Solutions to Chapter 1 Theoretical exercises 2, 12(a), and 16

2) The total number of outcomes can be divided according the m outcomes of the first experiment. Then looking at the possible outcomes for the second experiment we find that the total is $n_1 + n_2 + ... + n_m = \sum_{i=1}^{m} n_i$.

12a) If we first pick the committee and then the chair, for each possible committee size k, there are $\binom{n}{k}$ posssible choices for the committee, and for each choice of committee of there are k choices for the chair. Summing over all possible sizes (starting at 1 since the committee can't be empty - it must have a chair), we get $\sum_{k=1}^{n} k\binom{n}{k}$, which is the left-hand side.

On the other hand, if we first pick the chair and then the rest of the committee, there are $\binom{n}{1}$ or n choices for the chair, and for each choice of chair there are 2^{n-1} choices for the rest of the committee since the rest of the committee may consist of any subset of the other $n-1$ people (including the empty subset). So there are a total of $n2^{n-1}$ choices, which is the right-hand side.

Since these two numbers are counts of the same thing, they must be equal.

16a) Either all three are tied (1 way), the first two are tied and one is last ($\binom{3}{1} = 3$ ways), the last two are tied and one is first ($\binom{3}{1} = 3$ ways), or all three are in separate places ($3! = 6$ ways). So there are a total of $1 + 3 + 3 + 6 = 13$ possibilities.

(b) We divide the outcomes according to the number of people who are tied for last. The number of possible outcomes with exactly i people tied for last can be determined by multiplying the number of ways to choose the i people who tie for last, which is $\binom{n}{i}$, and the number of possible outcomes for the remaining $n-i$ people, which is $N(n-i)$ (this works even if there re no other people, since we have defined $N(0)$ to be 1.) Summing over i, we see that the total number of possible outcomes is $N(i) = \sum_{i=1}^{n} \binom{n}{i} N(n-i)$.

(c) Using the substitution $j = n - i$, we can rewrite the formula as

$$N(i) = \sum_{i=1}^{n} \binom{n}{i} N(n-i) = \sum_{j=0}^{n-1} \binom{n}{n-j} N(j) = \sum_{j=0}^{n-1} \binom{n}{j} N(j) = \sum_{i=0}^{n-1} \binom{n}{i} N(i)$$

(since as j ranges over the set $\{0, 1, ..., n-1\}$, i ranges over the set $\{n, n-1, ..., 1\}$.

(d) Using the formula from part (c) and the fact that $N(0) = 1$, $N(1) = 1$, and $N(2) = 3$, we get $N(3) = \binom{3}{0}N(0) + \binom{3}{1}N(1) + \binom{3}{2}N(2) = 1 \cdot 1 + 3 \cdot 1 + 3 \cdot 3 = 13$, and $N(4) = \binom{4}{0}N(0) + \binom{4}{1}N(1) + \binom{4}{2}N(2) + \binom{4}{3}N(3) = 1 \cdot 1 + 4 \cdot 1 + 6 \cdot 3 + 4 \cdot 13 = 75$.