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A METRIC FOR THE PRODUCT OF COUNTABLY MANY SETS

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INTRODUCTION. In a topology seminar at Guilford College, Dr. Elwood Parker asked the question: is the product of countably many metric spaces metrizable? While working on this question, the author discovered a metric that did not induce the usual product topology in all cases, but was a metric that could be applied to the product of countably many sets. Further investigation of this metric revealed many interesting properties (for example, the metric is an ultrametric and the space is totally disconnected, perfect, complete, and homogeneous). Also, different ways of obtaining known results were found (for example, the Cantor set is homogeneous and the irrationals are topologically complete). In addition, it was found that the topology generated by the metric agrees with the usual product topology when each of the countably many sets is assumed to have the discrete topology. For definitions not in this paper, refer to [2].

DEFINITION OF THE METRIC. For any infinite sequence of nondegenerate sets $\{X_i\}$, let $X = \Pi X_i$. An element x in X will be denoted by its coordinates in the following manner: $x = (x_i) = (x_1, x_2, x_3, \ldots)$ where x_n is in X_n for each natural number n.

If x and y are distinct elements in X, then define the distance between x and y by $d(x,y) = 1/n$ where n is the first natural number such that x_n differs from y_n. If x and y are the same element in X, then define $d(x,y) = 0$. In Theorem 1 it will be shown that d is a metric for X. Throughout this paper, (X,d) and $(\Pi X_i,d)$ will refer to any metric space defined in this manner.

Note that in (X,d) an ε-neighborhood $N(x,\varepsilon)$ is simply a "1/n-neighborhood" $N(x,1/n)$ where n is the largest natural number such that $1/n \geq \varepsilon$. Also, $N(x,1/n)$ contains any element y in X such that $x_i = y_i$ whenever $1 \leq i \leq n$.

DEFINITION 1. A metric ρ on a set S is an *ultrametric* if $\rho(x,z) \leq \max\{\rho(x,y), \rho(y,z)\}$ for any x,y,z in S.

Ultrametrics are an interesting topic in themselves and the reader is refered to [3] for an excellent introduction to the subject by J. E. Vaughan.

THEOREM 1. d *is an ultrametric for* X.

PROOF. For any x,y in X it is clear that:

1. $d(x,y) \geq 0$;
2. $d(x,y) = 0$ if and only if $x = y$; and,
3. $d(x,y) = d(y,x)$.

Observe that the triangle inequality is implied by the ultrametric inequality: $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for any x,y,z in X. Clearly this inequality holds if $x = z$. If $x \neq z$, then $d(x,z) = 1/n$ for some natural number n. Hence, if $d(x,y) \geq 1/n$, the inequality holds. If $d(x,y) < 1/n$, then $x_i = y_i$ whenever $1 \leq i \leq n$. Since $x_n = y_n$ and $x_n \neq z_n$, it follows that $y_n \neq z_n$. Therefore,

$d(y,z) \geq 1/n$ and the inequality holds.

The following theorem gives one of the odd properties of ultrametrics.

THEOREM 2. *If* ρ *is an ultrametric on a set* S, *then for any* x,y *in* S *and any* $\varepsilon,\delta > 0$ *such that* $\varepsilon \geq \delta$, *either* $N(x,\varepsilon) \cap N(y,\delta) = \phi$ *or* $N(x,\varepsilon) \cap N(y,\delta) = N(y,\delta)$.

PROOF. It will be shown that if $N(x,\varepsilon) \cap N(y,\delta) \neq \phi$, then $\rho(q,x) < \varepsilon$ for any q in $N(y,\delta)$. Let z be in $N(x,\varepsilon) \cap N(y,\delta)$. Then $\rho(q,x) \leq \max\{\rho(q,y), \rho(y,x)\} \leq \max\{\rho(q,y), \max\{\rho(y,z), \rho(z,x)\}\} \leq \max\{\rho(q,y), \rho(y,z), \rho(z,x)\} < \max\{\delta,\delta,\varepsilon\} = \varepsilon$.

Also, observe that the ε-neighborhoods in (S,ρ) are closed.

For any sequence of topological spaces $\{X_i\}$, the product space of $\{X_i\}$ with the usual (Tychonoff) topology $\mathcal{T}$ will be denoted as Π^*X_i. The basis for Π^*X_i consists of the sets Π^*U_i where each U_i is open in X_i and $U_i = X_i$ for all but finitely many indexing elements i.

THEOREM 3. *The space* $(\Pi X_i,d)$ *is homeomorphic to* Π^*X_i *where each* X_i *is assumed to have the discrete topology.*

PROOF. It will be shown that if U is an ε-neighborhood in $(\Pi X_i,d)$, then $U \in \mathcal{T}$; and, if V is a basis element for $\mathcal{T}$, then V is the union of ε-neighborhoods in $(\Pi X_i,d)$.

Let $U = N(p,1/n)$ for some p in ΠX_i. Then $U = \Pi U_i$ where $U_i = \{p_i\}$ whenever $1 \leq i \leq n$ and $U_i = X_i$ whenever $i > n$. Thus, $U \in \mathcal{T}$.

Let $V = \Pi V_i$ where each V_i is open in X_i and $V_i = X_i$ for all but finitely many natural numbers i. Let $n = \max\{ i : V_i \neq X_i\}$. Then $V = \bigcup\{N(p,1/n) : p \in V\}$.

An immediate result of Theorem 3 is that the order of the factors of ΠX_i does not matter. This is true because the order of the factors of Π^*X_i does not matter.

DEFINITION 2. A topological space S is *totally disconnected* if S has no nondegenerate connected subsets.

DEFINITION 3. A topological space S is *perfect* if every point of S is a limit point of S.

DEFINITION 4. A metric space S is *complete* if every Cauchy sequence of points in S converges to a point in S.

THEOREM 4. (X,d) *is (a) totally disconnected, (b) perfect, and (c) complete.*

PROOF. (a) It is shown that any nondegenerate subset M of X is the union of two mutually separated sets. Let n be the first natural number such that there exist p,q in M with $p_n \neq q_n$. Let $A = M \cap N(p,1/n)$ and $B = M - A$. Clearly A and B are disjoint and, since $p \in A$ and $q \in B$, each is non-empty. Observe that $A \subseteq N(p,1/n)$ and if $b \in B$, then $N(b,1/n) \cap N(p,1/n) = \phi$. Since $M = A \cup B$, it follows that M is the union of two mutually separated sets.

(b) For any p in X and any natural number n, it must be shown that $N(p,1/n)$ contains some $q \neq p$ in X. Pick q such that $q_i = p_i$ for all natural numbers i except $i = n+1$. Let q_{n+1} be different from p_{n+1}. It follows that $q \in N(p,1/n)$ and $q \neq p$.

(c) Let $\{x_k\} = \{(x_i)_k\}$ be a Cauchy sequence in X. For each natural number j, there eists a natural number N(j) such that if $n \geq N(j)$, then $d(x_n,x_{N(j)}) < 1/j$; and hence, $(x_j) = (x_j)_{N(j)}$ whenever $n \geq N(j)$. There exists a point p in X such that $p_j = (x_j)_{N(j)}$ for each natural number j. The sequence $\{x_k\}$ converges to p because for each natural number j, $N(p,1/j)$ contains x_k for all $k \geq \max\{N(i) : 1 \leq i \leq j\}$. In words, for any natural number j, the terms of $\{x_k\}$ can be made to agree in the first j coordinates by going sufficently far out in the sequence.

DEFINITION 5. A topological space S is *countably compact* if every infinite subset of S has a limit point in S.

In metric spaces, compactness is equivalent to countable compactness (see [1], page 19).

THEOREM 5. *(X,d) is compact if and only if X_i is a finite set for each natural number* i.

PROOF. *(only if)* Suppose there exists a natural number n such that X_n is infinite. Choose a sequence of distinct terms $\{(x_i)_k\}$ such that $(x_n)_k \neq (x_n)_j$ whenever $k \neq j$. Since for any p in X, N(p,1/n) contains at most one term of $\{(x_i)_k\}$, it follows that $\{(x_i)_k\}$ does not have a limit point in X, which contradicts the countable compactness of (X,d).

(if) If X_i is a finite set for each natural number i, let $\{(x_i)_k\}$ be any sequence of distinct terms in X. There exists p_1 in X_1 such that $p_1 = (x_1)_k$ for countably many natural numbers k. Assuming there exist p_1 in X_1, p_2 in $X_2, \ldots, p_n$ in X_n such that $p_i = (x_i)_k$ whenever $1 \leq i \leq n$ for countably many k, it is clear that there exists p_1 in X_1, p_2 in $X_2, \ldots, p_{n+1}$ in X_{n+1} such that $p_i = (x_i)_k$ whenever $1 \leq i \leq n+1$ for countably many k. It follows that $p = (p_i)$ is in X and that p is a limit point of $\{(x_i)_k\}$ because, for each natural number n, N(p,1/n) contains countably many terms of $\{(x_i)_k\}$. Thus, every infinite subset of X has a limit point in X.

DEFINITION 6. A topological space S is *locally compact* if for every point in S there exists an open set U containing that point such that the closure of U is compact.

THEOREM 6. *(X,d) is locally compact if and only if there exists a natural number* n *such that X_i is a finiet set for all* $i \geq n$.

PROOF. *(only if)* A proof by contradiction can be constructed similar to the one in Theorem 5. *(if)* Again following the proof for Theorem 5, it can be shown that N(x,1/n) is compact for each x in X.

THEOREM 7. *(X,d) is separable if and only if X_i is a countable set for each natural number* i.

PROOF. *(only if)* Suppose there exists a natural number n such that X_n is uncountable. Let C be a countable dense subset of X. There exists p in X-C such that $p_n \neq c_n$ for all c in C. Since N(p,1/n) contains no point of C, it follows that p is not a limit point of C which is a contradiction to the denseness of C.

(if) If X_i is countable for each i, then fix a point p in X. Let x belong to the set D if and only if there exists a natural number N such that if $n \geq N$, then $x_n = p_n$. It is clear that D is countable. D is dense because N(q,1/n) contains the point $(q_1, q_2, \ldots, q_n, p_{n+1}, p_{n+2}, \ldots)$ which is in D where q is any point in X and n is any natural number.

DEFINITION 7. A topological space S is *locally separable* if for every point p in S there exists an open set U containing p such that U is separable.

THEOREM 8. *(X,d) is locally separable if and only if there exists a natural number* n *such that X_i is countable for all* $i \geq n$.

PROOF. *(only if)* As in Theorem 7, it can be shown that no neighborhood has a countable dense subset by assuming there exist infinitely many natural numbers i such that X_i is infinite. *(if)* By following the methods of Theorem 7, a countable dense subset of N(x,1/n) can be constructed for each x in X.

The following definitions and lemma are introduced in order to prove a strong homogeneity property of (X,d).

DEFINITION 8. A topological space S is *homogeneous* if given any two points x and y in S, there exists a homeomorphism h from S onto S such that $h(x) = y$.

DEFINITION 9. A function f from a set S onto S is an *involution* if $f \circ f(x) = x$ for each x in S.

DEFINITION 10. A function f from a metric space (S,ρ) into a metric space (S',ρ') is an *isometric embedding* if $(x,y) = \rho'(f(x),f(y))$ for each x and y in S. If f is an isometric embedding of S onto S', then f is an *isometry*.

Notice that if f is an isometry, then f is a homeomorphism.

LEMMA 1. *For each natural number* i, *let* X_i *and* Y_i *be nondegenerate sets and let* $f_i : X_i \to Y_i$ *be a function. Define a function* f *from* $(\Pi X_i, d)$ *into* $(\Pi Y_i, d)$ *by* $f(x) = (f_i(x_i))$ *for each* x *in* ΠX_i. *Then*

(*a*) $d(x,y) \geq d(f(x),f(y))$ *for each* x *and* y *in* ΠX_i;

(*b*) f *is continuous;*

(*c*) *if each* f_i *is onto, then* f *is onto; and,*

(*d*) *if each* f_i *is* 1-1, *then* f *is an isometric embedding.*

PROOF. (*a*) If $x = y$, then $f(x) = f(y)$ and $d(x,y) = d(f(x),f(y)) = 0$. If $x \neq y$, then $d(x,y) = 1/n$ for some natural number n. It follows that $x_i = y_i$ whenever $1 \leq i \leq n-1$. Thus, $f_i(x_i) = f_i(y_i)$ whenever $1 \leq i \leq n-1$ and $d(f(x),f(y)) \leq 1/n$.

(*b*) follows directly from (*a*).

(*c*) If y is in ΠY_i, then for each i there exists x_i such that $f_i(x_i) = y_i$. Thus $f(x) = y$.

(*d*) If $x = y$, then $f(x) = f(y)$ and $d(x,y) = d(f(x),f(y)) = 0$. If $x \neq y$, then $d(f(x),f(y)) \leq d(x,y) = 1/n$ for some natural number n. Since $x_n \neq y_n$, it follows that $f_n(x_n) \neq f_n(y_n)$ and $d(f(x),f(y)) = 1/n$.

THEOREM 9. *For any* x *and* y *in* (X,d) *there exists a function* f *from* X *onto* X *that takes* x *to* y *such that* f *is both an involution and an isometry.*

PROOF. For any p in X define $f(p) = (f_i(p_i))$ by

$$f_i(p_i) = \begin{cases} p_i & \text{if } x_i \neq p_i \neq y_i \\ x_i & \text{if } p_i = y_i \\ y_i & \text{if } p_i = x_i \end{cases}$$

It is easy to see that $f(x) = y$ and $f \circ f(p) = p$ for all p in X. Since each f_i is onto and 1-1, it follows from Lemma 1 that f is an isometry.

An immediate consequence of Theorem 9 is that (X,d) is always homogeneous. The following corollary is a consequence of Lemma 1.

COROLLARY 1. *For each natural number* i, *let* X_i *and* Y_i *be nondegenerate sets such that* $\mathrm{card}(X_i) \leq \mathrm{card}(Y_i)$. *Then there exists an isometric embedding of* $(\Pi X_i, d)$ *into* $(\Pi Y_i, d)$ *and a continuous function from* $(\Pi Y_i, d)$ *onto* $(\Pi X_i, d)$.

PROOF. Since $\mathrm{card}(X_i) \leq \mathrm{card}(Y_i)$, it is clear that there exists a 1-1 function $f_i : X_i \to Y_i$ for each natural number i. By Lemma 1(*d*), $f : (\Pi X_i, d) \to (\Pi Y_i, d)$ defined by $f(x) = (f_i(x_i))$ is an isometric embedding. Also, for each natural number i, there exists an onto function $g_i : Y_i \to X_i$.

By Lemma 1(*b*) and (*c*), $g:(\Pi Y_i,d) \to (\Pi X_i,d)$ defined by $g(x) = (g_i(x_i))$ is a continuous onto function.

THEOREM 10. *Let* $\times$ *be the product with the usual topology. Then* $(\Pi X_i,d) \times (\Pi Y_i,d)$ *is homeomorphic to* $(\Pi(X_i \times Y_i),d)$.

PROOF. If z is in $(\Pi X_i,d) \times (\Pi Y_i,d)$, then $z = [x,y] = [(x_1, x_2,\ldots),(y_1, y_2,\ldots)]$. If z is in $(\Pi(X_i \times Y_i),d)$, then $z' = [(x_1,y_1), (x_2,y_2),\ldots]$. Observe that open sets of the form $N(x,1/n) \times N(y,1/n)$ form a basis for the topology on $(\Pi X_i,d) \times (\Pi Y_i,d)$. Define $h:(\Pi X_i,d) \times (\Pi Y_i,d) \to (\Pi(X_i \times Y_i),d)$ by $h(z) = z'$. Obviously, h is 1-1 and onto. If $U = N(z',1/n)$, then $h^{-1}(U) = N(x,1/n) \times N(y,1/n)$. Thus, h is continuous. If $V = N(x,1/n) \times N(y,1/n)$, then $h(V) = N(h[x,y],1/n) = N(z',1/n)$. Thus, h is open.

The following two lemmas are needed to show that, if X_i is countably infinite for each natural number i, then (X,d) is homeomorphic to the irrationals.

LEMMA 2. *If* D *is acountable dense subset of the real line* R, *then* R-D *is homeomorphic to the set of irrational numbers.*

This was a result obtained in acourse taught by J. R. Boyd at Guilford College (see [1],p.42).

LEMMA 3. *There exists a countable collection* $\mathcal{J} = \{I^n(m_1,m_2,\ldots,m_n)$:n *is a natural number,* $m_1,m_2,\ldots,m_n$ *are integers} each element of which is a closed interval in the real line* R *having the following properties:*

(*a*) $I^{n+1}(m_1,m_2,\ldots,m_n,m_{n+1})$ *is a proper subset of* $I^n(m_1,m_2,\ldots,m_n)$ *and if* $I^{n+1}(j_1,j_2,\ldots,j_{n+1}) \cap I^n(m_1,m_2,\ldots,m_n) \neq \phi$, *then* $j_i = m_i$ *whenever* $1 \le i \le n$.

(*b*) *The set* E *of all endpoints of members of* $\mathcal{J}$ *is a countable dense subset of* R.

(*c*) *For each* n, *the collection* $\{I^n(m_1,m_2,\ldots,m_n):m_1,m_2,\ldots m_n$ *are integers}* *covers* R-E.

(*d*) *For any infinite sequence of integers* $\{m_1,m_2,\ldots\}$, $\bigcap\{I^n(m_1,m_2,\ldots,m_n)$:n *is a natural number} exists and is a singleton set not in* E *and is distinct from* $\bigcap\{I^n(j_1,j_2,\ldots,j_n)$:n *is a natural number where* $\{j_1,j_2,\ldots\}$ *is any infinite sequence of integers different from* $\{m_1,m_2,\ldots\}$.

(*e*) *If* x *is in* R-E, *then there exists an infinite sequence of integers* $\{x_1,x_2,\ldots\}$ *such that* $x = \bigcap\{I^n(x_1,x_2,\ldots,x_n)$:n *is a natural number}.*

(*f*) *The members of* $\mathcal{J}$ *intersected with* R-E *form a basis for the usual topology on* R-E *as a subspace of* R.

PROOF. For n = 1 the members of $\mathcal{J}$ will consist of the closed intervals having succesive integers as endpoints. That is, $I^1(m_1) = [m_1,m_1+1]$ for any integer m_1 (see Figure 1). For n = 2 and any integers m_1 and m_2, let the interval $I^2(m_1,m_2)$ be determined in the following manner:

$I^2(m_1,0) = [m_1+1/2,m_1+3/4]$, $I^2(m_1,1) = [m_1+3/4,m_1+7/8]$, $I^2(m_1,2) = [m_1+7/8,m_1+15/16],\cdots$

$I^2(m_1,-1) = [m_1+1/4,m_1+1/2]$, $I^2(m_1,-2) = [m_1+1/8,m_1+1/4],\cdots$

For higher values of n, the process becomes more complex to describe, but it is clear from the figure how the intervals can be determined. The following are examples of the intervals for n = 3:

$I^3(m_1,0,0) = [m_1+1/2+1/8,m_1+1/2+1/8+1/16] = [m_1+5/8,m_1+11/16]$.

$I^3(m_1,1,0) = [m_1+1/2+1/4+1/16,m_1+1/2+1/4+1/16+1/32] = [m_1+13/16,m_1+27/32]$.

$I^3(m_1,0,-1) = [m_1+1/2+1/16,m_1+1/2+1/8] = [m_1+9/16,m_1+5/8]$.

It is easily seen that properties (*a*) through (*c*) and (*f*) hold.

(*d*) If $\{m_1,m_2,\ldots\}$ is an infinite sequence of integers, then $\bigcap\{I^n(m_1,m_2,\ldots,m_n)$:n is a natural number} exists because the intersection of nested closed intervals always exists. The intersection is a singleton set because the diameter of $I^n(m_1,m_2,\ldots,m_n)$ approaches zero as n approaches

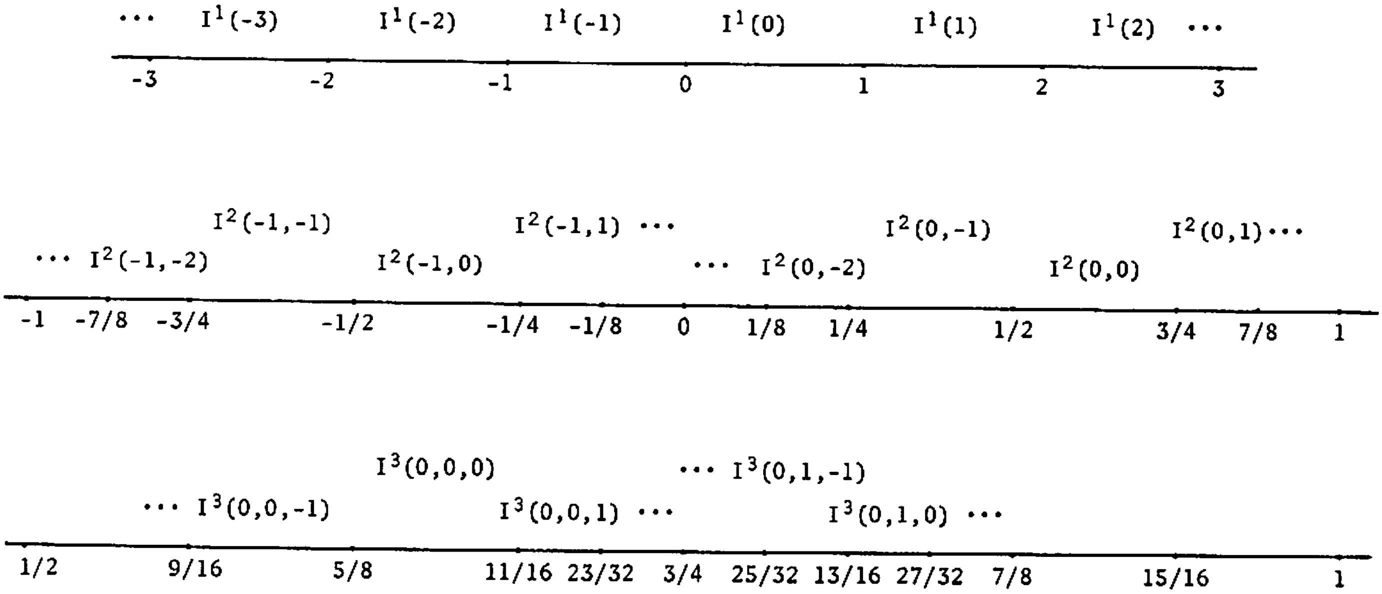

FIGURE 1

infinity. The intersection is not in E and is distinct from $\cap\{I^n(j_1,j_2,\ldots,j_n)$:n is a natural number} where $\{j_1,j_2,\ldots\}$ is any infinite sequence of integers different from $\{m_1,m_2,\ldots\}$ because of *(a)*.

(e) Because of *(c)*, for each n there exist integers $x_1,x_2,\ldots,x_n$ such that x is in $I^n(x_1,x_2,\ldots,x_n)$ where x is in R-E. Because of *(d)*, $x = \cap\{I^n(x_1,x_2,\ldots,x_n):n = 1,2,\ldots\}$.

THEOREM 11. *For each natural number* i, *let* X_i *be a countably infinite set. Then* (X,d) *is homeomorphic to the irrational numbers.*

PROOF. Since each X_i is countably infinite, X_i is thought of as consisting of the integers. Thus, the points in X are sequences of integers $(x_1,x_2,\ldots)$. Let $\mathcal{I}$ and E be as defined in Lemma 3. A homeomorphism h is defined from X onto R-E. Because of Lemma 2, R-E = S is homeomorphic to the irrationals. If $(x_1,x_2,\ldots)$ is a point in X, define $h((x_1,x_2,\ldots)) = \cap\{I^n(x_1,x_2,\ldots x_n)$: $n = 1,2,\ldots\}$. Because of Lemma 3*(e)*, h is onto. Because of Lemma 3*(d)*, h is 1-1. To show that h is continuous, by Lemma 3*(f)* it suffices to show $h^{-1}(S \cap I^n(p_1,p_2,\ldots,p_n))$ is open in X for any natural number n and any integers $p_1,p_2,\ldots,p_n$. It is clear that $h^{-1}(S \cap I^n(p_1,p_2,\ldots,p_n)) = \{x \text{ in } X: x_i = p_i \text{ for } 1 \le i \le n\} = N(p,1/n)$ where p is a point such that the first n coordinates of p are $p_1,p_2,\ldots,p_n$. To show h is open, it suffices to show h(N(p,1/n)) is open in S where n is a natural number and p is a point in X. It is clear that $h(N(p,1/n)) = \{h(x): x_i = p_i \text{ for } 1 \le i \le n\} = S \cap I^n(p_1,p_2,\ldots,p_n)$ which is open in S by Lemma 3*(f)*.

COROLLARIES ABOUT THE CANTOR SET AND THE IRRATIONALS. Let C be the Cantor set and let I be the irrationals. Recall that × denotes the product with the usual topology.

(1). (X,d) *is homeomorphic to* C *if and only if* X_i *is finite for each natural number* i.

PROOF. Any compact totally disconnected perfect metric space is homeomorphic to C ([2], Corollary 2-98, page 100). Recall that Theorem 5 states that (X,d) is compact if and only if X_i is finite for each natural number i.

(2). *The usual product of countably many finite discrete spaces is homeomorphic to* C, *and the usual product of countably many countably infinite discrete spaces is homeomorphic to* I.

PROOF. This is a result of Theorem 3.

(3). C *is homogeneous.*

PROOF. See Theorem 9 and (1).

(4). *There exist metrics for* C *and* I *such that* C *can be isometrically embedded in* I.

PROOF. See Corollary 1.

(5). *There exists a continuous function from* I *onto* C.

PROOF. See Corollary 1.

(6). *The irrationals are topologically complete (i.e. homeomorphic to a complete space).*

PROOF. See Theorem 4.

(7). C×C *is homeomorphic to* C.

(8). I×I *is homeomorphic to* I.

(9). I×C *is homeomorphic to* I.

PROOFS OF (7), (8), AND (9). By Theorem 10 we have $(\Pi X_i, d) \times (\Pi Y_i, d)$ is homeomorphic to $(\Pi(X_i \times Y_i), d)$. Letting each X_i and Y_i be a nondegenerate finite set gives (7). Letting each X_i and Y_i be a countably infinite set gives (8). Letting each X_i be a countably infinite set and each Y_i be a nondegenerate finite set gives (9).

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EMORY UNIVERSITY, *Atlanta, Georgia*
FORT LEWIS COLLEGE, *Durango, Colorado*
GRINNELL COLLEGE, *Grinnell, Iowa*
GUILFORD COLLEGE, *Greensboro, North Carolina*
HAMILTON COLLEGE, *Clinton, New York*
HARVEY MUDD COLLEGE, *Claremont, California*
HENDRIX COLLEGE, *Conway, Arkansas*
HOPE COLLEGE, *Holland, Michigan*
UNIVERSITY OF HOUSTON, *Houston, Texas*
UNIVERSITY OF KENTUCKY, *Lexington, Kentucky*
KENYON COLLEGE, *Gambier, Ohio*
KNOX COLLEGE, *Galesburg, Illinois*
LAKE FOREST COLLEGE, *Lake Forest, Illinois*
NEW COLLEGE, *Sarasota, Florida*
OAKLAND UNIVERSITY, *Oakland, Michigan*
RUTGERS UNIVERSITY, *New Brunswick, New Jersey*
UNIVERSITY OF SANTA CLARA, *Santa Clara, California*
UNIVERSITY OF TORONTO, *Toronto, Canada*
WESLEYAN UNIVERSITY, *Middletown, Connecticut*
WILLIAMS COLLEGE, *Williamstown, Massachusetts*
UNIVERSITY OF WISCONSIN, *Madison, Wisconsin*