Journal of

UNDERGRADUATE MATHEMATICS

CONTENTS

VOLUME 8
NUMBER 2

(see back cover)

MARCH
1976

Journal of
UNDERGRADUATE MATHEMATICS

The primary purpose of the *Journal of* UNDERGRADUATE MATHEMATICS is to provide for the publication of research or expository papers written by undergraduate students. A section, *Proposals for Research*, is a source of topics for undergraduate research; solutions are on the level of papers published in this journal - contributors to this section are both faculty and students.

The *Journal of* UNDERGRADUATE MATHEMATICS is published biannually. The price for a subscription per volume (two issues - March and September) is $3.00 for individuals and $5.00 for libraries and departments.

Subscriptions, manuscripts and all other correspondence should be mailed to:

Journal of UNDERGRADUATE MATHEMATICS
Guilford College
Greensboro, North Carolina 27410

Published by:
Department of Mathematics
Guilford College

IRREDUCIBLE CONTINUA AND SOME CHARACTERIZATIONS OF ARCS

C. Bruce Hughes[1]

INTRODUCTION. The purpose of this paper is to study local connectivity and some related properties in irreducible continua (see Section 1 for definitions). These properties are shown to be equivalent in irreducible continua. An arc is characterized as an irreducible continuum having any one of these properties at each of its points. We also obtain a new proof for the classical characterization of an arc as a continuum with exactly two non-separating points. For terms and concepts not explained in this paper the reader is refered to [2].

The author is indebted to Dr. G. R. Gordh, Jr. for his many helpful suggestions and comments during the development of this paper.

1. BASIC DEFINITIONS AND EXAMPLES. A *continuum* is a compact, connected metric space. If A is a subset of a continuum we denote the closure of A by $\overline{A}$. The *interior* of A, denoted int(A), is the set of all points x in A such that there is an open set U which contains x and is a subset of A. The *boundary* of A, denoted bd(A), is the set of all points x such that every open set containing x contains both a point in A and a point not in A.

DEFINITION 1. A continuum M is *irreducible from* p *to* q if no proper subcontinuum of M contains both p and q. A continuum M is *irreducible* if there exists points p and q in M such that M is irreducible from p to q.

DEFINITION 2. An *arc* is a homeomorphic image of the closed unit interval.

An important characterization of arcs used in this paper is the following: an arc is a connected, separable, linearly ordered topological space having a first point and a last point (see [1]). An element in the basis for the topology of a linearly ordered space consists of all points that are between two given points, the set of all points which precede a given point, or the set of all points which follow a given point.

DEFINITION 3. A continuum M is *freely decomposable* if for any two distinct points p and q in M there exists subcontinua A and B such that p is in A-B, q is in B-A, and M = A ∪ B.

DEFINITION 4. The point p is a *non-separating point* of the continuum M if M-{p} is connected.

The following concept is due to F. B. Jones (see [3],[4]).

DEFINITION 5. If A is a subset of a continuum M, then M is *aposyndetic at a point* p *with respect to* A if there exists a subcontinuum H of M such that $p \in \text{int}(H) \subseteq H \subseteq M-A$.

The following seven properties are those which are under investigation in this paper.

PROPERTY 1. A continuum M is *semi-aposyndetic at a point* p if for all q in M-{p}, either M is aposyndetic at p with respect to q or M is aposyndetic at q with respect to p.

PROPERTY 2. A continuum M is *semi-locally connected at a point* p if for any open set U containing p there exists an open set V such that $p \in V \subseteq U$ and M-V has finitely many components.

PROPERTY 3. A continuum M is *aposyndetic at a point* p if M is aposyndetic at p with respect to q for all q in M-{p}.

PROPERTY 4. A continuum M is *finitely-aposyndetic at a point* p if, given a set F containing finitely many points all of which are distinct from p, M is aposyndetic at p with respect to F.

PROPERTY 5. A continuum M is *continuum-aposyndetic at a point* p if given any subcontinuum H of M-{p}, M is aposyndetic at p with respect to H.

PROPERTY 6. A continuum M is *connected im Kleinen at a point* p if for every open set U containing p there exists a subcontinuum H such that $p \in \mathrm{int}(H) \subseteq H \subseteq U$.

PROPERTY 7. A continuum M is *locally connected at a point* p if for every open set U containing p there exists an open set V such that $p \in V \subseteq U$ and V is connected.

For each of Properties 1-7 if the words "at a point" are deleted, we mean that the continuum has that property at each of its points. An arc and a circle are examples of continua which are freely decomposable and have all of Properties 1-7 at each of their points. An arc is irreducible, but a circle is not. In Section 3 it is shown that the only irreducible continua which have any one of Properties 1-7 at each of their points are arcs. The following are examples of continua having some of Properties 1-7 at certain points and not having some of the Properties at other points. None of these continua are irreducible. The details of the examples are left to the reader.

EXAMPLE 1. In the plane with a cartesian coordinate system, let $M = \{(x,y) : 0 \le x \le 1$ and $y = 0$ or $y = x/n,\ n = 1,2,3,\cdots\}$. Let p be the point whose coordinates are (0,0) and let q be the point whose coordinates are (1,0). It can be seen that M is semi-aposyndetic at p and aposyndetic at p, but M is not semi-locally connected at p. Also, M is semi-locally connected at q and semi-aposyndetic at q, but M is not aposyndetic at q.

EXAMPLE 2. In the plane with a cartesian coordinate system, let $M = \{(x,y) : 0 \le x \le 1$ and $y = 0$ or $y = 1/n,\ n = 1,2,3,\cdots\} \cup \{(x,y) : x = 0 \text{ and } 0 \le y \le 1\}$. Let p be the point whose coordinates are (1/2,0). Then M is aposyndetic at p, but M is not finitely-aposyndetic at p.

EXAMPLE 3. This example is the 3-dimensional analogue of Example 2. In Euclidean 3-space with a cartesian coordinate system, let $M = \{(x,y,z) : 0 \le x \le 1$ and $0 \le y \le 1$ and $z = 0$ or $z = 1/n,\ n = 1,2,3,\cdots\} \cup \{(x,y,z) : x = 0 \text{ and } 0 \le y \le 1 \text{ and } 0 \le z \le 1\} \cup \{(x,y,z) : x = 1 \text{ and } 0 \le y \le 1 \text{ and } 0 \le z \le 1\}$. Let p be the point whose coordinates are (1/2,1/2,0). It can be seen that M is finitely-aposyndetic at p, but M is not continuum-aposyndetic at p.

EXAMPLE 4. Let M be the plane continuum pictured in Figure 1. Then M is continuum-aposyndetic at the point p, but M is not connected im Kleinen at p.

EXAMPLE 5. Let M be the plane continuum pictured in Figure 2. Then it can be seen that M is connected im Kleinen at the point p, but M is not locally connected at p.

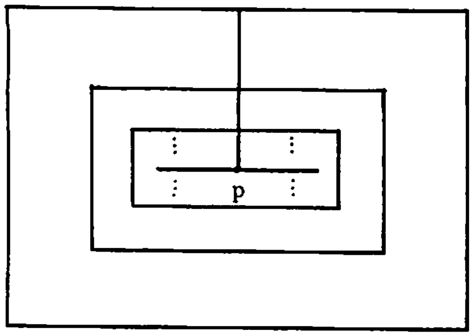

FIGURE 1

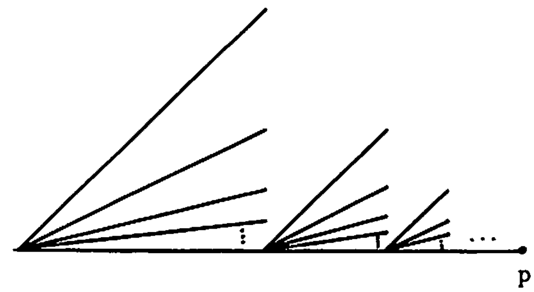

FIGURE 2

2. PRELIMINARY RESULTS. In this section we establish propositions that will be important in proving the theorems of Section 3. We begin by stating one of the most basic theorems of continua theory (for a proof, see [2], page 47).

LEMMA 1. *(Boundary Bumping Theorem) If* U *is an open subset of a continuum* M *and* C *is a component of* U, *then* $\overline{U}$-U *contains a limit point of* C *(that is,* $bd(U) \cap \overline{C} \neq \phi$*).*

PROPOSITION 1. *A continuum* M *is semi-locally connected at* p *if and only if* M *is aposyndetic at* q *with respect to* p *for each* q *in* M-{p}.

PROOF. *(if)* Let U be an open set containing p. An open set V must be found such that $p \in V \subseteq U$ and M-V has finitely many components. For each q in M-{p} there exists a subcontinuum H_q and an open set O_q such that $q \in O_q \subseteq H_q \subseteq M-\{p\}$. The collection $\{O_q : q \in M-U\}$ covers M-U. Since M-U is compact, there is a finite subcollection $\{O_{q_i} : 1 \leq i \leq n\}$ of $\{O_q : q \in M-U\}$ covering M-U, where each O_{q_i} is contained in the corresponding continuum H_{q_i}. Let $H = \cup\{H_{q_i} : 1 \leq i \leq n\}$ and let V = M-H. Then $p \in V \subseteq U$ and M-V has at most n components.

(only if) Let q be a point different from p in M. Then a continuum H must be found such that $q \in int(H) \subseteq H \subseteq M-\{p\}$. There exists an open set U containing p such that q is not in $\overline{U}$. Since M is semi-locally connected at p, there is an open set V such that $p \in V \subseteq U$ and M-V has finitely many components. Let H be the component of M-V at q. Since M-V is closed, it follows that H is a continuum. Because V is contained in U and q is not in $\overline{U}$, we have that q is not a limit point of V. Since M-V is the union of finitely many closed components, it follows that q is not a limit point of M-H. Therefore, $q \in int(H) \subseteq H \subseteq M-\{p\}$.

PROPOSITION 2. *If an irreducible continuum* I *is aposyndetic at* p *with respect to* q, *then* I *is aposyndetic at* q *with respect to* p.

PROOF. Let a and b be points in I such that I is irreducible from a to b. There exists a continuum H such that $p \in int(H) \subseteq H \subseteq I-\{q\}$. Let V = I-H and let K_a and K_b be the components of V at a and b, respectively. By the Boundary Bumping Theorem, it can be shown that $\overline{K}_a \cap H \neq \phi$ and $\overline{K}_b \cap H \neq \phi$ because $bd(V) \subseteq H$. Suppose that there exists a point c in I-H such that c is not in K_a or K_b. Then $\overline{K}_a \cup H \cup \overline{K}_b$ would be a proper subcontinuum of I containing a and b, which contradicts the fact that I is irreducible from a to b. Thus, q is in either K_a or K_b. Assume without loss of generality that q is in K_b. It follows that q is in $\text{int}(\overline{K_b})$ and $\overline{K_b}$ is a continuum not intersecting int(H). Therefore, $q \in \text{int}(\overline{K_b}) \subseteq \overline{K_b} \subseteq I-\{p\}$ and I is aposyndetic at q with respect to p.

LEMMA 2. *If a continuum* I *is irreducible from* a *to* b *and* H_1 *and* H_2 *are disjoint subcontinua of* I *containing* a *and* b, *respectively, then* $I-H_1$, $I-H_2$, *and* $I-(H_1 \cup H_2)$ *are connected.*

PROOF. To show $I-H_1$ is connected, it can be assumed that b is not in H_1; otherwise, $H_1 = I$. Suppose that $I-H_1 = A \cup B$ is a separation with b in B. Since $H_1 \cup B$ is connected and $\overline{B} \cap A = \emptyset$, it follows that $H_1 \cup \overline{B}$ is a proper subcontinuum of I containing a and b, which contradicts the fact that I is irreducible from a to b. Similarly, it can be shown that $I-H_2$ is connected. To show that $I-(H_1 \cup H_2)$ is connected, it can be assumed without loss of any generality that $H_1 \cup H_2 \neq I$ and that b is not in H_1. Observe that $I-(H_1 \cup H_2) = (I-H_1)-H_2$. Suppose that $(I-H_1)-H_2$ is not connected; that is, there exists a separation $S \cup T = (I-H_1)-H_2$. Since $I-H_1$ and H_2 are both connected, it follows that $H_2 \cup S$ and $H_2 \cup T$ are connected. Since I is connected, either $\overline{S} \cap H_1 \neq \emptyset$ or $\overline{T} \cap H_1 \neq \emptyset$. Therefore, either $H_1 \cup \overline{S} \cup H_2$ or $H_1 \cup \overline{T} \cup H_2$ is a proper subcontinuum of I containing a and b, which contradicts the fact that I is irreducible from a to b.

PROPOSITION 3. *If* H *is a subcontinuum of an irreducible continuum* I, *then* int(H) *is connected.*

PROOF. Let a and b be points such that I is irreducible from a to b. Let H_a and H_b be the components of I-int(H) at a and b, respectively. It is clear that $\text{int}(H) \subseteq I-(H_a \cup H_b)$. If $p \in I-(H_a \cup H_b)$, then $p \in H$. For if p is not in H, then $H_a \cup H \cup H_b$ would be a proper subcontinuum of I containing a and b, which contradicts the fact that I is irreducible from a to b. Since $I-H \subseteq H_a \cup H_b$, it follows that p is not a limit point of I-H and $p \in \text{int}(H)$. Therefore, $\text{int}(H) = I-(H_a \cup H_b)$ which is connected from Lemma 2.

PROPOSITION 4. *If* H_1 *and* H_2 *are subcontinua of an irreducible continuum* I *such that* $\text{int}(H_1) \cap \text{int}(H_2) \neq \emptyset$, *then* $H_1 \cap H_2$ *is a continuum.*

PROOF. Let a and b be points such that I is irreducible from a to b. Since I is irreducible, every subcontinuum of I with non-empty interior separates I into two sets, one containing a and one containing b (possibly empty). Let $I-H_1 = A_1 \cup B_1$ and $I-H_2 = A_2 \cup B_2$ be separations where a is in A_1 and A_2 and b is in B_1 and B_2. Let $A = \overline{(A_1 \cup A_2)}$ and let $B = \overline{(B_1 \cup B_2)}$. By Lemma 2, it follows that $I-(A \cup B)$ is connected. It is easy to verify that $H_1 \cap H_2 = \overline{(I-(A \cup B))}$, which is a continuum.

PROPOSITION 5. *If an irreducible continuum* I *is aposyndetic at* p *with respect to* S *and* I *is aposyndetic at* p *with respect to* T, *then* I *is aposyndetic at* p *with respect to* $S \cup T$.

PROOF. There exists continua H_1 and H_2 such that $p \in \text{int}(H_1) \subseteq H_1 \subseteq I-S$ and $p \in \text{int}(H_2) \subseteq H_2 \subseteq I-T$. Therefore, $p \in \text{int}(H_1 \cap H_2) \subseteq H_1 \cap H_2 \subseteq I-(S \cup T)$ and, by Proposition 4, $H_1 \cap H_2$ is a continuum.

PROPOSITION 6. *A continuum* M *is freely decomposable if and only if* M *is aposyndetic.*

PROOF. (*only if*) Given p and q in M, it must be shown that M is aposyndetic at p with respect to q. Since M is freely decomposable, there exists continua A and B such that p is in A-B and q is in B-A and $A \cup B = M$. Since A-B is an open subset of A which contains p, it follows that $p \in \text{int}(A) \subseteq A \subseteq M-\{q\}$.

(*if*) Given p and q in M, it must be shown that there exists subcontinua A and B such that

$p \in A-B$ and $q \in B-A$ and $A \cup B = M$. For each x in $M-\{p\}$ there exists a continuum H_x and an open set U_x such that $x \in U_x \subseteq H_x \subseteq M-\{p\}$. Also, there is a continuum H and an open set U such that $p \in U \subseteq H \subseteq M-\{q\}$. The collection $\mathcal{S} = \{U\} \cup \{U_x : x \in M-p\}$ covers M and therefore there exists a finite subcollection $\{U_i : i = 1,2,\ldots n\}$ of $\mathcal{S}$ that covers M. Let H_i be the continuum associated with U_i for $i = 1,2,\ldots n$. Let $A_1 = \cup\{H_i : 1 \le i \le n \text{ and } p \in U_i\}$ and let $B_1 = \cup\{H_i : 1 \le i \le n$ and $q \in U_i\}$. For each m such that $1 < m \le n$ let $A_m = \cup\{H_i : 1 \le i \le n \text{ and } H_i \cap A_{m-1} \ne \phi\} \cup A_{m-1}$, and let $B_m = \cup\{H_i : 1 \le i \le n \text{ and } H_i \cap B_{m-1} \ne \phi\} \cup B_{m-1}$. Let $A = \cup\{A_m : 1 \le m \le n\}$ and let $B = \cup\{B_m : 1 \le m \le n\}$. It follows from the connectedness of M and the fact that $\{H_i : i = 1,2,\ldots n\}$ covers M that $A \cup B = M$. It is clear that A and B are continua and that $p \in A-B$ and $q \in B-A$, since each H_i for $1 \le i \le n$ contains at most one of p and q.

PROPOSITION 7. *If* C *is a connected proper subset of a continuum* M, *then there exists a point* p *in* $M-C$ *such that* p *is a non-separating point of* M.

PROOF. On the contrary, suppose that C is a connected proper subset of M and every point in $M-C$ separates M. A transfinite sequence of open subsets of M will be constructed such that the collection of these open sets covers M but contains no finite subcollection covering M. This will be a contradiction to the fact that M is compact. Choose $\{x_0\}$ in $M-C$ and let $A_0 \cup B_0$ be a separation of $M-\{x_0\}$ where $C \subseteq A$. Notice that A_0 is open and $A_0 \cup \{x_0\}$ is a continuum. We proceed to define by transfinite induction an open set A_λ for each ordinal λ. If λ is an ordinal such that $\lambda-1$ exists, then let $X_\lambda = \cup\{A_\gamma : \gamma \lneq \lambda\} \cup \{x_\gamma : \gamma < \lambda\}$. If $X_\lambda \ne M$, then choose x_λ in $M-X_\lambda$. Since $C \subseteq A_0 \subseteq X_\lambda$, we have by assumption that x_λ separates M. Let $A_\lambda \cup B_\lambda$ be a separation of $M-\{x_\lambda\}$ where $A_{\lambda-1} \cup \{x_{\lambda-1}\} \subseteq A_\lambda$. If $X_\lambda = M$, then let $A_\lambda = M$. If $\lambda-1$ does not exist (i.e. λ is a limit ordinal), let $A_\lambda = \cup\{A_\gamma : \gamma < \lambda\}$. Notice that $A_\alpha \subseteq A_\beta$ whenever $\alpha < \beta$. Let μ be the first ordinal such that $A_\mu = M$. It follows that μ is a limit ordinal, for suppose that $\mu-1$ exists. Then $A_{\mu-1} \ne M$ and $X_\mu = M$. In this case $X_\mu = A_{\mu-1} \cup \{x_{\mu-1}\} = M$, but $B_{\mu-1} \ne \phi$ which is a contradiction. Therefore, μ is a limit ordinal and $\mathcal{U} = \{A_\lambda : \lambda < \mu\}$ is a cover of M by open sets since $M = A_\mu = \cup\{A_\lambda : \lambda < \mu\}$. Suppose that there is a finite subcollection of $\mathcal{U}$ which covers M. Then let α be the greatest ordinal such that $A_\alpha \in \mathcal{U}$. Since $A_\alpha \ne M$, there exists an x in $M-A_\alpha$. But $A_\delta \subseteq A_\alpha$ for each ordinal δ such that $A_\delta \in \mathcal{U}$. Thus $\mathcal{U}$ does not cover $\{x\}$ and no finite subcollection of $\{A_\lambda : \lambda < \mu\}$ covers M. This is the desired contradiction.

COROLLARY 1. *Every non-degenerate continuum* M *contains at least two non-separating points.*

PROOF. Choose x in M. It follows that $\{x\}$ is a connected proper subset of M and by Proposition 7 there exists p in $M-\{x\}$ such that p is a non-separating point of M. Also $\{p\}$ is a connected proper subset of M and therefore, there exists q in $M-\{p\}$ such that q is a non-separating point of M. Hence, p and q are non-separating points of M.

In Theorem 3, it will be shown that a continuum with exactly two non-separating points is an arc. Thus, an arc has the minimal number of non-separating points for a non-degenerate continuum.

3. MAIN RESULTS.

THEOREM 1. *If* I *is a continuum irreducible from* a *to* b, *then the following are equivalent:*

(a) I *is semi-aposyndetic at* p;

(b) I *is semi-locally connected at* p;

(c) I *is aposyndetic at* p*;*

(d) I *is finitely-aposyndetic at* p*;*

(e) I *is continuum-aposyndetic at* p*;*

(f) I *is connected im kleinen at* p*; and,*

(g) I *is locally connected at* p.

PROOF. (a) implies (b). By Proposition 1, the continuum I is semi-locally connected at p if and only if I is aposyndetic at q with respect to p for each q in I-{p}. From Proposition 2, it follows that I is aposyndetic at q with respect to p for each q in I-{p}.

(b) implies (c). Again this is a consequence of Propositions 1 and 2.

(c) implies (d). This is a consequence of applying induction to the result obtained in Proposition 5.

(d) implies (e). Let H be a subcontinuum of I-{p}. Since I is aposyndetic at p, it follows from Proposition 5 that I is aposyndetic at x with respect to p for each x in H. Thus for each x in H there exists a continuum H_x such that $x \in \text{int}(H_x) \subseteq H_x \subseteq I-\{p\}$. If $U_x = \text{int}(H_x)$ for each x in H, then $\{U_x : x \in H\}$ is an open cover of H which has a finite subcover $\{U_{x_i} : i = 1,2,\ldots,k\}$. For $i = 1,2,\ldots,k$ let C_i be the component of $I-U_{x_i}$ at p. For each $i = 1,2,\ldots,k$ there are at most two components of $I-U_{x_i}$, else H together with the components of $I-U_{x_i}$ at a and b would be a proper subcontinuum of I containing a and b. Thus, since p is not in H, it follows that $p \in \text{int}(C_i)$ for $i = 1,2,\ldots,k$. Applying induction to the result obtained in Lemma 4, we find that $C = \cap\{C_i : i = 1,2,\ldots,k\}$ is a continuum containing p in its interior and $C \subseteq I-H$. Hence, I is aposyndetic at p with respect to H.

(e) implies (f). Since I is continuum-aposyndetic at p, it is clear that I is semi-aposyndetic at p and, hence, I is semi-locally connected at p. Therefore, if U is any open set containing p there is an open set V such that I-V has finitely many components $\{H_i : i = 1,2,\ldots,k\}$ and $p \in V \subseteq U$. Each H_i $(i = 1,2,\ldots,k)$ is a continuum so I is aposyndetic at p with respect to H_i $(i = 1,2,\ldots,k)$. Applying induction to Proposition 5, we have that I is aposyndetic at p with respect to $\cup\{H_i : i = 1,2,\ldots,k\}$. Thus, there exists a continuum H such that $p \in \text{int}(H) \subseteq H \subseteq I-\cup\{H_i : i = 1,2,\ldots,k\} = V \subseteq U$.

(f) implies (g). If U is any open set containing p, then there exists a continuum H such that $p \in \text{int}(H) \subseteq H \subseteq U$. From Proposition 4, it follows that int(H) is an open connected subset of U containing p.

(g) implies (a). Let q be a point different from p in I. There exists an open set U such that $p \in U$ and $\overline{U} \subseteq I-q$ and an open set V such that $p \in V \subseteq U$ and V is connected. Thus, $\overline{V}$ is the required continuum to show that I is aposyndetic at p with respect to q.

THEOREM 2. *If* I *is an irreducible continuum, then the following are equivalent:*

(a) I *is an arc;*

(b) I *is semi-aposyndetic;*

(c) I *is semi-locally connected;*

(d) I *is aposyndetic;*

(e) I *is finitely-aposyndetic;*

(f) I *is continuum-aposyndetic;*

(g) I *is connected im kleinen; and,*

(h) I *is locally connected.*

PROOF. It is clear that every arc is semi-aposyndetic. The equivalency of statements *(b)* through *(h)* follows immediately from Theorem 1. It remains to be shown that a continuum I which is irreducible from a to b and locally connected is an arc. We will first define a binary relation "$\prec$" between each pair of points of I and show that this relation is a linear order on I (see [1], Axiom 1, page 5). It will then be shown that the topology generated by the relation "$\prec$" agrees with the usual topology on I and that "$\prec$" defines a first point and a last point in I. Since every continuum is separable, we will have established that I is a connected, separable, linearly ordered topological space having a first point and a last point. Hence, recalling the remark that was made after Definition 2 in Section 1, it follows that I is an arc.

From Proposition 6 we have that I is freely decomposable. Therefore, given two distinct points x and y in I, there exists subcontinua H_x and H_y such that $I = H_x \cup H_y$ and $x \in H_x - H_y$ and $y \in H_y - H_x$. Since I is irreducible from a to b either $a \in H_x$ and $b \in H_y$ or $a \in H_y$ and $b \in H_x$. If $a \in H_x$, then define $x \prec y$ and if $a \in H_y$, then define $y \prec x$. Suppose $x \prec y$. This means there exists subcontinua A_x and B_y such that $I = A_x \cup B_y$ and x and a are in $A_x - B_y$ and y and b are in $B_y - A_x$. Suppose also that $y \prec x$. Then there exists subcontinua A_y and B_x such that $I = A_y \cup B_x$ and y and a are in $A_y - B_x$ and x and b are in $B_x - A_y$. It follows that $A_x \cup B_x$ is a proper subcontinuum containing a and b which contradicts the fact that I is irreducible from a to b. Thus, if $x \prec y$, then it is not true that $y \prec x$. It is clear that for any distinct x and y in I, either $x \prec y$ or $y \prec x$. In addition, if $x \prec y$, then x is diferent from y. It remains to be shown that if $x \prec y$ and $y \prec z$, then $x \prec z$. If $x \prec y$, then there exists subcontinua A_x and B_y such that a and x are in $A_x - B_y$ and b and y are in $B_y - A_x$, and $I = A_x \cup B_y$. If $y \prec z$, then there exist subcontinua A_y and B_z such that a and y are in $A_y - B_z$ and b and z are in $B_z - A_y$, and $I = A_y \cup B_z$. If x is in $A_y - B_z$, then it follows that $x \prec z$. But if x is in B_z, then $A_x \cup B_z$ is a subcontinuum that contains a and b and does not contain y. This contradicts the fact that I is irreducible from a to b.

It will now be shown that the topology generated by the relation "$\prec$" agrees with the usual topology on I. For any x and y in I let $R_{xy} = \{p : x \prec p \prec y\}$. If p is in R_{xy}, then there exists continua H_{ax} and H_{pb} such that a and x are in $H_{ax} - H_{pb}$ and p and b are in $H_{pb} - H_{ax}$ with $I = H_{ax} \cup H_{pb}$. Thus $p \in \text{int}(H_{pb})$ and if $q \in H_{pb}$ then $x \prec q$. Also, there exists continua H_{ap} and H_{yb} such that a and p are in $H_{ap} - H_{yb}$ and y and b are in $H_{yb} - H_{ap}$ with $I = H_{ap} \cup H_{yb}$. Thus $p \in \text{int}(H_{ap})$ and if $q \in H_{ap}$ then $q \prec y$. It follows that $p \in \text{int}(H_{pb}) \cap \text{int}(H_{ap}) \subseteq R_{xy}$ is open in I.

It remains to be shown that if p is an open subset U of I, then there exist x and y in I such that $p \in R_{xy} \subseteq U$. First, if H is a subcontinuum of I and $x \prec y$ in H, then $R_{xy} \subseteq H$. For suppose there exists p in R_{xy} such that p is not in H. Since $x \prec p \prec y$, there exists a continuum H_{ax} containing a and x that does not contain p and a continuum H_{yb} containing y and b that does not contain p. It follows that $H_{ax} \cup H \cup H_{yb}$ is a proper subcontinuum of I containing a and b, which contradicts the fact that I is irreducible from a to b. Also, if H is a subcontinuum of I, then H has at laest two non-interior points. For suppose there exist distinct x,y,z in H-int(H). Without loss of generality, assume that $x \prec y \prec z$. Then $y \in R_{xz} \subseteq H$ and since R_{xz} is open, it follows that $y \in \text{int}(H)$. In addition, if p is in I, then p is a limit point of $R = \{x : p \prec x\}$. For suppose p is not a limit point of R and let $L = \{x : x \prec p \text{ or } x = p\}$. Since no point of L is a limit point of R and R is open, it follows that $L \cup R$ is a separation of I which contradicts the fact that I is connected. Similarly, p is a limit point of $\{x : x \prec p\}$.

Let p be a point in an open set U of I. From Theorem 1, I is semi-locally connected and thus there exists an open subset V of U containing p such that I-V has finitely many components $\{H_i : i = 1,2,\ldots,n\}$. If $z \in bd(V)$, then $z \in I-V$ and z is not in $int(H_i)$ for $1 \le i \le n$. Since $bd(V) = \bigcup\{bd(H_i) : i = 1,2,\ldots,n\}$ and $card(bd(H_i)) \le 2$ for $1 \le i \le n$, it follows that bd(V) is finite. Therefore, let x and y be the points in bd(V) such that $x \prec p \prec y$ and if z is in bd(V) and $x \ne z \ne y$, then $z \prec x$ or $y \prec z$. It follows that $R_{xy} \subseteq V$. For suppose on the contrary that there exists z in R_{xy} such that z is in H_i for some $i \le n$. Since $z \in R_{xy}$, then $z \in bd(H_i)$. Assume without loss of generality that if $q \in H_i$, then $q \prec p$. For each h in H_i let $U_h = \{q : q \prec h\}$. Each U_h is open. For each h in $H_i \cap R_{xy}$ there exists $h' \in H_i$ such that $h \prec h' \prec p$. For if no point in H_i is between h and p, then h would be in $bd(H_i)$ and thus bd(V), which is a contradiction. The collection $\{U_h : h \in H_i\}$ covers H_i, for if $h \in H_i$ then there exists h' such that $h \prec h'$. Suppose $\{U_{h_k} : k = 1,2,\ldots,j\}$ is a finite subcollection of $\{U_h : h \in H_i\}$ which covers H_i. Then there exists $m \le j$ such that $h_k \prec h_m$ or $h_k = h_m$ for $k \le j$. It would follow that $h_m \in R_{xy} \cap bd(H_i)$, which is a contradiction. Therefore, no finite subcollection of $\{U_h : h \in H_i\}$ covers H_i which contradicts the fact that H_i is compact. Thus $p \in R_{xy} \subseteq V \subseteq U$ and the theorem is proved.

THEOREM 3. *A continuum* M *is an arc if and only if* M *has exactly two non-separating points.*

PROOF. *(only if)* It is easy to see that an arc has exactly two non-separating points, namely its two endpoints.

(if) Let p and q be the two non-separating points of M. It follows that M is irreducible from p to q. For if on the contrary there was a proper subcontinuum I of M containing p and q, then by Proposition 7 M-I would contain a non-separating point of M, which is a contradiction. It will be shown that M is aposyndetic and it will then follow from Theorem 2 that M is an arc. Given x and y in M, it must be shown that M is aposyndetic at x with respect to y. If x is different from p and q, then let $A_1 \cup B_1$ be a separation of M-{x} where $p \in A_1$ and $q \in B_1$. If x is the same as p or q, then let $A_1 = \phi$ and $B_1 = M-\{x\}$ or let $A_1 = M-\{x\}$ and $B_1 = \phi$, respectively. Assume, for the moment, that $y \in B_1$. If y is different from p and q, then let $A_2 \cup B_2$ be a separation of M-{y} where $p \in A_2$ and $q \in B_2$. If y is the same as p or q, then let $A_2 = \phi$ and $B_2 = M-\{y\}$ or let $A_2 = M-\{y\}$ and $B_2 = \phi$, respectively. It follows that $(A_1 \cup \{x\}) \cap (B_2 \cup \{y\}) = \phi$, for if not, since $A_1 \cup \{x\}$ is connected, $A_1 \cup \{x\}$ would have to lie entirely in B_2. Then $B_2 \cup \{y\}$ would be a proper subcontinuum of M containing p and q, which contradicts the fact that M is irreducible from p to q. Hence, there exists z in $M-(A_1 \cup \{x\} \cup B_2 \cup \{y\})$. Let $A_3 \cup B_3$ be a separation of M-{z} where $A_1 \cup \{x\} \subseteq A_3$ and $B_2 \cup \{y\} \subseteq B_3$. Thus, $A_3 \cup \{z\}$ is a continuum with x in A_3 and A_3 is an open set. Therefore, let $H = A_3 \cup \{z\}$ and it follows that $x \in int(H) \subseteq H \subseteq M-\{q\}$. A similar arguement holds if y is in A_1 instead of B_1. Hence, M is aposyndetic at x with respect to y and M is an arc.

REFERENCES

1. Boyd, J. R., *Introduction to Elementary Point Set Theory*, Department of Mathematics, Guilford College, Greensboro, N. C.
2. Hocking, J. G. and Young, G. S., *Topology*, Addison-Wesley, Reading, Mass., 1961.
3. Jones, F. B., "Concerning aposyndetic and non-aposyndetic continua", *Bulletin of the Amer-*

ican Mathematical Society, 58(1952), 137-151.

4. Jones, F. B., "Aposyndesis revisited", *Proceedings of the University of Oklahoma Topology Conference*, (1972), 64-78.

C. Bruce Hughes *(senior)*
Guilford College

This paper was written under the direction of Dr. G. R. Gordh, Jr.

[1] This paper was presented at the FIRST ANNUAL CONFERENCE ON UNDERGRADUATE MATHEMATICS held at Guilford College, November 13-15, 1975.

The *Journal of* UNDERGRADUATE MATHEMATICS is published in cooperation with the mathematics departments of the following institutions.

AMHERST COLLEGE, *Amherst, Massachusetts*
BOWDOIN COLLEGE, *Brunswick, Maine*
CARLETON COLLEGE, *Northfield, Minnesota*
DARTMOUTH COLLEGE, *Hanover, New Hampshire*
DENISON UNIVERSITY, *Granville, Ohio*
EMORY UNIVERSITY, *Atlanta, Georgia*
FORT LEWIS COLLEGE, *Durango, Colorado*
GRINNELL COLLEGE, *Grinnell, Iowa*
GUILFORD COLLEGE, *Greensboro, North Carolina*
HAMILTON COLLEGE, *Clinton, New York*
HARVEY MUDD COLLEGE, *Claremont, California*
HENDRIX COLLEGE, *Conway, Arkansas*
HOPE COLLEGE, *Holland, Michigan*
UNIVERSITY OF HOUSTON, *Houston, Texas*
UNIVERSITY OF KENTUCKY, *Lexington, Kentucky*
KENYON COLLEGE, *Gambier, Ohio*
KNOX COLLEGE, *Galesburg, Illinois*
LAKE FOREST COLLEGE, *Lake Forest, Illinois*
NEW COLLEGE, *Sarasota, Florida*
OAKLAND UNIVERSITY, *Oakland, Michigan*
RUTGERS UNIVERSITY, *New Brunswick, New Jersey*
UNIVERSITY OF SANTA CLARA, *Santa Clara, California*
UNIVERSITY OF TORONTO, *Toronto, Canada*
WESLEYAN UNIVERSITY, *Middletown, Connecticut*
WILLIAMS COLLEGE, *Williamstown, Massachusetts*
UNIVERSITY OF WISCONSIN, *Madison, Wisconsin*