Math 247 Exam 1 Solutions

1) There are $2^6 = 64$ subsets of the group; excluding the empty set (since the committee must have at least one member) leaves 63 choices.

2) a) Since there are 3 "A"s, 2 "N"s, and 1 each "R", "H", and "C", the number of distinct words is given by the multinomial coefficient $\binom{8}{3,3,1,1} = \frac{8!}{3!2!1!1!1!} = 3360$.

b)If the two "N"s must appear together, they may be considered as a single letter, so the number of distinct words is $\binom{7}{3,1,1,1} = 840$.

3) a) If $A = \{1\}$, the $P(A) + P(A^C) = \frac{1+1}{3+1} + \frac{2+1}{3+1} = \frac{5}{4} \neq P(A \cup A^C) = \frac{3+1}{3+1} = 1$, so it is not a valid probability space. (This would work with any event). Alternatively, note that $P(\emptyset) = \frac{1}{4}$, which is impossible.

b) $P(S) = \frac{1}{10} \sum_{x=1}^{3} x^2 = \frac{14}{10} \neq 1$, so it not a valid probability space.

c) This is a valid probability space. (For example, imagine a wheel with three outcomes, 1, 2, or 3, but which is biased so that it ALWAYS lands on 2.)

4) Let R be the event that a ball is red. Initially there are six red balls and seven not red, and then four more balls are added. a)$P(-R) = P(RR) + P(R^C R) = P(-R|R)P(R) + P(-R|R^C)P(R^C) \frac{6}{13}\frac{10}{17} + \frac{7}{13}\frac{6}{17} = \frac{6}{13}$.

b) $P(R|-R) = \frac{P(R \cap -R)}{P(-R)} = \frac{P(RR)}{P(-R)} = \frac{\frac{6}{13}\frac{10}{17}}{\frac{6}{13}} = \frac{10}{17}$.

5) Let p_n be the probability that at least one of n shots hits the can. Then $p_n = 1 - (.9)^n$. Setting $p_n = 1 - (.9)^n \geq .99$ gives $n \geq \frac{ln(.01)}{ln(.9)} = 43.7$, so he would need 44 bullets to have a 99% chance of hitting the can.

6) Let D, I, R be the events that a random person called is a Democrat, Independent, or Republican, respectively. Let Z be the event that s/he supports Proposition Z. Then by Bayes' formula, $P(R|Z) = \frac{P(Z|R)P(R)}{P(Z|R)P(R) + P(Z|I)P(I) + P(Z|D)P(D)} = \frac{.45 \cdot .35}{.45 \cdot .35 + .19 \cdot .25 + .17 \cdot .4} = .577$ (so if the respondent supports Proposition Z, there is a 58% chance s/he is a Republican.

7) In the space of outcome for two flips of a fair coin, let A be the event that the first flip is heads (so $A = \{HT, HH\}$), B be the event that the second flip is heads (so $B = \{TH, HH\}$), and C be the event that the flips are different (so $C = \{TH, HT\}$). Then $P(A) = P(B) = P(C) = \frac{1}{2}$. Also $A \cap B = \{HH\}, A \cap C = \{HT\}$, and$B \cap C = \{TH\}$, so $P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$. So A, B, C are pairwise independent since $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$, etc. But they are not all independent since $A \cap B \cap C$ is the empty set, so $P(A \cap B \cap C) = 0 \neq P(A)P(B)PC)$.

8) For any given roll, the probabilities of 3, even, and neither are $\frac{1}{6}, \frac{1}{2}$, and $\frac{1}{3}$, respectively. (a) The probability that the first roll which is either 3 or even will actually be an even is $\dfrac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{6}} = \dfrac{3}{4}$.

(b) The probability that the first two rolls which are either 3 or even are actually both even will be $(\dfrac{3}{4})^2 = \dfrac{9}{16}$.

9) For the fair wheel, the outcomes 1,2,3 each have probability $\frac{1}{3}$. For the other wheel, the probability of the outcome "i" is given by ci, and must satisfy $c1 + c2 + c3 = c6 = 1$, so $c = \frac{1}{6}$, and the probabilities of the outcomes 1,2,3 are $\frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}$, respectively. If each wheel is rolled once independently, there are 9 possible outcomes whose probabilities can be obtained by multiplication: $P(1,1) = \frac{1}{18}, P(1,2) = \frac{2}{18}, P(1,3) = \frac{3}{18}, P(2,1) = \frac{1}{18}, P(2,2) = \frac{2}{18}, P(2,3) = \frac{3}{18}, P(3,1) = \frac{1}{18}, P(3,2) = \frac{2}{18}, P(3,2) = \frac{3}{18}$.

Then if X is the sum of the rolls, $Ran(x) = \{2,3,4,5,6\}$ and the probability mass function is obtained by tallying the probability for each outcome: $p_X(2) = P(1,1) = \frac{1}{18}, p_X(3) = P(1,2) + P(2,1) = \frac{3}{18}, p_X(4) = P(3,1) + P(1,3) + P(2,2) = \frac{6}{18}, p_X(5) = P(2,3) + P(3,2) = \frac{5}{18}, p_X(6) = P(3,3) = \frac{3}{18}$. Then $E(X) = \sum_{i=2}^{6} i \cdot p_X(i) = 2 \cdot \frac{1}{18} + 3 \cdot \frac{3}{18} + 4 \cdot \frac{6}{18} + 5 \cdot \frac{5}{18} + 6 \cdot \frac{3}{18} = \frac{13}{3}$, and $E(X^2) = \sum_{i=2}^{6} i^2 \cdot p_X(i) = 4 \cdot \frac{1}{18} + 9 \cdot \frac{3}{18} + 16 \cdot \frac{6}{18} + 25 \cdot \frac{5}{18} + 36 \cdot \frac{3}{18} = 20$.

10) (a) Since there are 5 choices, exactly one of which is correct, the probability is $\frac{1}{5}$.

(b) By part(a), there is a $\frac{1}{5}$ chance of guessing P.M. 1 correctly, in which case there are four remaining choices for P.M. 2. If she guesses, P.M. 2 correctly, there are 3 remaining choices for P.M. 3. So the probability she guesses the first 3 correctly is $\frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3}$.

(c) Let A_i be the probability she guesses the P.M. of Country "i" correctly, and let A be the event that she guesses the P.M. of at least one country correcly. Then by the inclusion-exclusion principle

$$P(A) = P(\cup_{i=1}^{5})$$

$$= \binom{5}{1} P(A_1) - \binom{5}{2} P(A_1 \cap A_2) + \binom{5}{3} P(A_1 \cap A_2 \cap A_3) - \binom{5}{4} P(A_1 \cap A_4 \cap A_3 \cap A_4) + \binom{5}{5} P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

$$= 5 \cdot \frac{1}{5} 10 \cdot \frac{1}{5 \cdot 4} + 10 \cdot \frac{1}{5 \cdot 4 \cdot 3} - 5 \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} + 1 \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{19}{30}$$

where we have used the fact that every triple intersection is equally likely with the probability computed in part (b), and similarly for single intersections, double intersections, etc.