z_{2.7} \times \{-2\} \times I^n)$, then $y \sim h_1(y)$. Let $\bar{M} = Y/\sim$ and let $q : Y \to \bar{M}$ denote the quotient map.

ASSERTION 1. The relation $\sim$ induces an upper semi-continuous decomposition of Y.

PROOF. The nondegenerate elements are of the form $\{y,h_1(y)\}$ where y is in $Y \cap f^{-1}(Z_{2.7} \times \{-2\} \times I^n)$. Thus, the union of the nondegenerate elements is $(Y \cap f^{-1}(Z_{2.7} \times \{-2\} \times I^n)) \cup h_1(Y \cap f^{-1}(Z_{2.7} \times \{-2\} \times I^n))$ which is closed in Y. Therefore, it suffices to show that the elements $\{y,h_1(y)\}$ have arbitrarily small open saturated neighborhoods. Let U be a small open neighborhood of y in $M \times I^n$ such that $U \cap E_+ = \emptyset = h_1(U) \cap f^{-1}(Z \times \{-2\} \times I^n)$. Set $W = (U \cap Y) \cup (h_1(U) \cap Y)$. Then W is the required small open saturated neighborhood of $\{y,h_1(y)\}$ in Y.

ASSERTION 2. There exists a map $\alpha : Y \to Z$ such that

i) $\alpha(y) = \alpha(y')$ if $y \sim y'$,

ii) $\alpha|[f^{-1}(Z \times [-2,1.99] \times I^n) \cap Y] = p_1 f|$,

iii) α is δ'-close to $p_1 f|Y$,

iv) α is ν'-close to $p_1 f|Y$ over Z^{r_1} where r_1 is fixed so that $1/2 < r_1 < 2/3$.

PROOF. Define a homotopy $g_s : [f^{-1}(Z \times [-2,1.99] \times I^n) \cap Y] \cup E_+ \to Z$, $0 \le s \le 1$, by $g_s|[f^{-1}(Z \times [-2,1.99] \times I^n) \cap Y] = p_1 f|$ and $g_s|E_+ = p_1 f h_{1-s} h_1^{-1}|E_+$. Note that g_0 extends to $p_1 f| : Y \to Z$. By the homotopy extension property there is an extension $\tilde{g}_s : Y \to Z$ of g_s such that $\tilde{g}_0 = p_1 f|$. Using the estimated homotopy extension property and the control on the homotopy h_s, we may assume that the homotopy $\tilde{g}_s$ is controlled in the $p_1 f$ direction. Then define $\alpha = \tilde{g}_1$.

ASSERTION 3. If we are given the compact Q-manifold F as in the hypothesis, then the map α of Assertion 2 can be chosen so that $\alpha|(F \times Z \times \mathbb{R} \times I^n) \cap Y = p_1 f| = p_1$.

PROOF. In the proof of Assertion 2 extend g_s to $(F \times Z \times \mathbb{R} \times I^n) \cap Y$ by setting $g_s = p_1 f| = p_1$ on this set. This is well-defined because $p_1 f h_{1-s} h_1^{-1}|F \times Z \times \mathbb{R} \times I^n = p_1$.

ASSERTION 4. There is a map $\beta : Y \to [-2,2]$ such that

i) $\beta(E_-) = -2$,

ii) $\beta(E_+) = +2$,

iii) $\beta|[f^{-1}(Z \times [-2,1.99] \times I^n) \cap Y] = p_2 f|$,

iv) β is δ'-close to $p_2 f|$,

v) β is ν'-close to $p_2 f|$ on $f^{-1}(Z^{r_1} \times \mathbb{R} \times I^n) \cap Y$,

vi) if F is a compact Q-manifold given as in the hypothesis, then $\beta|(F \times Z \times \mathbb{R} \times I^n) \cap Y = p_2f| = p_2$.

PROOF. Define a homotopy $g_s : ([f^{-1}(Z \times [-2,1.99] \times I^n) \cup (F \times Z \times \mathbb{R} \times I^n)] \cap Y) \cup E_+ \to \mathbb{R}$, $0 \leq s \leq 1$, as follows: first $g_s|[f^{-1}(Z \times [-2,1.99] \times I^n) \cup (F \times Z \times \mathbb{R} \times I^n)] \cap Y = p_2f|$. On E_+ define g_s so that $g_0|E_+ = p_2f|$ and as s goes from 0 to 1, g_s shrinks $p_2f(E_+)$ to +2 so that $g_1(E_+) = +2$. Note that this can be done so that it does not conflict with the definition of $g_s|(F \times Z \times \mathbb{R} \times I^n) \cap Y$. Now g_0 extends to $p_2f| : Y \to \mathbb{R}$ and so we may use the estimated homotopy extension property to extend g_s to $\tilde{g}_s : Y \to \mathbb{R}$. Let $r : \mathbb{R} \to [-2,2]$ be the retraction such that $r((-\infty,-2]) = -2$ and $r([+2,+\infty)) = +2$. Then define $\beta = r\tilde{g}_1$. This completes the proof of Assertion 4.

Identify S^1 with the quotient space $[-2,2]/\{-2,2\}$ and let $u : [-2,2] \to S^1$ be the quotient map. Do this in such a way that $u|[-1,1] = e|[-1,1]$.

Define $\bar{f} : \bar{M} \to Z \times S^1 \times I^n$ by $\bar{f}(q(y)) = (\alpha(y), u\beta(y), p_3(y))$ for y in Y. This map is well-defined. Let $\tilde{M} = \bar{f}^{-1}(\mathring{Z}_{2.5} \times S^1 \times I^n)$ and let $\tilde{f} : \tilde{M} \to \mathring{Z}_{2.5} \times S^1 \times I^n$ denote the restriction of $\bar{f}$ to $\tilde{M}$. Define $\pi : \tilde{M} \to I^n$ by $\pi(q(y)) = p_3(y)$ for y in $q^{-1}(\tilde{M}) \subset Y$. The remainder of the proof consists of showing that $\tilde{f}$ and π satisfy the conclusions of the theorem.

ASSERTION 5. $\tilde{f}$ is a proper map.

PROOF. Let $C \subset \mathring{Z}_{2.5} \times S^1 \times I^n$ be a compactum. Then $\tilde{f}^{-1}(C) = \bar{f}^{-1}(C)$. Let $K \subset \mathring{Z}_{2.5}$ be a compactum such that $C \subset K \times S^1 \times I^n$. It suffices to show that $q^{-1}\bar{f}^{-1}(C) \subset f^{-1}(Z_6 \times [-3,3] \times I^n)$. To this end let $y \in Y$ such that $\bar{f}(q(y)) \in C$. We need only show that $p_1 f(y) \in Z_6$. Since $\alpha(y) \in \mathring{Z}_{2.5}$ and $p_1 f(y)$ is close to $\alpha(y)$, the result follows.

ASSERTION 6. $\pi : \tilde{M} \to I^n$ is a submersion.

PROOF. First let $y \in E_-$ such that $q(y) \in \tilde{M}$. Thus $\bar{f}q(y) \in \mathring{Z}_{2.5} \times S^1 \times I^n$ and from this we may conclude that $y \in f^{-1}(\mathring{Z}_{2.5} \times \{-2\} \times I^n)$. Let $U = h_1 f^{-1}(\mathring{Z}_{2.6} \times (-\infty,-1.8) \times I^n) \setminus f^{-1}(Z \times (-\infty,1.8] \times I^n)$. Define $q' : U \to \bar{M}$ by $q' = q$ on $U \cap Y$ and $q' = qh_1^{-1}$ on $U \setminus Y$. Note that q' is an open embedding and $q(y) \in q'(U)$. Let $\tilde{U} = U \cap (q')^{-1}(\tilde{M})$. Then $\tilde{U}$ is an open subset of $M \times I^n$ and $q'|\tilde{U} : \tilde{U} \to \tilde{M}$ is an open embedding onto a neighborhood of $q(y)$ such that $\pi q'|\tilde{U} = p_3$. It follows that there are product charts about $q(y)$ for π.

Next let $y \in Y$ such that $q(y) \in \tilde{M} \setminus q(E_-)$. Since $q(y) \in \tilde{M}$, we have $y \in f^{-1}(\mathring{Z}_{2.55} \times [-3,3] \times I^n)$. Since $q(y)$ is not in $q(E_-)$, we have $y \in V = h_1 f^{-1}(\mathring{Z}_{2.6} \times (-\infty,-2) \times I^n) \setminus f^{-1}(Z \times (-\infty,-2] \times I^n)$. And $q|V : V \to \bar{M}$ is an open embedding. By setting $\tilde{V} = V \cap q^{-1}(\tilde{M})$ we get product charts about $q(y)$ for π by using $q|\tilde{V} : \tilde{V} \to \tilde{M}$. This completes the proof of Assertion 6.

Notice that the proof of Assertion 6 shows that $\tilde{M}$, as well as $\pi^{-1}(t)$ for each t in I^n, is a Q-manifold.

ASSERTION 7. The submersion $\pi : \tilde{M} \to I^n$ has nice cross sections on $\tilde{f}^{-1}(Z_{2.1} \times S^1 \times I^n)$.

PROOF. Let $x \in \tilde{f}^{-1}(Z_{2.1} \times S^1 \times I^n)$. Suppose $x = q(y)$ where $f(y) \in \mathring{Z}_{2.2} \times \{-2\} \times I^n$. Let $y \in G \subset f^{-1}(\mathring{Z}_{2.2} \times (-2.1,-1.9) \times \{p_3(y)\})$ where G is open such that $G \times I^n \subset f^{-1}(\mathring{Z}_{2.2} \times (-2.1,-1.9) \times I^n)$. Adopting the notation of Assertion 6, we see that $q'h_1(G \times I^n) \subset \tilde{M}$ provides the nice cross sections about x. The case when $y \in h_1 f^{-1}(\mathring{Z}_{2.2} \times (-\infty,-2) \times I^n) \setminus f^{-1}(Z \times (-\infty,-2] \times I^n)$ is handled similarly.

ASSERTION 8. $\tilde{f}_t : \pi^{-1}(t) \to \mathring{Z}_{2.5} \times S^1$ is an ε-fibration over $Z_2 \times S^1$ for each t in I^n.

PROOF. Extend the previously defined quotient map $u : [-2,2] \to S^1$ to a covering projection $u : \mathbb{R} \to S^1$ of period 4. It will be helpful to keep the following commutative diagram in mind:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\ \tilde{f}\ } & \mathring{Z}_{2.5} \times S^1 \times I^n \\
\cap & & \cap \\
\bar{M} & \xrightarrow{\ \bar{f}\ } & Z \times S^1 \times I^n \\
{\scriptstyle q}\uparrow & & \uparrow{\scriptstyle \mathrm{id} \times u \times \mathrm{id}} \\
Y & \xrightarrow{\ \alpha \times \beta \times p_3\ } & Z \times \mathbb{R} \times I^n
\end{array}$$

Choose ε' so that any map to $\mathring{Z}_{2.5} \times S^1$ which is an ε'-fibration over both $Z_{2.1} \times u([1.9,2.1])$ and $Z_{2.1} \times u([-1.95,1.95])$ is an $(\varepsilon/2)$-fibration over $Z_2 \times S^1$.

Note that if $(\mathrm{id} \times u)f_t | f_t^{-1}(Z \times \mathbb{R}) : f_t^{-1}(Z \times \mathbb{R}) \to Z \times S^1$ is an ε'-fibration over $Z_{2.1} \times u[-1.95,1.95]$, then so is $\tilde{f}_t$. This is because of the following commutative diagram:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\ \tilde{f}\ } & \mathring{Z}_{2.5} \times S^1 \times I^n \\
{\scriptstyle q|}\uparrow & & \uparrow{\scriptstyle \mathrm{id} \times u \times \mathrm{id}} \\
f^{-1}(\mathring{Z}_{2.5} \times [-2,1.99] \times I^n) & \xrightarrow{\ f|\ } & \mathring{Z}_{2.5} \times [-2,1.99] \times I^n
\end{array}$$

And of course, f_t will have this property if δ is small enough.

We are only left with the problem of showing that $\tilde{f}_t$ is an ε'-fibration over $Z_{2.1} \times u([1.9,2.1])$. To this end define $U = h_1 f^{-1}(\mathring{Z}_{2.4} \times (-\infty,-1.8) \times I^n) \setminus f^{-1}(Z \times (-\infty,1.8] \times I^n)$. Define a map $g : U \to Z \times (1.8,2.2) \times I^n$ componentwise as follows:

$$p_1 g = \begin{cases} \alpha & \text{on } U \cap Y \\ \alpha \circ h_1^{-1} & \text{on } U \setminus Y , \end{cases}$$

$$p_2 g = \begin{cases} \beta & \text{on } U \cap Y \\ p_2\theta(p_1 f \times \beta \times p_3 f)h_1^{-1} & \text{on } U \setminus Y , \end{cases}$$

$$p_3 g = p_3 \ .$$

Define $\tilde{q} : U \to \tilde{M}$ so that $\tilde{q} = q$ on $U \cap Y$ and $\tilde{q} = qh_1^{-1}$ on $U \setminus Y$ and consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \mathring{Z}_{2.5} \times S^1 \times I^n \\ \uparrow{\scriptstyle \tilde{q}} & & \uparrow{\scriptstyle \mathrm{id} \times u \times \mathrm{id}} \\ U & \xrightarrow{g} & \mathring{Z}_{2.5} \times (1.8,2.2) \times I^n \end{array}$$

Note that $\tilde{q}$ is an open embedding. Thus we will be done if we show that g_t is an ε'-fibration over $Z_{2.1} \times [1.9,2.1]$. This is accomplished by observing that g is as close to $f|U$ as we need by choosing δ small. This completes the proof of Assertion 8.

ASSERTION 9. $\tilde{f}_t : \pi^{-1}(t) \to \mathring{Z}_{2.5} \times S^1$ is a μ-fibration over $(Z_2 \setminus \mathring{Z}_{2/3}) \times S^1$ for each t in I^n.

The proof of Assertion 9 goes almost word-for-word like the proof of Assertion 8 and we omit the details.

To complete the proof of Theorem 8.2, it only remains to consider the embellishment when we are given the compact Q-manifold F as in the hypothesis.

ASSERTION 10. There is a sliced Z-embedding $g : F \times \mathring{Z}_{2.5} \times S^1 \times I^n \to \tilde{M}$ for which $\tilde{f}g$ = projection and for which the following diagram commutes:

$$\begin{array}{ccc} F \times \mathring{Z}_{2.5} \times S^1 \times I^n & \xrightarrow{\quad g \quad} & \tilde{M} \\ \Big\uparrow{\scriptstyle id \times e \times id} & & \Big\uparrow{\scriptstyle \psi \,=\, \text{inclusion}} \\ F \times \mathring{Z}_1 \times (-1,1) \times I^n & \subset & f^{-1}(\mathring{Z}_1 \times (-1,1) \times I^n) \end{array}$$

PROOF. Note that $F \times Z_{2.7} \times (-\infty,2] \times I^n \subset f^{-1}(Z_{2.7} \times (-\infty,2] \times I^n)$ and that $h_1 | F \times Z \times \mathbb{R} \times I^n = id_F \times \theta$. Therefore, $Y \cap (F \times Z \times \mathbb{R} \times I^n) = F \times Z_{2.7} \times [-2,2] \times I^n$ and $q | F \times Z_{2.7} \times [-2,2] \times I^n = id \times u \times id$. Then g is defined so that the following diagram commutes:

$$\begin{array}{ccc} F \times \mathring{Z}_{2.5} \times S^1 \times I^n & \xrightarrow{\quad g \quad} & \tilde{M} \\ \Big\uparrow{\scriptstyle id \times u \times id} & & \cap \\ F \times \mathring{Z}_{2.5} \times [-2,2] \times I^n & \xrightarrow{\quad q| \quad} & \bar{M} \end{array}$$

To show that g is a sliced Z-embedding we must make some adjustments. Specifically, we must use a collar on $F \times B$ in M to modify h_1 (and h_s) so that it behaves like h_1 (or h_s) does on $F \times B \times I^n$. With this our construction yields a fibered collar on $F \times \mathring{Z}_{2.5} \times S^1 \times I^n$ in $\tilde{M}$. This completes the proof of Assertion 10 and hence the proof of Theorem 8.2. □

SECTION 9:
HANDLE LEMMAS

In this section we establish three handle lemmas needed for the results of Section 10. For notation B and X will denote ANRs where X is compact. Let $n \geq 0$ be an integer and let C be a closed subset of ∂I^n which is collared in I^n. Thus, we consider $C \times [0,1)$ as an open subset of I^n with C identified with $C \times \{0\}$ (the possibility that C is empty is not ruled out). This notation will be used throughout this section.

PROPOSITION 9.1. Suppose $\mathring{c}(X)$ is an open subset of B. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

> if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map such that f is a sliced δ-fibration over $c_3(X) \times I^n$ and a sliced ν-fibration over $[c_3(X) \setminus \mathring{c}_{1/3}(X)] \times I^n$, and $f_t : M \to B$ is an approximate fibration for each t in C, then there is a proper f.p. map $\tilde{f} : M \times I^n \to B \times I^n$ which is a sliced μ-fibration over $c_1(X) \times I^n$ and is f.p. ε-homotopic to f rel $[(M \times I^n) \setminus f^{-1}(\mathring{c}_{2/3}(X) \times I^n)] \cup [M \times C]$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we may additionally conclude that $\tilde{f}|F \times B \times I^n$ is projection and that the homotopy from f to $\tilde{f}$ is rel $F \times B \times I^n$.

PROOF. To simplify matters we will only show that f is ε-close to such an $\tilde{f}$. Further epsilonics would produce the homotopy. Using Theorem 5.3 we are only required to show that each $\tilde{f}_t$ is a μ-fibration over $c_2(X)$.

Let $\varepsilon > 0$ be given and choose $r_0 > 0$ so small that the diameter of $c_{r_0}(X)$ is less than $\varepsilon/2$. Let $\theta : [0,+\infty) \times I^n \to [0,+\infty) \times I^n$ be a f.p. homeomorphism such that

i) θ is supported on $[r_0/2, 7/12] \times I^n$,

ii) for each t in $I^n \setminus (C \times [0,1/2))$, θ_t is a PL homeomorphism which takes $3r_0/4$ to $1/2$,

iii) for t in $C \times [0,1/2]$, θ_t is linearly phased out to the identity so that $\theta | [0,+\infty) \times C = \text{id}$.

Then θ induces a f.p. homeomorphism on $\mathring{c}(X) \times I^n$ which extends via the identity to a f.p. homeomorphism $\bar{\theta} : B \times I^n \to B \times I^n$.

Choose r_1 such that $r_0 < r_1 < 1/2$ and the diameter of $c_{r_1}(X)$ is less than $\varepsilon/2$. Choose $\varepsilon_1 > 0$ so that the ε_1-neighborhood of $c_{r_0}(X)$ in B is contained in $c_{r_1}(X)$. Let $\delta_1 = \delta(\varepsilon_1)$ be given by Theorem 4.8 so that if we start with a map $g : E \to B$ which is a δ_1-fibration over $c_3(X)$ and a ν'-fibration over $c_{2.25}(X) \setminus \mathring{c}_{3r_0/4}(X)$, then g is an (ε_1, μ')-fibration over $(c_{2.5}(X), c_2(X) \setminus \mathring{c}_{r_0}(X))$.

Choose $\delta_2 > 0$ so that if $g : E \to B$ is a δ_2-fibration over $c_3(X)$, then $\bar{\theta}_t^{-1} g$ is a δ_1-fibration over $c_3(X)$ for each t in I^n.

Choose $\delta_3 > 0$ by the engulfing result Theorem 7.3 so that if $g : M \times I^n \to B \times I^n$ is a sliced δ_3-fibration over $c_3(X) \times I^n$, then $\bar{\theta}$ can be $(\varepsilon/2)$-covered by a f.p. homeomorphism of $M \times I^n$ (this is described in more detail below).

Let $\delta = \min\{\delta_2, \delta_3\}$ and let $\mu > 0$ be given. Choose s_1 so that $0 < s_1 < r_1$ and the diameter of $c_{s_1}(X)$ is less than μ. Define a f.p. homeomorphism $\gamma : [0,+\infty) \times I^n \to [0,+\infty) \times I^n$ such that

i) γ is supported on $[0, 2/3] \times I^n$,

ii) for t in $I^n \setminus (C \times [0,1/2))$, γ_t is a PL homeomorphism which takes r_1 to s_1,

iii) for t in $C \times [0,1/2]$, γ_t is linearly phased out to the identity so that $\gamma|[0,+\infty) \times C = id$.

Then γ induces a f.p. homeomorphism on $\mathring{c}(X) \times I^n$ which extends via the identity to a f.p. homeomorphism $\bar{\gamma} : B \times I^n \to B \times I^n$. Choose $\mu_1 > 0$ so that if $g : E \to B$ is an (ε_1,μ_1)-fibration over $(c_{2.5}(X), c_2(X) \setminus \mathring{c}_{r_0}(X))$, then $\bar{\gamma}_t g$ is a μ-fibration over $c_2(X)$ for each t in $I^n \setminus (C \times [0,1/2))$.

Let $\nu_1 = \nu(\mu_1)$ be given by Theorem 4.8 so that if $g : E \to B$ is a δ_1-fibration over $c_3(X)$ and a ν_1-fibration over $c_{2.25}(X) \setminus \mathring{c}_{3r_0/4}(X)$, then g is an (ε_1,μ_1)-fibration over $(c_{2.5}(X), c_2(X) \setminus \mathring{c}_{r_0}(X))$. Choose $\nu > 0$ so that if $g : E \to B$ is a ν-fibration over $c_3(X) \setminus \mathring{c}_{1/3}(X)$, then $\bar{\theta}_t^{-1} g$ is a ν_1-fibration over $c_3(X) \setminus \mathring{c}_{3r_0/4}(X)$ for each t in $I^n \setminus (C \times [0,1/2))$.

Now let $f : M \times I^n \to B \times I^n$ be given as in the hypothesis. Choose $\alpha > 0$ so that if $g : E \to B$ is an α-fibration over $c_3(X)$, then $\bar{\gamma}_t \bar{\theta}_t^{-1} g$ is a μ-fibration over $c_3(X)$ for each t in I^n. Choose β so that $0 < \beta < 1/2$ and $f_t : M \to B$ is an α-fibration over $c_3(X)$ whenever t is in $C \times [0,\beta]$. Let $\psi : I^n \to I^n$ be the homeomorphism which is supported on $C \times [0,3/4]$ which takes each interval $\{c\} \times [0,\beta]$ linearly onto $\{c\} \times [0,1/2]$ and takes $\{c\} \times [\beta,3/4]$ linearly onto $\{c\} \times [1/2,3/4]$.

Let $\tilde{\theta} : M \times I^n \to M \times I^n$ be the f.p. homeomorphism given by engulfing with the following properties:

i) $\tilde{\theta}|M \times C$ is the identity;

ii) $\bar{\theta}_t^{-1} \circ f_{\psi^{-1}(t)} \circ \tilde{\theta}_t$ is $(\varepsilon/2)$-close to $f_{\psi^{-1}(t)}$ for each t in I^n;

iii) $\tilde{\theta}$ is supported on $(M \times I^n) \setminus f^{-1}(\mathring{c}_{2/3}(X) \times I^n)$.

Define $\tilde{f} : M \times I^n \to B \times I^n$ by $\tilde{f}_{\psi^{-1}(t)} = \bar{\gamma}_t \circ \bar{\theta}_t^{-1} \circ f_{\psi^{-1}(t)} \circ \tilde{\theta}_t$ for each t in I^n. This $\tilde{f}$ satisfies the conclusions of the proposition. We need only indicate how to modify the proof to take care of the embellishment when we are given a compact Q-manifold F as in the hypothesis. By Theorem 7.3 $\tilde{\theta}$ can be chosen so that $\bar{\theta}^{-1}f\tilde{\theta}|F \times B \times I^n$ is projection. The problem is that $\bar{\gamma}\bar{\theta}^{-1}f\tilde{\theta}|F \times B \times I^n$ is not projection. However, it is easy to see that by using a collar on $F \times B$ in M a small f.p. isotopy on $M \times I^n$ will correct the action of $\bar{\gamma}$. □

LEMMA 9.2. Suppose m is a positive integer. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if M is a Q-manifold and $f : M \times I^n \to \mathbb{R}^m \times I^n$ is a proper f.p. map such that

i) f_t is a δ-fibration for each t in I^n,

ii) f_t is a ν-fibration over $\mathbb{R}^m \setminus \mathring{B}_3^m$ for each t in I^n,

iii) f_t is an approximate fibration for each t in C,

then there is a proper f.p. map $\tilde{f} : M \times I^n \to \mathbb{R}^m \times I^n$ such that

i) $\tilde{f}$ is ε-close to f,

ii) $\tilde{f}_t$ is a μ-fibration over B_2^m for each t in I^n,

iii) $\tilde{f}_t = f_t$ for each t in C.

Moreover, if we are additionally given a compact Q-manifold F and a sliced Z-embedding $g : F \times \mathbb{R}^m \times I^n \to M \times I^n$ such that $fg : F \times \mathbb{R}^m \times I^n \to \mathbb{R}^m \times I^n$ is projection, then we can additionally conclude that $\tilde{f}g$ is projection.

PROOF. Define a f.p. homeomorphism $\theta : \mathbb{R}^m \times I^n \to \mathbb{R}^m \times I^n$ so that

i) θ is supported on $B_8^m \times I^n$,

ii) θ_t affects only the first coordinate of any point in $\mathbb{R}^m$ for each t in I^n,

iii) $\theta_t(x_1,x_2,\ldots,x_m) = (x_1 + 5,x_2,\ldots,x_m)$ for each $(x_1,x_2,\ldots,x_m)$ in B_2^m and t in $I^n \setminus (C \times [0,1/2))$,

iv) for t in $C \times [0,1/2]$, θ_t is linearly phased out to the identity so that $\theta | \mathbb{R}^m \times C = \text{id}$.

Given $\varepsilon > 0$ choose $\delta > 0$ by the engulfing result Theorem 7.8 so that θ will be able to be covered as described below. Given $\mu > 0$ choose $\nu > 0$ so that if x and y are any two ν-close points of $\mathbb{R}^m$, then $\theta_t^{-1}(x)$ is μ-close to $\theta_t^{-1}(y)$ for each t in I^n (thus, if $g : E \to \mathbb{R}^m$ is a ν-fibration over $\theta_t(B_2^m)$, then $\theta_t^{-1}g$ is a μ-fibration over B_2^m).

Let $f : M \times I^n \to \mathbb{R}^m \times I^n$ be given as in the hypothesis. Choose α such that $0 < \alpha < 1/2$ and f_t is a ν-fibration for each t in $C \times [0,\alpha]$. Let $\psi : I^n \to I^n$ be the homeomorphism supported on $C \times [0,3/4]$ which takes each interval $\{c\} \times [0,\alpha]$ linearly onto $\{c\} \times [0,1/2]$ and $\{c\} \times [\alpha,3/4]$ linearly onto $\{c\} \times [1/2,3/4]$.

By engulfing there is a f.p. homeomorphism $\tilde{\theta} : M \times I^n \to M \times I^n$ such that

i) $\tilde{\theta} | M \times C = \text{id}$,

ii) $\theta_t^{-1} f_{\psi^{-1}(t)} \tilde{\theta}_t$ is ε-close to $f_{\psi^{-1}(t)}$ for each t in I^n.

Define $\tilde{f} : M \times I^n \to \mathbb{R}^m \times I^n$ by $\tilde{f}_{\psi^{-1}(t)} = \theta_t^{-1} f_{\psi^{-1}(t)} \tilde{\theta}_t$ for each t in I^n. This $\tilde{f}$ satisfies the conclusions of the lemma. If we are additionally given a sliced Z-embedding $g : F \times \mathbb{R}^m \times I^n \to M \times I^n$ where F is a compact Q-manifold and fg is projection, then just require $\tilde{\theta}$ to additionally satisfy $\tilde{\theta}g = g(id_F \times \theta)$. Then $\tilde{f}g$ is projection. □

LEMMA 9.3. Suppose m is a positive integer. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mu > 0$, M is a Q-manifold, and $f : M \times I^n \to \mathbb{R}^m \times I^n$ is a proper f.p. map such that f_t is a δ-fibration for each t in I^n and f_t is an approximate fibration for each t in C, then there is a proper f.p. map $\bar{f} : M \times I^n \to \mathbb{R}^m \times I^n$ such that

i) $\bar{f}|f^{-1}(B_2^m \times I^n)$ is ε-close to $f|f^{-1}(B_2^m \times I^n)$,

ii) $\bar{f}_t$ is a μ-fibration over B_2^m for each t in I^n,

iii) $\bar{f}_t = f_t$ for each t in C.

Moreover, if we are additionally given a compact Q-manifold F and a sliced Z-embedding $g : F \times \mathbb{R}^m \times I^n \to M \times I^n$ such that $fg : F \times \mathbb{R}^m \times I^n \to \mathbb{R}^m \times I^n$ is projection, then we can additionally conclude that $\bar{f}g|F \times B_2^m \times I^n : F \times B_2^m \times I^n \to B_2^m \times I^n$ is projection.

PROOF. Given $\varepsilon > 0$ let $\delta = \delta(\varepsilon)/3$ where $\delta(\varepsilon)$ comes from Lemma 9.2. Given $\mu > 0$, choose $\nu = \nu(\mu)$ by Lemma 9.2. Choose K to be a large number (in fact, $K = 3\delta/\nu$ suffices). The usual PL norm on $\mathbb{R}^m$ will be denoted by $\|\cdot\|$; thus, $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_m|\}$. An elementary construction provides a radially defined f.p. isotopy $\gamma : \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ with the following properties:

i) $\gamma_0 = id$;

ii) $\gamma_t|B_2^m = id$ for each t in I^n;

iii) $\|\gamma_t(x) - \gamma_t(y)\| \leq 3\|x - y\|$ for all x,y in $\mathbb{R}^m$ and t in I^n;

iv) if x is in $\mathbb{R}^m \setminus \mathring{B}^m_K$ and $||x - y|| < \delta$, then $||\gamma_1(x) - \gamma_1(y)|| < \nu$;

v) $\gamma_1(B^m_K) = B^m_3$.

Let $f : M \times I^n \to \mathbb{R}^m \times I^n$ be given as in the hypothesis. Construct a map $u : I^n \to [0,1]$ such that $u^{-1}(0) = C$ and if $u(t) < 1$, then f_t is a $(\nu/3)$-fibration. Define $\bar{\gamma} : \mathbb{R}^m \times I^n \to \mathbb{R}^m \times I^n$ to be the f.p. homeomorphism such that $\bar{\gamma}_t = \gamma_{u(t)}$. Define $f' : M \times I^n \to \mathbb{R}^m \times I^n$ to be the f.p. map such that $f'_t = \bar{\gamma}_t \circ f_t$. Then we have the following properties:

i) $f'_t = f_t$ for each t in C;

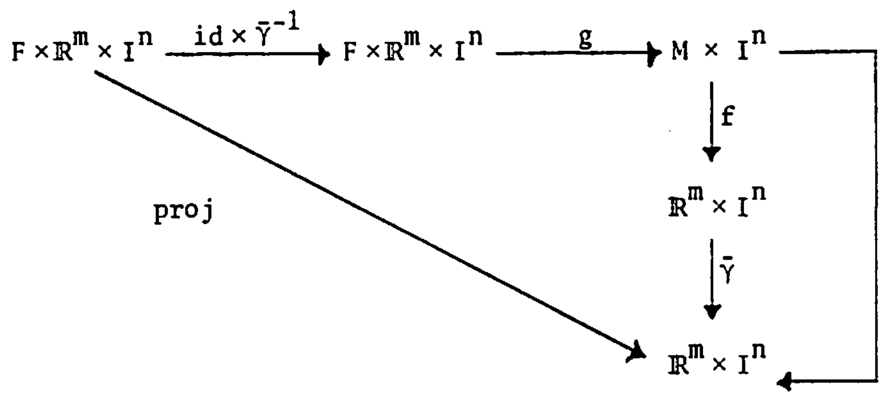
ii) $f'_t = f_t$ over B^n_2 for each t in I^n;

iii) f'_t is a ν-fibration over $\mathbb{R}^m \setminus \mathring{B}^m_3$ for each t in I^n;

iv) f'_t is a 3δ-fibration for each t in I^n.

An application of Lemma 9.2 to f' yields the desired $\bar{f} : M \times I^n \to \mathbb{R}^m \times I^n$.

If we are additionally given a sliced Z-embedding $g : F \times \mathbb{R}^m \times I^n \to M \times I^n$ where F is a compact Q-manifold and fg is projection, then consider the following commutative diagram:



The composition along the top row is a sliced Z-embedding, so when we produce $\bar{f}$ from f' using Lemma 9.2 we can assume that $\bar{f}g(\mathrm{id} \times \bar{\gamma}^{-1})$ is projection. But $\bar{\gamma}^{-1} | B^m_2 \times I^n = \mathrm{id}$. This shows that $\bar{f}g | F \times B^m_2 \times I^n$ is projection. □

PROPOSITION 9.4. Suppose m is a positive integer and $\mathbb{R}^m \hookrightarrow B$ is an open embedding. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mu > 0$, M is a Q-manifold, and $f : M \times I^n \to B \times I^n$ is a proper f.p. map such that f is a sliced δ-fibration over $B^m_3 \times I^n$ and f_t is an approximate fibration for each t in C, then there is a proper f.p. map $\tilde{f} : M \times I^n \to B \times I^n$ which is a sliced μ-fibration over $B^m_1 \times I^n$ and which is f.p. ε-homotopic to f rel $[(M \times I^n) \setminus f^{-1}(\mathring{B}^m_3 \times I^n)] \cup [M \times C]$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $\tilde{f}|F \times B \times I^n$ is projection and that the homotopy from f to $\tilde{f}$ is rel $F \times B \times I^n$.

PROOF. Let $e : \mathbb{R} \to S^1$ denote the standard covering projection of period 8 as defined in Section 8 and let $e^m = e \times \cdots \times e : \mathbb{R}^m \to T^m$ be the product covering projection. As in [11, Section 8] we regard $T^{m-1} \times \mathbb{R}$ as an open subset of $\mathring{B}^m_3$ so that the composition $e^{m-1} \times \mathrm{id} : B^m_2 = B^{m-1}_2 \times [-2,2] \to T^{m-1} \times [-2,2] \subset T^{m-1} \times \mathbb{R} \subset \mathring{B}^m_3$ is the inclusion.

Let $\mu > 0$ and $f : M \times I^n \to B \times I^n$ be given. Since f is a sliced δ-fibration over $T^{m-1} \times [-3,3] \times I^n$ we can use Proposition 8.1 to find a compact Q-manifold M', a f.p. map $f' : M' \times I^n \to T^{m-1} \times S^1 \times I^n = T^m \times I^n$ which is a sliced δ'-fibration, and a f.p. open embedding $\psi : f^{-1}(T^{m-1} \times (-2,2) \times I^n) \to M' \times I^n$ so that the following diagram commutes:

$$\begin{array}{ccc} M' \times I^n & \xrightarrow{\quad f' \quad} & T^m \times I^n \\ \uparrow \psi & & \uparrow \mathrm{id} \times e \times \mathrm{id} \\ f^{-1}(T^{m-1} \times (-2,2) \times I^n) & \xrightarrow{\quad f| \quad} & T^{m-1} \times (-2,2) \times I^n \end{array}$$

Here δ' is small if δ is and we regard ψ as an inclusion map.

Now form the following pull-back diagram:

$$\begin{array}{ccc} M'' & \xrightarrow{\ f''\ } & \mathbb{R}^m \times I^n \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle e^m \times id} \\ M' \times I^n & \xrightarrow{\ f'\ } & T^m \times I^n \end{array}$$

It follows that p is a bundle projection so that we may write $M'' = M''' \times I^n$ where M''' is a Q-manifold ($M''' \cong p^{-1}(M' \times \{pt.\})$ and assume that p and f'' are f.p.

It follows from elementary facts about pull-backs that f'' is a sliced δ''-fibration (where the size of δ'' depends on the size of δ') and that f''_t is an approximate fibration for each t in C.

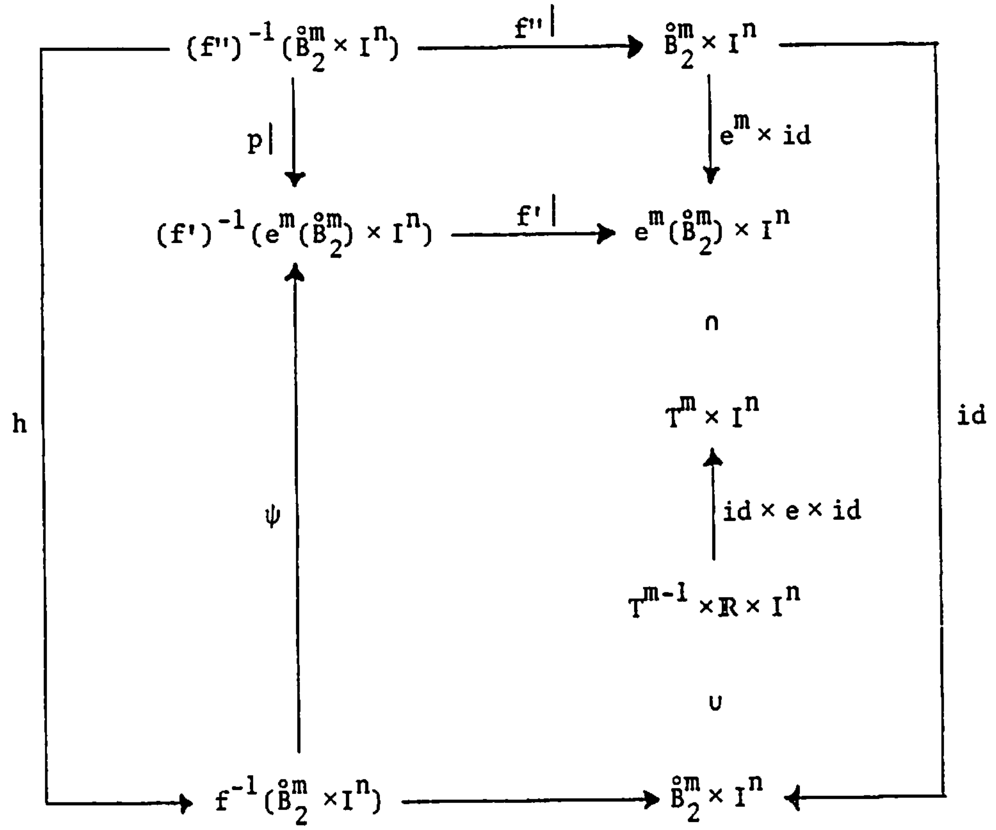
Using Lemma 9.3 we can find a f.p. map $\bar{f} : M''' \times I^n \to \mathbb{R}^m \times I^n$ with the following properties:

i) $\bar{f}_t = f''_t$ for each t in C;

ii) $\bar{f}_t$ is a μ-fibration over B^m_2 for each t in I^n;

iii) there is a f.p. δ'''-homotopy G from $f''|(f'')^{-1}(B^m_2 \times I^n)$ to $\bar{f}|(f'')^{-1}(B^m_2 \times I^n)$ which is rel $(f'')^{-1}(B^m_2 \times C)$.

Consider the following commutative diagram:



The right-hand vertical composition is the identity by our choice of notation. The left-hand composition is a f.p. homeomorphism by using elementary facts about pull-backs. Let $u : \mathbb{R}^m \to [0,1]$ be a map such that $u^{-1}(1) = B^m_{1.5}$ and $u^{-1}(0) = \mathbb{R}^m \setminus \mathring{B}^m_2$. Define $\tilde{f} : M \times I^n \to B \times I^n$ as follows:

$$\tilde{f}(x,t) = \begin{cases} f(x,t) & \text{if } f(x,t) \in (B \setminus \mathring{B}^m_2) \times I^n \\ G(h^{-1}(x,t), uf_t(x)) & \text{if } f(x,t) \in \mathring{B}^m_2 \times I^n . \end{cases}$$

This $\tilde{f}$ satisfies the conclusions of the Proposition.

We now discuss the case when we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n$ is projection. We want to conclude that $\tilde{f}|F \times B \times I^n$ is projection. First note when we use Proposition 8.1 to wrap f up to get $f' : M' \times I^n \to T^m \times I^n$, we are provided with a sliced Z-embedding $g : F \times T^m \times I^n \to M' \times I^n$ for which $f'g$ is projection and for which the following diagram commutes:

$$\begin{array}{ccc} F \times T^m \times I^n & \xrightarrow{\quad g \quad} & M' \times I^n \\ \uparrow{\scriptstyle id \times e \times id} & & \uparrow{\scriptstyle \psi} \\ F \times T^{m-1} \times (-2,2) \times I^n & \subset & f^{-1}(T^{m-1} \times (-2,2) \times I^n) \end{array}$$

Next observe that there is a sliced Z-embedding $\tilde{g} : F \times \mathbb{R}^m \times I^n \to M''$ such that $f''\tilde{g}$ is projection and the following diagram commutes:

$$\begin{array}{ccc} F \times \mathbb{R}^m \times I^n & \xrightarrow{\quad \tilde{g} \quad} & M'' \\ \downarrow{\scriptstyle id \times e^m \times id} & & \downarrow{\scriptstyle p} \\ F \times T^m \times I^n & \xrightarrow{\quad g \quad} & M' \times I^n \end{array}$$

To define $\tilde{g}$ recall that $M'' = \{(x,t,y,t) \in M' \times I^n \times \mathbb{R}^m \times I^n \mid f'(x,t) = (e^m(y),t)\}$. Then set $\tilde{g}(x,y,t) = (g(x,e^m(y),t),(y,t))$ for each (x,y,t) in $F \times \mathbb{R}^m \times I^n$. That $\tilde{g}$ is a sliced Z-embedding follows from the construction and the fact that p is a covering projection. Now when Lemma 9.3 is used to construct $\bar{f}$, simply insist that $\bar{f}\tilde{g} | F \times B_2^m \times I^n$ is projection. This will imply that $\tilde{f} | F \times B \times I^n$ is projection. □

The next result generalizes Propositions 9.1 and 9.4. It is the main result of this section.

THEOREM 9.5. Suppose $m \geq 0$ is an integer and $\mathring{c}(X) \times \mathbb{R}^m \hookrightarrow B$ is an open embedding. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:

if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map such that f is a sliced δ-fibration over $c_3(X) \times B_3^m \times I^n$ and a sliced ν-fibration over $[c_3(X) \setminus \mathring{c}_{1/3}(X)] \times B_3^m \times I^n$ and f_t is an approximate fibration for each t in C, then there is a proper f.p. map $\tilde{f} : M \times I^n \to B \times I^n$ which is a sliced μ-fibration over $c_1(X) \times B_1^m \times I^n$ and which is f.p. ε-homotopic to f rel $[(M \times I^n) \setminus f^{-1}(\mathring{c}_{2/3}(X) \times \mathring{B}_3^m \times I^n)] \cup [M \times C]$.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $\tilde{f}|F \times B \times I^n$ is projection and that the homotopy from f to $\tilde{f}$ is rel $F \times B \times I^n$.

REMARKS ON PROOF. First note that Proposition 9.1 is the $m = 0$ case of this theorem. For $m \geq 1$ the proof is similar to the proof of Proposition 9.4. The major changes are as follows. In wrapping up Theorem 8.2 must be used instead of Proposition 8.1 because of the extra $\mathring{c}(X)$-factor. This introduces a submersion into the proof and therefore the engulfing result Thoerem 7.9 must be used instead of Theorem 7.8 for the analogue of Lemma 9.2. Finally the radial squeeze $\bar{\gamma}$ of $\mathbb{R}^m \times I^n$ in Lemma 9.3 must now be followed by a radial squeeze of $\mathring{c}(X) \times I^m$ towards the vertex. This is similar to the procedure of Proposition 9.1. □

SECTION 10:
THE MAIN RESULTS

In this section we present our main result on deforming a parameterized family of ε-fibrations to a parameterized family of approximate fibrations (Theorem 10.2). It will follow from this that the space of approximate fibrations from a compact Q-manifold to a compact polyhedron is uniformly LC^n for every $n \geq 0$. We begin with a key lemma.

LEMMA 10.1. Let B be a polyhedron, $n \geq 0$ an integer, and C a closed subset of ∂I^n which is collared in I^n. For every open cover α of B there exists an open cover β of B so that the following statement is true:

> if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map such that f_t is a β-fibration for each t in I^n and an approximate fibration for each t in C, then for every open cover γ of B there is a proper f.p. map $f' : M \times I^n \to B \times I^n$ such that f'_t is a γ-fibration α-close to f_t for each t in I^n and $f'_t = f_t$ for each t in C.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $f'|F \times B \times I^n$ is projection.

PROOF. We first treat the case where B is an m-dimensional compact polyhedron. Assume that B has a fixed triangulation and label the barycenters of B as $b^0_1, b^0_2, \ldots, b^1_1, b^1_2, \ldots, b^m_1, b^m_2, \ldots$, where b^i_j is the barycenter of some i-simplex of B. Let C^i_j denote the closed star of b^i_j in the second barycentric subdivision of B. We identify C^i_j with $c_1(A^i_j) \times B^i_1$

where A_j^i is a compact polyhedron and we consider $\mathring{c}(A_j^i) \times \mathbb{R}^i$ as an open subset of B so that the following two properties are satisfied (here $1.2 = r_0 < r_1 < \cdots < r_m = 1.3$):

i) $[c_3(A_j^i) \setminus \mathring{c}_{1/3}(A_j^i)] \times B_3^i \subset \cup\{c_{r_{i+1}}(A_q^p) \times B_{r_{i+1}}^p \mid i+1 \le p \le m, 1 \le q\}$;

ii) $c_{2/3}(A_j^i) \times B_3^i$ misses $\cup\{c_{r_i}(A_q^p) \times B_{r_i}^p \mid i+1 \le p \le m, 1 \le q\}$.

(In the course of the proof we will often produce a map which is an ε-fibration over some compacta C_i, $i = 1,2,\ldots,k$ and then declare that the map is an ε-fibration over $\cup C_i$. Of course this is incorrect, but in each case it is easy to see that we could introduce some more notation and produce a map which is an ε'-fibration over compact neighborhoods of the C_i. The desired conclusion would follow from the remark following Proposition 2.2 in [2].)

Let $\varepsilon > 0$ be given (since B is compact, we will replace the open covers α, β, and γ of B in the hypothesis with positive numbers ε, δ, and μ, respectively). We can inductively define small positive numbers $\varepsilon_0,\delta_0,\varepsilon_1,\delta_1,\ldots,\varepsilon_m,\delta_m$ with the following properties (choose $0 < \varepsilon_0 < \varepsilon/(m+1)$:

i) $\delta_i < \delta(\varepsilon_i)$ where $\delta(\varepsilon_i)$ is given by the handle lemma (Theorem 9.5) for the open embeddings $\mathring{c}(A_j^i) \times \mathbb{R}^i \hookrightarrow B$ (we apply the handle lemma independently for each j);

ii) $\delta_i < \delta_{i-1}/2$;

iii) $\varepsilon_i < \varepsilon/(m+1)$;

iv) any map to B which is ε_i-close to a $(\delta_{i-1}/2)$-fibration is itself a δ_{i-1}-fibration.

Set $\delta = \delta_m$ and let $\mu > 0$ be given. Let $f : M \times I^n \to B \times I^n$ be given as in the hypothesis where we assume that f is in fact a sliced δ-fibration. We will produce a f.p. map $f' : M \times I^n \to B \times I^n$ ε-close to f such that f'_t is a μ-fibration over each $c_{1.1}(A_j^i) \times B_{1.1}^i$ for each t in I^n and $f'_t = f_t$ for each t in C. It suffices to construct a sequence of maps $f = f^{m+1}, f^m, \ldots, f^1, f^0 = f'$ such that f^i is ε_i-close to f^{i+1}, f_t^i is a μ-fibration over $c_{r_i}(A_q^p) \times B_{r_i}^p$ for $i \le p \le m$, each q, and t in I^n, and $f_t^i = f_t$ for t in C. First inductively define small positive numbers $\nu_{-1}, \nu_0, \ldots, \nu_m$ by setting $\nu_{-1} = \mu$ and for $i = 0,\ldots,m-1$ choosing $\nu_i < \mu$ such that $\nu_i < \nu(\nu_{i-1})$ where $\nu(\nu_{i-1})$ is given by the handle lemma for the open embeddings $\mathring{c}(A_j^i) \times \mathbb{R}^i \hookrightarrow B$. Using the appropriate handle lemma we inductively produce the maps f^i (starting with $i = m$) so that

i) f^i is a sliced ν_{i-1}-fibration over $c_{r_i}(A_j^i) \times B_{r_i}^i \times I^n$ for each j,

ii) f^i is ε_i-close to f^{i+1},

iii) $f^i = f^{i+1}$ over $B \setminus \cup_{j\ge1}[\mathring{c}_{2/3}(A_j^i) \times \mathring{B}_3^i]$,

iv) $f_t^i = f_t^{i+1}$ for each t in C.

In order to apply the handle lemma inductively simply observe that f^i is a δ_{i-1}-fibration. Also observe that f^i is a sliced μ-fibration over $c_{r_i}(A_q^p) \times B_{r_i}^p \times I^n$ for $i \le p \le m$ and $q \ge 1$.

If B is not compact, then the procedure is similar. Assume that B has a fixed locally finite triangulation and for each barycenter of b let C_b denote the closed star of B in the second barycentric subdivision of B. Define subsets $S_1, S_2, S_3, \ldots$ as follows. Let $S_1 = \cup\{C_b | b$ is the barycenter of a principal simplex of B$\}$, $S_2 = \cup\{C_b | b$ is the barycenter of a top dimensional proper face of a principal simplex of B$\}$, etc. One then modifies the map f over a neighborhood of S_1, then over a neighborhood of S_2, etc., just like in the compact case. This process is "locally finite."

If we are additionally given a compact Q-manifold F as in the hypothesis, then using the full strength of the handle lemma in the construction above will allow us to conclude that $f'|F \times B \times I^n$ is projection. □

THEOREM 10.2. Let B be a polyhedron, $n \geq 0$ an integer, and C a closed subset of ∂I^n which is collared in I^n. For every open cover α of B there exists an open cover β of B so that if M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map such that f_t is a β-fibration for each t in I^n and an approximate fibration for each t in C, then there is a proper f.p. map $\tilde{f} : M \times I^n \to B \times I^n$ such that $\tilde{f}_t$ is an approximate fibration α-close to f_t for each t in I^n and $\tilde{f}_t = f_t$ for each t in C.

Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in M and $f|F \times B \times I^n : F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $\tilde{f}|F \times B \times I^n$ is projection.

PROOF. Just as in [2, Section 6] this theorem follows immediately from Lemma 10.1 by an induction argument. Briefly, one constructs $\tilde{f}$ as a limit of maps $f^i : M \times I^n \to B \times I^n$ which are chosen inductively by Lemma 10.1 so that f^i is close to f and each f^i_t is a β_i-fibration. □

COROLLARY 10.3. If M is a compact Q-manifold and B is a compact polyhedron, then the space of approximate fibrations from M to B endowed with the compact-open topology is LC^n for each non-negative integer n.

PROOF. Recall that since M and B are compact the compact-open topology coincides with the uniform topology. We consider B to have a fixed metric. Let $\varepsilon > 0$ and $n \geq 0$ be given and choose $\beta = \beta(\varepsilon/3) > 0$ by Theorem 10.2 with $C = \partial I^{n+1}$ so that any f.p. map $f : M \times I^{n+1} \to B \times I^{n+1}$ with f_t an approximate fibration for each t in ∂I^{n+1} and f_t a β-fibration for each t in I^{n+1} is f.p. $(\varepsilon/3)$-homotopic rel $M \times \partial I^{n+1}$ to a f.p. map $\tilde{f} : M \times I^{n+1} \to B \times I^{n+1}$ such that $\tilde{f}_t$ is an approximate fibration for each t in I^{n+1}. Now choose $0 < \gamma < \varepsilon/3$ so that any map to B which is γ-close to an approximate fibration is a β-fibration.

Choose $\delta > 0$ so that if $f : M \times \partial I^{n+1} \to B \times \partial I^{n+1}$ is any f.p. map with the property that for each s, t in ∂I^{n+1} f_s is δ-close to f_t, then there exists a f.p. extension $g : M \times I^{n+1} \to B \times I^{n+1}$ of f such that g_s is γ-close to g_t for all s, t in I^{n+1}.

To complete the proof we claim that if $f : M \times \partial I^{n+1} \to B \times \partial I^{n+1}$ is a f.p. map such that f_s is δ-close to f_t for all s, t in ∂I^{n+1} and f_t is an approximate fibration for each t in ∂I^{n+1}, then there exists a f.p. extension $\tilde{f} : M \times I^{n+1} \to B \times I^{n+1}$ of f with the property that $\tilde{f}_s$ is ε-close to $\tilde{f}_t$ for all s, t in I^{n+1} and f_t is an approximate fibration for each t in I^{n+1}. This is obvious from the choices made above. □

REMARK 10.4. A (possibly non-locally compact) metric space (X,d) is said to be *uniformly* LC^n if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every map $f : \partial I^{n+1} \to X$ with the diameter of $f(\partial I^{n+1})$ less than δ extends to a map $\tilde{f} : I^{n+1} \to X$ with the diameter of $f(I^{n+1})$ less than ε. If in the statement of Corollary 10.3 we fix a metric for B, then the proof shows that the space of approximate fibrations from M to B endowed with the uniform topology is uniformly LC^n for each non-negative integer n.

In [13] it is shown that if $f : E \to B$ is an approximate fibration between connected ANRs, then $f^{-1}(b)$ is shape equivalent to the homotopy fiber of f for each b in B. From this it follows that if $f,g : E \to B$ are homotopic approximate fibrations between (not necessarily connected) ANRs and b is in B, then $f^{-1}(b)$ is shape equivalent to $g^{-1}(b)$. For example, if $f,g : E \to B$ are homotopic approximate fibrations and f is cell-like, then g is cell-like. Or, if $f,g : E \to B$ are homotopic approximate fibrations and f is monotone, then g is monotone. With these facts in mind the following two corollaries follow immediately from Corollary 10.3.

COROLLARY 10.5. If M is a compact Q-manifold and B is a compact polyhedron, then the space of cell-like maps from M to B endowed with the compact-open topology is LC^n for each non-negative integer n.

COROLLARY 10.6. If M is a compact Q-manifold and B is a compact polyhedron, then the space of monotone approximate fibrations from M to B endowed with the compact-open topology is LC^n for each non-negative integer n.

As a final offering we remark that the following weak version of Theorem 10.2 holds when B is any ANR. The proof is exactly like the analogous result in [2, Section 6].

COROLLARY 10.7. Let B be an ANR and let $n \geq 0$ be an integer. For every open cover α of B there exists an open cover β of B so that if

M is a Q-manifold and $f : M \times I^n \to B \times I^n$ is a proper f.p. map such that f_t is a β-fibration for each t in I^n, there there is a proper f.p. map $\tilde{f} : M \times I^n \to B \times I^n$ such that $\tilde{f}_t$ is an approximate fibration α-close to f_t for each t in I^n.

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