$\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$. Here $f^n(x)$ is the nth iterate of $f(x)$. Prove that for each $x \in [a, b]$, $L_x \subset \overline{C(f)}$. Is this true in R^2?

6134. *Proposed by Barbara Osofsky, Rutgers University*

Let R be a ring, not necessarily with identity, and let R^n be the subring generated by n-fold products of elements of R. Prove: If R has descending chain condition (d.c.c.) on right ideals, then so does R^n. Does this result hold if d.c.c. is replaced by ascending chain condition (a.c.c.)?

6135*. *Proposed by Paul Erdös, Hungary*

Denote by $P(n)$ the greatest prime factor of n and put $A(x, y) = \Pi_{1 \leqq i \leqq y-x}(x + i)$. An integer n is called *exceptional* if for some $x \leqq n \leqq y$, $(P(A(x, y)))^2$ divides $A(x, y)$, i.e., the greatest prime factor of $A(x, y)$ occurs with an exponent greater than one.

Prove that the density of exceptional numbers is 0 and estimate the number $E(x)$ not exceeding x as well as you can.

6136. *Proposed by H. L. Montgomery, University of Michigan*

Let $P(z, w) = \Sigma c_{mn} z^m w^n$ be a polynomial in $\mathbb{C}[z, w]$. Suppose that $Q(z, w) = P(z, w/z)$ is also a polynomial: that is $c_{mn} = 0$ whenever $n > m$. Show that

$$\{P(z, w): |z| < 1, |w| < 1\} = \{Q(z, w): |z| < 1, |w| < 1\}.$$

6137*. *Proposed by I. J. Good, Virginia Polytechnic Institute and State University*

Let $p(n)$ denote the number of partitions of n $(n = 1, 2, \ldots)$, and let k denote an integer greater than 3. Prove that $\Delta^k p(n)$ $(n = 1, 2, \ldots)$ is a sequence of alternating terms.

SOLUTIONS OF ADVANCED PROBLEMS

Distance to the Boundary of a Set

6025 [1975, 409]. *Proposed by S. F. Wong and B. B. Winter, University of Ottawa*

Let (X, d) be a metric space, T an arbitrary subset of X, and t an arbitrary element of T. As usual, $d(t, A) = \inf\{d(t, a): a \in A\}$ is $-\infty$ if $A = \varnothing$; ∂T and T^c are, respectively, the boundary and the complement of T.

(a) Is it always true that $d(t, x) < d(t, \partial T)$ implies $x \in T$?

If not, find a condition on (X, d) which is necessary and sufficient for the validity of this implication.

(b) Is it always true that $d(t, \partial T) = d(t, T^c)$?

If not, find a condition of (X, d) which is necessary and sufficient for the validity of this equality.

Solution by C. Bruce Hughes, Guilford College. Conditions (a) and (b) are equivalent, and are also equivalent to:

(c) Every ε-neighborhood $N_\varepsilon(x)$ is connected.

Proof. (a) implies (b). Using the triangle inequality it is easy to verify that $d(t, T^c) \leqq d(t, \partial T)$ is true in any metric space. If $d(t, T^c) < d(t, \partial T)$, then there exists $x \in T^c$ with $d(t, x) < d(t, \partial T)$. It follows from (a) that $x \in T$, which contradicts $x \in T^c$.

(b) implies (c). Suppose that there exists an ε-neighborhood $N_\varepsilon(x)$ which is not connected. Let $N_\varepsilon(x) = A \cup B$ be a separation, and assume that $x \in A$. Since A and B are open, $\partial A \cap N_\varepsilon(x) = \partial A \cap (A \cup B) = \varnothing$. By (b) $\partial A \neq \varnothing$, hence $d(x, \partial A) = \varepsilon$. Since $B \subseteq A^c$, it follows that $d(x, A^c) \leqq d(x, B) < \varepsilon$. But $d(x, \partial A) = d(x, A^c) < \varepsilon$.

(c) implies (a). Suppose that there exists $x \in T^c$ and $t \in T$ such that $d(t, x) < d(t, \partial T)$. Then there exists $\varepsilon > 0$ such that $x \in N_\varepsilon(t)$ and $N_\varepsilon(t) \cap \partial T = \varnothing$. Since $N_\varepsilon(t)$ is connected and intersects both T and T^c, $N_\varepsilon(t) \cap \partial T \neq \varnothing$.

Also solved by Bethany College Problems Group, David Browder, Victor Manjarrez & Louise Moser, David Ritter, Thomas Sellke, Carlton Woods, and the proposers.

Editor's Note. Several solvers pointed out that $d(x, \varnothing)$ is usually taken to be $+\infty$ rather than $-\infty$. This convention is required in the above solution.

Number of Elements in a Group Inverted by an Automorphism

6026 [1975, 409]. *Proposed by Fred Commoner, Cambridge, Massachusetts*

Prove the theorem: Let p be an odd prime. If G is a finite non-abelian group such that p is less than or equal to the least prime dividing $|G|$, then no automorphism of G can send more than $|G|/p$ elements of G to their inverses. There is a non-abelian group G of order p^3 and an automorphism of G sending exactly $|G|/p$ elements of G to their inverses.

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands. (1) If $|G|$ is even, then there is nothing to prove, so suppose $|G|$ is odd. Let $\sigma \in \operatorname{Aut}(G)$ and put $S = \{x \in G \mid \sigma x = x^{-1}\}$, $C_i = \{x \in G \mid \sigma^i x = x\}$, $i = 1, 2$. Then C_2 contains S and $|C_2|$ divides $|G|$ by Lagrange's theorem. Now suppose by way of contradiction that $|S| > |G|/p$. Then $C_2 = G$; i.e., σ has order two. Since in addition G is non-abelian, σ cannot be fixed-point-free (see [**1**], p. 336, thm. 1.4). That is $|C_1| > 1$, and by our assumption on $|S|$ we have $|S| > [G: C_1]$. Hence at least one coset of C_1 contains more than one element of S. Say, $x, y \in S$ with $x = yc$, $c \in C_1$, $c \neq e$. Then $c^{-1}y^{-1} = x^{-1} = \sigma x = (\sigma y)(\sigma c) = y^{-1}c$; or $y^{-1}cy = c^{-1}$. Consequently $y^{-2}cy^2 = c$. Hence, either $c^{-1} = c$, or the centralizer of c in $\langle y \rangle$ is a proper subgroup of $\langle y \rangle$ generated by y^2. However, $|G|$ is odd, so the order of both c and y is odd: a contradiction in either case.

(2) Up to isomorphism there are precisely two nonabelian groups of order p^3 (cf. [**2**], p. 93), viz.

$$G = \langle n, s \mid s^{-1}ns = n^{1+p}, n^{p^2} = s^p = e \rangle,$$

resp.

$$G = \langle a, b, c \mid a^p = b^p = c^p = e,\ a^{-1}ba = bc,\ a^{-1}ca = b^{-1}cb = c \rangle.$$

In either case G is a semidirect product of an abelian normal subgroup N and a p-cyclic subgroup, and so an automorphism ϕ with the desired properties can be obtained by extending the isomorphism $x \to x^{-1}$ $(x \in N)$ to G. Explicitly, put $\phi n = n^{-1}$, $\phi s = s$ in the first case and $\phi a = a$, $\phi b = b^{-1}$, $\phi c = c^{-1}$ in the second case.

References

1. D. Gorenstein, Finite Groups, Harper & Row, New York, 1968.
2. B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.

Also solved by Peter Borwein & Martin Schechter (Canada), Allan Coppola, M. G. Greening (Australia), Kenneth Klinger, O. P. Lossers (Netherlands), R. C. Lyndon, Victor Manjarrez & Louis Moser, M. R. Modak (India), Ram Murty & Kumar Murty (Canada), Barbara Osofsky, S. G. Updikar (India), and the proposer.

Editor's Comment. Desmond MacHale (Ireland) has advised that a solution of the above problem appears in a joint paper by Hans Liebeck and himself, *Groups of odd order with automorphisms inverting many elements*, J. London Math. Soc. (2), 6 (1973), 215–223. He has also obtained the following generalization: G finite, p least prime dividing $|G|$, $\alpha \in \operatorname{Aut}(G)$, $T_n = \{g \in G \mid g\alpha = g^n\}$, $n \in \mathbb{Z}$; then if $|T_n| > |G|/p$ we have $T_n = G$. See J. London Math. Soc., 11 (1975), 366–368.