A MANDELBROT SET WHOSE BOUNDARY IS PIECEWISE SMOOTH

M. F. BARNESLEY AND D. P. HARDIN

ABSTRACT. It is proved that the Mandelbrot set associated with the pair of maps
\[ w_{1,2} : \mathbb{C} \to \mathbb{C}, \quad w_1(z) = sz + 1, \quad w_2(z) = s^*z - 1, \]
with parameter \( s \in \mathbb{C} \), is connected and has piecewise smooth boundary.

INTRODUCTION

The discovery [1] of the Mandelbrot set \( M \) for the iterated complex polynomial \( z^2 + s \) has generated considerable research activity [2, 3], especially because of its relation to cascades of bifurcations and universal phenomena [4].

The Mandelbrot set \( M \) consists of those values of \( s \in \mathbb{C} \) such that the Julia set \( J(s) \) for \( z^2 - s \) is connected. Barnsley and Harrington [5] considered an analogous Mandelbrot set \( D \) associated with the two affine maps \( T_{1,2} : \mathbb{C} \to \mathbb{C} \) defined by

\[ T_1(z) = sz + 1, \quad T_2(z) = sz - 1 \]

for \( s \in \mathbb{C} \) and \( |s| < 1 \). There is a unique nonempty compact set \( A(s) \) which is invariant under \( T_1 \) and \( T_2 \) (i.e., \( T_1(A(s)) \cup T_2(A(s)) = A(s) \)) [5, 6]. Generically, \( A(s) \) is a fractal. \( D \) is defined to be the set of \( s \in \mathbb{C}, \quad |s| < 1 \) for which \( A(s) \) is disconnected. The boundary of \( D \) contains self-similar structures (see Figure 2) and appears to be a fractal. It is not known whether \( D \) is connected; however, new pictures of this set presented here indicate that it is.

In this paper we study the Mandelbrot set \( G \) associated with the two affine maps \( w_{1,2} : \mathbb{C} \to \mathbb{C} \) defined by

\[ w_1(z) = sz + 1, \quad w_2(z) = s^*z - 1 \]

for \( s \in \mathbb{C} \) with \( |s| < 1 \). (Here \( s^* \) denotes the conjugate of \( s \).) As in the previous case, there is a unique invariant compact set \( A(s) \) which is generically a fractal. Despite the apparent similarity between the two pairs of maps, \( G \) is easier to analyze than \( D \). We will show among other things that \( G \) is connected and, remarkably, has a piecewise smooth boundary. Pictures of the associated fractals as one travels around the boundary of \( G \) are given.
1. Preliminaries

Let \((X, d)\) be a compact metric space or \(\mathbb{R}^n\) and let \(H\) denote the set of all nonempty compact subsets of \(X\). If \(B, C \subset X\), then define

\[
d(B, C) = \inf\{d(b, c) \mid b \in B, c \in C\}
\]

and define the Hausdorff metric \(h\) on \(H\) by

\[
h(B, C) = \sup\{d(\{b\}, C), d(\{c\}, B) \mid b \in B, c \in C\}.
\]

It is known that \((H, h)\) is a complete metric space [6].

Let \(0 \leq c < 1\) and let the mappings \(w_i : X \to X, i = 1, \ldots, N\), be such that \(d(w_i(x), w_i(y)) \leq cd(x, y)\) for all \(x, y \in X\). Following Barnsley and Demko [7] we call \(\{X, w_i : i = 1, \ldots, N\}\) a hyperbolic iterated function system (HIFS). Define \(w : H \to H\) by

\[
w(B) = \bigcup_{i=1}^N w_i(B) = \bigcup \{w_i(x) \mid x \in B, i \in [1, \ldots, N]\}
\]

for all \(B \in H\). From the definition of \(h\) it is immediate that \(w\) is a contraction on \(H\) with \(h(w(B_1), w(B_2)) \leq ch(B_1, B_2)\) for \(B_1, B_2 \in H\). Since \(H\) is complete, the Banach fixed point theorem implies

**Theorem 1.** (1) \(w\) has a unique fixed point \(A \in H\). (A is called the attractor for the HIFS \((X, w)\).)

(2) \(\lim_{n \to \infty} w^{o_n}(B) = A\) (i.e., \(\lim_{n \to \infty} h(w^{o_n}(B), A) = 0\)) for any \(B \in H\), where we define \(w^{o_0}(B) = B\) and \(w^{o_n}(B) = w(w^{o_{n-1}}(B))\) for \(n \in \mathbb{N}\).

We will need the following lemma.

**Lemma 2.** If \(B \in H\) and \(B \supset w(B)\) then \(w^{o_n}(B) \supset A\) for all \(n \in \mathbb{N}\), where \(A\) is the attractor for \(\{X, w\}\). If \(w^{o_n}(B)\) is connected for all \(n \in \mathbb{N}\) and some \(B \subset H\) then \(A\) is connected.

**Proof.** If \(w(B) \subset B\), then \(w^{o_n}(B) \subset w^{o_{n-1}}(B)\) for all \(n \in \mathbb{N}\). Thus \(A = \lim_{n \to \infty} w^{o_n}(B) = \bigcap_{n=1}^\infty w^{o_n}(B)\), because the sequence of compact sets \(\{w^{o_n}(B)\}\) is decreasing.

Suppose \(A\) is disconnected; then \(A = B_1 \cup B_2\) with \(B_1, B_2 \in H\) and \(B_1 \cap B_2 = \emptyset\). Thus \(d(B_1, B_2) > 0\) and so for any set \(C\) such that \(h(A, C) < d(B_1, B_2)/2\) then \(C\) is also disconnected. Since \(\lim_{n \to \infty} h(A, w^{o_n}(B)) = 0\), \(w^{o_n}(B)\) is eventually disconnected for any \(B \in H\). \(\square\)

The following corollary generalizes a result of Barnsley and Harrington [5].

**Corollary 3.** Let \((X, w_1, w_2)\) be an HIFS with attractor \(A\) such that there exists a nonempty connected \(B \in H\) with \(w(B) \subset B\). \(A\) is disconnected if and only if \(w_1(A) \cap w_2(A) = \emptyset\).

**Proof.** If \(w_1(A) \cap w_2(A) = \emptyset\) then \(w_1(A)\) and \(w_2(A)\) form a disconnection of \(A\).
Suppose \( w_1(A) \cap w_2(A) \neq \emptyset \). By Lemma 2, \( w^{\infty}(B) \supset A \). Suppose \( w^{\infty}(B) \) is connected; then \( w_1(w^{\infty}(B)) \cap w_2(w^{\infty}(B)) \supset w_1(A) \cap w_2(A) \neq \emptyset \). By continuity \( w_1(w^{\infty}(B)) \) and \( w_2(w^{\infty}(B)) \) are connected so \( w^{\infty(n+1)}(B) \) is connected. Since \( B \) is connected, \( w^{\infty(n)}(B) \) is connected for all \( n \in \mathbb{N} \) by induction. By Theorem 1, \( \lim_{n \to \infty} h(w^{\infty}(B), A) = 0 \) and so by Lemma 2, \( A \) is connected. \( \square \)

Note that if \( X = \mathbb{R}^d \) then we can always find a nonempty connected \( B \in H \) such that \( w(B) \subset B \); for instance, if we pick the radius large enough we can take \( B \) to be a closed ball centered at the origin.

If \( (X, w(\lambda, \cdot)) \) is an HIFS for each \( \lambda \) in an index set \( \Lambda \), then we define the M
delbrot set for the family \( \{(X, w(\lambda, \cdot)) \mid \lambda \in \Lambda\} \) to be the set of \( \lambda \in \Lambda \) for which \( A(\lambda) \) (i.e., the attractor for \( (X, w(\lambda, \cdot)) \)) is disconnected.

2. A previously considered Mandelbrot set

Consider the family of pairs of maps \( T_i : C \to C, \ i = 1, 2 \), defined by

\[
T_1(s, \cdot) : z \to sz + 1, \quad T_2(s, \cdot) : z \to sz - 1
\]

for \( s \in C \) and \( |s| < 1 \). Note that \( T_1 \) and \( T_2 \) are similitudes. Let \( \theta(s) = \arg(s) \) and suppose \( B \subset K \). Geometrically, \( T(s, B) = T_1(s, B) \cup T_2(s, B) \) is generated by shrinking \( B \) by \( |s| \) toward 0, rotating by \( \theta(s) \) about 0, and translating one such copy by \( 1 + i0 \) and another by \(-1 + i0\).

Since \( |T_i(s, a) - T_i(s, b)| = |s||a-b| \) for \( i = 1, 2 \) and \( a, b \in C \), we see that \((C, T_1(s, \cdot), T_2(s, \cdot))\) is an HIFS for \( |s| < 1 \). Let \( A(s) \) denote the attractor for this HIFS. Figure 1 shows \( A(s) \) for several values of \( s \). It is instructive to identify \( T_1(s, A(s)) \) and \( T_2(s, A(s)) \) and to note that \( A(s) \) is indeed the fixed point of \( T(s, \cdot) \).

Barnsley and Demko [7] investigated the Mandelbrot set for the family of HIFSs \( \{(C, T(s, \cdot)) \mid s \in C, \ |s| < 1\} \). We will denote this Mandelbrot set by \( D \). Figure 2 shows a computer-generated picture of \( D \) (from [7]) along with several blowups of portions of the boundary of \( D \). They hypothesized that \( D \) may be disconnected; however, Figure 2 suggests that the opposite may be true.

They found inner and outer bounds for \( D \) using the fact that if \( s \in D \) then the Hausdorff dimension \( d \) of \( A(s) \) is given by

\[
d = \log(\frac{1}{2}) / \log(|s|).
\]

We will prove the same bounds for \( D \) using the results we developed in the previous section. In the following, we will suppress the \( s \) dependence of \( T_1 \) and \( T_2 \).

Proposition 4. If \( |s| < .5 \) then \( s \in D \).

Proof. Let \( R_s = 1/(1 - |s|) \) and \( B(x, r) = \{z \in C \mid |z - x| \leq r\} \). Then \( T_1(B(0, R_s)) = B(1, |s|R_s) \subset B(0, R_s) \) and \( T_2(B(0, R_s)) = B(-1, |s|R_s) \) (see Figure 3). Thus, \( T(B(0, R_s)) \subset B(0, R_s) \) and, by Lemma 2, \( A(s) \subset B(0, R_s) \). If \( |s| < .5 \) then \( |s|R_s < 1 \) and so \( B(-1, |s|R_s) \cap B(1, |s|R_s) = \emptyset \). Thus \( T_1(A(s)) \cap T_2(A(s)) = \emptyset \) and \( A(s) \) is disconnected. \( \square \)
\begin{figure}
\centering
\includegraphics[width=\textwidth]{attractor.png}
\caption{The attractor $A(s)$ for $I$ shown for various values of $s$}
\end{figure}

We can calculate successively better inner bounds for $D$ by determining the values of $s$ for which it is true that $T_1(I^{\infty}(B(0, R_1))) \cap T_2(I^{\infty}(B(0, R_2))) = \emptyset$ for successively larger values of $n$. In fact, all of $D$ can be calculated in this manner.

\textbf{Theorem 5.} \ $s \in D$ if and only if $T_1(I^{\infty}(B(0, R_1))) \cap T_2(I^{\infty}(B(0, R_2))) = \emptyset$ for some $n \in \mathbb{N}$.

\textit{Proof.} Let $B = B(0, R_1)$ and $B_n = I^{\infty}(B(0, R_2))$ for $n \in \mathbb{N}$.

By Lemma 2, $A(s) \subset B_n$ for all $n \in \mathbb{N}$, so $A(s)$ is disconnected if $T_1(B_n) \cap T_2(B_n) = \emptyset$ for some $n \in \mathbb{N}$.

If $T_1(B_n) \cap T_2(B_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then since $T_1$ and $T_2$ are continuous and $B$ is connected, we get, via an induction, that $B_n$ is connected. By Lemma 2, $A(s)$ is connected. \qed
(a) The Mandelbrot set $D$

(b) Blowup of part of $D$, where $0.49 \leq \text{Re}[s] \leq 0.55$ and $0.35 \leq \text{Im}[s] \leq 0.45$

(c) Blowup of part of $D$, where $0.572 \leq \text{Re}[s] \leq 0.593$ and $0.352 \leq \text{Im}[s] \leq 0.378$

FIGURE 2
We wrote a computer program based on Theorem 5 which checks if any of the \(2^n\) balls making up \(T_1(B_n)\) intersect any of the \(2^n\) balls making up \(T_2(B_n)\) for \(n = 1, \ldots, N\). The pictures of the boundary shown in Figure 2 were generated by this program. The different shadings show successive approximations to \(D\). For more details, see the appendix.

Note that Theorem 5 suggests a method for proving that \(D\) is connected: show that the \(n\)th approximation to \(D\) is connected assuming that the \((n-1)\)th approximation to \(D\) is connected.

**Proposition 6.** If \(|s| < 1/\sqrt{2}\) then \(A(s)\) is connected.

**Proof.** Let \(B = B(0, R_0)\) and \(B_n = T^{on}(B)\) again. If \(|s| > 1/\sqrt{2}\) then \(\text{area}(T_1(B)) > .5\text{area}(B)\) and \(\text{area}(T_2(B)) > .5\text{area}(B)\). Since \(T(B) \subset B\) we must have \(T_1(B) \cap T_2(B) \neq \emptyset\). Since \(T(B_n) \subset B_n\) we must have, in the same way, that \(T_1(B_n) \cap T_2(B_n) \neq \emptyset\). By Proposition 5, \(A(s)\) is connected. \(\Box\)

Proposition 6 can be generalized to get

**Proposition 7.** If \((\mathbb{R}^n, w_1, w_2)\) is an HIFS such that \(\text{vol}_n(w_1(B)) > \frac{1}{2} \text{vol}_n(B)\) for every set \(B \in \mathbb{R}^n\) with finite and nonzero \(n\)-dimensional volume \(\text{vol}_n(B)\), then the attractor \(A\) is connected.
3. A piecewise smooth Mandelbrot set

Now consider the family of HIFSs arising from the pair of maps $w_i: \mathbb{C} \to \mathbb{C}$, $i = 1, 2$, defined by

$$w_1(s, \cdot): z \to sz + 1, \quad w_2(s, \cdot): z \to s^*z - 1$$

for $s \in \mathbb{C}$ and $|s| < 1$.

\[
\begin{align*}
\text{Re}[s] &= 0.4900001 \\
\text{Im}[s] &= 0.37
\end{align*}
\]

\[
\begin{align*}
\text{Re}[s] &= 0.17 \\
\text{Im}[s] &= 0.5770999
\end{align*}
\]

\[
\begin{align*}
\text{Re}[s] &= -0.3 \\
\text{Im}[s] &= 0.5271
\end{align*}
\]

**Figure 4.** $A(s)$ for various values of $s$
Geometrically, \( w \) acts on a set \( B \) in almost the same way as \( T \), the difference being that one of the shrunken copies is rotated by \(-\theta(s)\) where \( \theta(s) = \arg(s) \) again. Figure 4 shows \( A(s) \) for various values of \( s \).

Let \( G \) denote the Mandelbrot set for \( \{ (C, w(s, \cdot)) | s \in C, |s| < 1 \} \). As we shall see, in contrast to all other known cases, \( G \) can be completely described in an elementary way. We will show that \( G \) is connected and that the boundary of \( G \) is a countable collection of pieces of polynomial curves in \( x = \Re[s] \) and \( y = \Im[s] \). Figure 5 shows a picture of \( G \). Note that the inner and outer bounds for \( D \) are also applicable to \( G \) by exactly the same arguments.

First we will prove that \( G \) is symmetric about the real axis.
PROPOSITION 8. $A(s^*) = -A(s)$ so $s \in G$ if and only if $s^* \in G$.

Proof. $A(s)$ satisfies $w(s, A(s)) = (sA(s) + 1) \cup (s^*A(s) - 1) = A(s)$. Thus, $-A(s) = (s(-A(s)) - 1) \cup (s^*(-A(s)) + 1) = w_2(s^*, -A(s)) \cup w_1(s^*, -A(s)) = w(s^*, -A(s))$. Since $w(s^*, \cdot)$ has a unique fixed point, $-A(s) = A(s^*)$. □

Let $z_1$ be the fixed point of the contraction $w_1 \circ w_2$ and $z_2$ be the fixed point of $w_2 \circ w_1$. We will need the following collection of facts, which follow directly from the definitions of $w_1$ and $w_2$. Hereafter we will suppress the $s$ dependence of $w_1$ and $w_2$.

Lemma 9.
(a) $z_1 = (1 - s)/(1 - |s|^2) = -z_2^* = w_1(z_2)$.
(b) $z_2 = (s^* - 1)/(1 - |s|^2) = -z_1^* = w_2(z_1)$.
(c) $-w_1(x)^* = w_2(-z^*)$ for $z \in \mathbb{C}$.
(d) From the above we get
   (i) $w_2(z_2) = -w_1(z_1)^*$ and
   (ii) $w_2 \circ w_2(z_2) = -(w_1 \circ w_1(z_1))^*$.

PROPOSITION 10. If $3\pi/4 \leq \theta(s) \leq \pi$ and $\text{Re}[w_1(z_1)] > 0$ then $A(s)$ is disconnected.

Proof. Since $|s| < 1$, it is clear that $\text{Re}[z_1] = \text{Re}[(1 - s)/(1 - |s|^2)] > 0$. Let $B$ denote the closed convex hull of $\{z_1, z_2, w_1(z_1), w_2(z_2)\}$. The idea of
the proof is to show that \( w(B) \subset B \) so that \( A \subset B \) and then to show that \( w_1(B) \cap w_4(B) = \emptyset \).

Lemma 9 gave \( z_2 = -z_1^* \) and \( w_2(z_2) = -(w_1(z_1))^* \) so \( B \) is a trapezoid as shown in Figure 6. Since \( w_1(z_2 z_1) = \frac{z_2}{z_1} w_1(z_1) \), then from the definition of \( w_1 \) we see that the vertex angle at \( w_1(z_1) \) is \( \theta(s) \) and the vertex angle at \( z_1 \) is \( \pi - \theta(s) \). By symmetry the vertex angle at \( z_2 \) is \( \pi - \theta(s) \) and the vertex angle at \( w_2(z_2) \) is \( \theta(s) \).

Consider Figure 7. Since the angle \( \angle z_1 z_2 w_2(z_2) \) measured from \( z_2 z_1 \) in a counterclockwise direction is \( \pi - \theta(s) \) (i.e., the vertex angle at \( z_2 \)) we see that
the angle $\angle w_1(z_1)z_1w_1 \circ w_2(z_2)$ measured from $\overline{z_1w_1(z_1)}$ in a ccw direction is $\pi - \theta(s)$. Thus $w_1 \circ w_2(z_2)$ lies on $\overline{z_1z_2}$. Similarly, $\angle w_1w_1(z_1)w_1(z_1)z_1$ measured from $\overline{z_1w_1(z_1)}$ in a ccw direction is $\pi - \theta(s)$. Let $P$ be a point on the line through $w_1(z_1)$ and $w_2(z_2)$ with $\text{Re}[P] > \text{Re}[w_1(z_1)]$; then $\overline{Pw_1(z_1)z_1}$ measured from $\overline{Pw_1(z_1)}$ in a ccw direction is $\pi - \theta(s)$. Thus $\angle Pw_1(z_1)w_1w_1(z_1)$ is $2(\pi - \theta(s))$, which is between 0 and $\pi/2$. Thus, $w_1 \circ w_1(z_1) \in B$ and $\text{Re}[w_1(z_1)] \leq \text{Re}[w_1(w_1(z_1))] \leq \text{Re}[w_1 \circ w_2(z_2)] \leq \text{Re}[z_1]$. Now $w_1(B)$ is the trapezoid with vertices $\{z_1, w_1(z_1), w_1 \circ w_1(z_1), w_1 \circ w_2(z_2)\}$, all of which we have shown to lie in $\overline{B}$. Thus, $w_1(B) \subset \overline{B}$. Furthermore, if $z \in w_1(B)$ then $\text{Re}[z] \geq \text{Re}[w_1(z_1)] > 0$. Lemma 9 implies $w_2(B) = -(w_1(B))^*$, so if $z \in w_2(B)$ then $\text{Re}[z] < 0$. Thus $w_2(B) \subset B$ and so $A(s)$ is disconnected. □

The converse is also true.

**Proposition 11.** If $3\pi/4 \leq \theta(s) \leq \pi$ and $\text{Re}[w_1(z_1)] \leq 0$ (equivalent $\text{Re}[s] \leq -0.5$) then $A(s)$ is connected.
Proof. Let $B_n = z_1z_2$ and $B_n = w_n^s(B_0)$ for $n \in \mathbb{N}$. Figure 8 shows $B_0$, $B_1$, $B_2$ for a typical $s$. We will first show that $B_n$ is connected.

Since $w_2(z_1) = z_2$, $w_1(z_1) = z_1$, and $z_1, z_2 \in B_0$ then $z_1, z_2 \in B_n$ and thus $w_1(z_1), z_1 \in w_1(B_n)$ for $n \in \mathbb{N}$.

Note that $-B_0^* = B_0$. Suppose $-B_n^* = B_n$; then $-B_{n+1}^* = (-w_1(B_n))^* \cup (-w_2(B_n))^* = w_2(-B_n^*) \cup w_1(-B_n^*) = w(B_n) = B_{n+1}$. By induction $-B_n^* = B_n$ for $n \in \mathbb{N}$. Thus $-(w_1(B_n))^* = w_2(-B_n^*) = w_2(B_n)$, so if $x \in w_1(B_n)$ and Re$x = 0$ then $x \in w_2(B_n)$.

Note that $B_0$ is connected. If $B_n$ is connected then $w_1(B_n)$ and $w_2(B_n)$ are connected. Recall that $w_1(z_1), z_1 \in w_1(B_n)$, and that Re$[z_1] > 0$ and by hypothesis Re$[w_1(z_1)] \leq 0$. By the intermediate value theorem there must be some $a \in w_1(B_n)$ with Re$[a] = 0$. But then $a \in w_2(B_n)$ so $w_1(B_n) \cap w_2(B_n) \neq \emptyset$ and $w(B_n)$ is connected. The proposition then follows from Lemma 2.

Figures 9 and 10 illustrate the case for $\theta(s) \in [\pi/2, 3\pi/4]$. Now $w_1 \circ w_1(z_1)$ plays the role that $w_1(z_1)$ played for $\theta(s) \in [3\pi/4, \pi]$.

**Proposition 12.** If $3\pi/4 \leq \theta(s) \leq \pi/2$ then $A(s)$ is disconnected if and only if Re$[w_1 \circ w_1(z_1)] > 0$.

**Proof.** Suppose Re$[w_1 \circ w_1(z_1)] > 0$. Again let $B$ be the trapezoid with vertices $\{z_1, z_2, w_1(z_1), w_2(z_2)\}$. From the proof of Proposition 11 it still follows that $w(B) \subset B$. Since $\frac{1}{P}w_1(z_1), w_1 \circ w_1(z_1) = 2(\pi - \theta(s)) \in [\pi/2, \pi]$ we see that Re$[w_1 \circ w_1(z_1)] \leq$ Re$[w_1(z_1)]$. Since $\sqrt{w_1 \circ w_1(z), w_1 \circ w_2(z_2)z_1 = \theta(s) \in [3\pi/4, \pi/2]}$ we see that Re$[w_1 \circ w_1(z_1)] \leq$ Re$[w_1 \circ w_2(z_2)] \leq$ Re$[z_1]$. Thus $0 <$ Re$[w_1 \circ w_1(z_1)] \leq$ Re$[z]$ for all $z \in w_1(B)$ and so $w_1(B) \cap w_2(B) = \emptyset$. 
Now suppose $\text{Re}[w_1 \circ w_1(z_1)] \leq 0$. Let $B_0 = \overline{w_1(z_1)z_1 \cup z_1z_2 \cup z_2w_2(z_2)}$ and let $B_n = w_2^{\circ n}(B_0)$. By an induction $B_n$ is connected and so by Lemma 2 $A(s)$ is connected. \qed

**Proposition 13.** If $n \in \mathbb{N}$ and $\pi/(2n + 2) \leq \theta(s) \leq \pi/(2n)$ then $A(s)$ is disconnected if and only if $\text{Re}[w_1 \circ w_2^{\circ (n+1)}(z_2)] > 0$.

**Proof.** Since the method of proof should be familiar by now, we will only outline the proof of this proposition. Figure 11 illustrates the case for $n = 1$ and $n = 2$.

Suppose $\text{Re}[w_1 \circ w_2^{\circ (n+1)}(z_2)] > 0$. Let $B$ be the closed convex hull of \{\(z_1, z_2, w_1(z_1), w_2(z_2), \ldots, w_1^{\circ (2n+1)}(z_1), w_2^{\circ (2n+1)}(z_2)\} \); then $w(B) \subset B$ and if $z \in w_1(B)$ then $\text{Re}(z) \geq \text{Re}(w_1 \circ w_2^{\circ (n+1)}(z_2)) > 0$. Thus $w_1(B) \cap w_2(B) = \emptyset$ and so $A(s)$ is disconnected.
Suppose $\Re[w_1 \circ w_2^o(n+1)(z_2)] \leq 0$. Let

$$B_0 = \overline{z_1 z_2} \cup \left\{ \bigcup_{i=0}^{n} w_1^o(i+1)(z_1) w_1^o(i)(z_1) \cup w_2^o(i+1)(z_2) w_2^o(i)(z_2) \right\}.$$ 

An induction shows that $w^o_k(B_0)$ is connected for all $k \in \mathbb{N}$, so $A(s)$ is connected. □

The conditions given in Propositions 10–13 that $s$ be on $\partial G$ can be expressed as polynomial curves in $x = \Re[s]$ and $y = \Im[s]$. For $|s| < 1$,
Re\(w_1(z_1)\) = 0 if and only if \(x = -.5\), and \(\text{Re}[w_1 \circ w_1(z_1)] = 0\) if and only if \(2x + 2x + 1 - 2y^2 = 0\). Note that \(\text{Re}[w_1 \circ w_2^{o(n+1)}(z_2)] = 0\) if and only if
\[
\text{Re} \left[ |s|^2 (s^n - s^{n+1}) + (1 - |s|^2) \left( |s|^2 \sum_{p=0}^{n} s^{p-1} - 1 \right) \right] = 0,
\]
which describes a polynomial curve for each \(n \in \mathbb{N}\). We will now use these conditions to prove our main result.

**Theorem 14.** \(G\) is connected.

**Proof.** We will show that for each \(\theta(s) \in [0, \pi]\) there is an \(r^* \in (0, 1)\) such that \(s \in G\) if and only if \(|s| < r^*\). Recall that we already know that \(s \in G\) if \(|s| < .5\) and that \(s \notin G\) if \(|s| > 1/\sqrt{2}\). Thus we need only show that the appropriate function (for instance \(\text{Re}[w_1 \circ w_1(z_1)]\) for \(\theta(s) \in [\pi/2, 3\pi/4]\)) can be zero at most once in the interval \(.5, 1/\sqrt{2}\).

\[\text{Re}(s) = -0.5\]
\[\text{Im}(s) = 0\]

\[\text{Re}(s) = -0.375\]
\[\text{Im}(s) = -3.75\]

\[\text{Re}(s) = -1.25\]
\[\text{Im}(s) = -3.375\]

\[\text{Re}(s) = -3.5\]
\[\text{Im}(s) = -3.375\]

\[\text{Re}(s) = -3.375\]
\[\text{Im}(s) = -3.375\]

**Figure 12.** \(A(s)\) as \(s\) varies along \(\partial G\)
Case 1. $\theta(s) \in [3\pi/4, \pi]$: Propositions 10 and 11 imply $s \in G$ if and only if $\Re[s] < .5$; however, $\Re[s] < .5$ if and only if $|s| < .5|\sec \theta(s)| \equiv r^*$. 

Case 2. $\theta(s) \in [\pi/2, 3\pi/4]$: By Proposition 12, $s \in G$ if and only if $\Re[w_1 \circ w_1(z_1)] = 0$. Define

$$f(r) = \Re[w_1 \circ w_1(z_1)] = [r^2/(1 - r^2)][\cos 2\theta(s) - r \cos 3\theta(s)] + r \cos \theta(s) + 1,$$

where $r = |s|$. Since $\cos \theta(s) < 0$, $\cos 2\theta(s) < 0$, and $\cos 3\theta(s) > 0$, it is clear that $f(r)$ is a decreasing function for $r \in (0, 1)$ and thus can be zero at most once in the interval $(.5, 1/\sqrt{2}]$. 

Figure 12 (continued)
Case 3. \( \theta(s) \in (0, \pi/2) \): Let \( n \) be such that \( \pi/2^{n+1} \leq \theta(s) \leq \pi/2^n \). Now \( s \in G \) if and only if \( \text{Re}[w_1 \circ w_2^{o(n+1)}(z_2)] > 0 \). Define
\[
 f(r) = \text{Re}[w_1 \circ w_2^{o(n+1)}(z_2)] \\
= (r^{(n+3)} \cos[(n + 1)\theta(s)] - r^{(n+2)} \cos[n\theta(s)])/(1 - r^2) \\
- \left( \sum_{p=0}^{n} r^{p+1} \cos[(p - 1)\theta(s)] \right) + 1.
\]
It is a short exercise in freshman calculus to show that \( f(r) \) is decreasing on \((.5, 1/\sqrt{2})\).

Case 4. \( \theta(s) = 0 \): \( A(s) \) is an interval if \( |s| \geq .5 \) and a Cantor set if \( |s| < .5 \), so \( s \in G \) if and only if \( |s| < .5 \).

Since \( G \) is symmetric about the real axis, we see that \( G \) is connected. \( \Box \)

A tour around the boundary of \( G \). The evolution of \( A(s) \) as \( s \) varies along \( \partial G \) is rather interesting. Figure 12 shows \( A(s) \) at various values of \( s \) on \( \partial G \). Note that \( A(s) \) consists of a family of Koch [8] curves as \( s \) varies from \(-.5\) to \(-.5 + i.5\), at which point \( A(s) \) is a right triangle. The other interesting point is \( s = i/\sqrt{2} \), where \( A(s) \) is a rectangle.

The family of attractors for \( \theta(s) \in (0, \pi/2) \) includes fractals which arise as natural boundaries in the complex \( t \)-plane for nonintegrable dynamical systems [9, 10]. In fact, these fractals provided our original motivation for studying this particular family.

APPENDIX

In this appendix we present a computer program which generates computer images of \( D \). The program can be used with minor modifications to find the Mandelbrot set for any family of pairs of similitudes on \( \mathbb{R}^2 \).

The program runs on the IBM PC microcomputer in compiled BASIC. A typical picture is produced in approximately 12 hours when the number of iterations is between 10 and 15. The program is much slower in regions which are near \( \partial D \) and which are near the real axis.
10 DIM AX(4),AY(4),X(2,2,30),Y(2,2,30),P(2,30),RSC(30),LN(30)
20 INPUT *window in parameter space a<Re[S]<b; c<Im[S]<d*;AA,BB,CC,DD
30 INPUT *pixel window; px1,px2,py1,py2 where 0<=px1<px2<320
   and 0<=py1<py2<200 (e.g. 40,279,0,199 gives a square)*;PX1,PX2,PY1,PY2
40 INPUT *file name for picture*;PICFILE$
50 INPUT *number of iterations<=30*;NUMIT
60 HX=(BB-AA)/(PX2-PX1) : HY=(DD-CC)/(PY2-PY1)
70 SCREEN 1,0:KEY OFF:CLS
80 P(1,0)=1:P(2,0)=2
90 FOR SY=CC TO DD STEP HY
100 FOR SX=AA TO BB STEP HX
110 PSX=(PX2-PX1)*(SX-AA)/(BB-AA)+PX1 : PSY=(PY2-PY1)*(DD-SY)/(DD-CC)+PY1
120 SC=SY*SX+SY*SY
130 IF SC>.5 THEN GOTO 300
140 RSC(0)=SC/(1-SQR(SC))$2
150 FOR K=1 TO NUMIT
160 RSC(K)=RSC(K-1)*SC
170 NEXT K
180 IF RSC(0)<1 THEN COLCODE=3:GOTO 370
190 N=1:COLCODE=1
200 LN(N)=1:WP=1:P(1,N)=1:GOSUB 380
210 IF COLCODE<N THEN COLCODE=N
220 P(2,N)=1:WP=2:GOSUB 380
230 A=1:B=1:GOSUB 520: IF DST<=RSC(N) THEN GOTO 290
240 LN(N)=2:WP=2:P(2,N)=2:GOSUB 380
250 A=1:B=2:GOSUB 520: IF DST<=RSC(N) THEN GOTO 290
260 LN(N)=3:WP=1:P(1,N)=2:GOSUB 380
270 A=2:B=1:GOSUB 520:IF DST<=RSC(N) THEN GOTO 290
280 LN(N)=4:A=2:B=2:P(2,N)=2:GOSUB 520:IF DST>RSC(N) THEN GOTO 550
290 IF N<NUMIT THEN N=N+1 : GOTO 200
300 NEXT SX
310 NEXT SY
320 DEF SEG = &H800 :BSAVE PICFILE$0,&H4000
330 INPUT WONT
340 IF WONT THEN GOTO 20
350 END
360 COLCODE = COLCODE MOD 3 +1
370 PSET (PSX,PSY),COLCODE : GOTO 300
380 XX=0:YY=0
390 FOR K=0 TO N
400 ON P(WP,N-K) GOSUB 440,480
410 NEXT K
420 X(WP,P(WP,N),N)=XX:Y(WP,P(WP,N),N)=YY
430 RETURN
440 XX=SX*XX-SY*YY+1
450 YY=SY*YY+SY*XX
460 XX=XX
470 RETURN
480 XX=SX*XX-SY*YY-1
490 YY=SY*YY+SY*XX
500 XX=XX
510 RETURN
520 DELX=X(1,A,N)-X(2,B,N):DELY=Y(1,A,N)-Y(2,B,N)
530 DST=.25*DELY*DELY+DELY*DELY)
540 RETURN
550 IF N=1 THEN GOTO 360
560 N=N-1 : ON LN(N) GOTO 240,260,280,550
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School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

Current address (D. P. Hardin): Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37235