Math 198, Section 4, Test 2

March 12, 2008

Name:

Pledged

Honor code: I have neither given nor received help on this test.

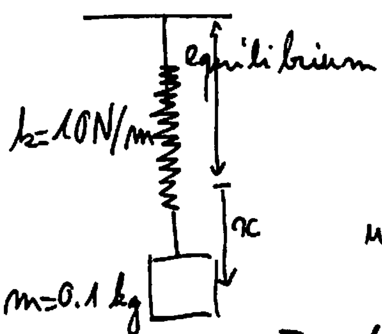
Note: the total number of points is 110 — anything above 100 is extra credit.

1. (15 pts) A mass $m = 0.1$ kg attached to a spring whose constant is $k = 10$ N/m is released from a point 30 cm below its equilibrium position with an upward velocity of 4 m/s.

   (a) (5 pts) Write the initial-value problem describing the motion.

   (b) (5 pts) Give the equation of the motion.

   (c) (5 pts) Determine the amplitude of the oscillations.



(a) The initial-value problem is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

with $x_0 = 0.3$, $x_1 = -4$, and $\frac{k}{m} = \frac{10}{0.1} = 100$.

In other words: $\underline{\frac{d^2x}{dt^2} + 100x = 0, \quad x(0) = 0.3, \quad x'(0) = -4.}$

(b) The equation of the motion is then:

$\underline{x(t) = 0.3 \cos(10t) - 0.4 \sin(10t)}$

(c) The amplitude of the oscillations is given by

$A = \sqrt{0.3^2 + 0.4^2} = \sqrt{(9+16) \cdot 10^{-2}} = 5 \cdot 10^{-1}$,

that is $\underline{A = 0.5 \text{ m}}$

2. (20 pts) Consider the differential equation

$$4x^2y'' + y = 0.$$

Using the reduction of order — given that $y_1 = x^{1/2}\ln x$ is a solution on $(0,\infty)$ — or any other method at your disposal, find the general solution on $(0,\infty)$.

We won't use the reduction of the order, we just recognize a Cauchy-Euler equation whose auxiliary equation is

$0 = 4m(m-1)+1 = 4m^2-4m+1 = (2m-1)^2$.

There is a double root at $m = 1/2$.

It then follows that the general solution is

$$\underline{y = c_1 x^{1/2} + c_2 x^{1/2}\ln x}.$$

3. (20 pts) Solve the differential equation

$$y'' - 2y' + y = x\exp(x).$$

1/ <u>Homogeneous equation</u>

Auxiliary equation: $0 = m^2 - 2m + 1 = (m-1)^2$

Complementary solution: $y_c = C_1 \exp(x) + C_2 x \exp(x)$.

2/ <u>Particular solution</u>

We look for y_p in the form $y_p = x^2 \left(A \exp(x) + Bx \exp(x)\right)$

$$y_p = A x^2 \exp(x) + B x^3 \exp(x)$$

$$y_p' = 2A x \exp(x) + (A+3B) x^2 \exp(x) + B x^3 \exp(x)$$

$$y_p'' = 2A \exp(x) + (4A+6B) x \exp(x) + (A+6B) x^2 \exp(x) + B x^3 \exp(x)$$

$$y_p'' - 2y_p' + y_p = x \exp(x)$$

$$\begin{aligned} = &\ (B - 2B + B)\ x^3 \exp(x) \\ &+ (A + 6B - 2A - 6B + A)\ x^2 \exp(x) \\ &+ (4A + 6B - 4A)\ x \exp(x) \\ &+ (2A)\ \exp(x) \end{aligned} \quad = \quad \begin{aligned} &/ \\ &/ \\ &6B\ x \exp(x) \\ &+2A \exp(x) \end{aligned}$$

So we take $2A = 0$ and $6B = 1$, that is $A = 0$ and $B = 1/6$.

$$y_p = \frac{1}{6} x^3 \exp(x)$$

3/ <u>General solution</u>

$$y = C_1 \exp(x) + C_2 x \exp(x) + \frac{1}{6} x^3 \exp(x)$$

4. (20 pts) Write the general solution on the interval $(0, \infty)$ of the following differential equation:

$$x^2y'' + xy' - y = \frac{x}{x+1}.$$

1/ Homogeneous equation

Auxiliary equation: $0 = m(m-1) + m - 1 = m^2 - 1 = (m-1)(m+1)$

$y_c = c_1 x + c_2 \frac{1}{x}$

2/ Particular solution : variation of parameters

$y_1 = x$
$y_2 = 1/x$, $W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = \frac{-2}{x}$, $W_1 = \begin{vmatrix} 0 & 1/x \\ \frac{1}{x(x+1)} & -1/x^2 \end{vmatrix} = -\frac{1}{x^2(x+1)}$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{1}{x(x+1)} \end{vmatrix} = \frac{1}{x+1}$

$u_1' = \frac{W_1}{W} = \frac{1}{2} \frac{1}{x(x+1)} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+1}\right)$

$u_1 = \frac{1}{2}\left(\ln|x| - \ln|x+1|\right) = \frac{1}{2}\left(\ln(x) - \ln(x+1)\right) = \ln\sqrt{\frac{x}{x+1}}$

$u_2' = \frac{W_2}{W} = -\frac{1}{2}\frac{x}{x+1} = -\frac{1}{2}\left(1 - \frac{1}{x+1}\right)$

$u_2 = -\frac{1}{2}\left(x - \ln(x+1)\right) = -\frac{x}{2} + \ln\sqrt{x+1}$

$y_p = u_1 y_1 + u_2 y_2 = \ln\left(\sqrt{\frac{x}{x+1}}\right) \times x - \frac{1}{2} + \ln\left(\sqrt{x+1}\right) \times \frac{1}{x}$

3/ General solution

$$\underline{y = c_1 x + c_2 \frac{1}{x} + x \ln\sqrt{\frac{x}{x+1}} - \frac{1}{2} + \frac{1}{x}\ln\sqrt{x+1}}$$

5. (15 pts) Let y_1 and y_2 be two solutions of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

on an interval I where the functions P and Q are continuous.

(a) (5 pts) Define the Wronskian of y_1 and y_2.

(b) (5 pts) Give a criterion for y_1 and y_2 to form a fundamental set of solutions.

(c) (5 pts) Find a first-order linear differential equation satisfied by the Wronskian.

(a) Wronskian of y_1 and y_2 : $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

(b) [(y_1, y_2) fundamental set of solutions] $\Longleftrightarrow$ [for all x, $W \neq 0$]

(c) From $W = y_1 y_2' - y_1' y_2$, we get

$$\begin{aligned} W' &= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} \\ &= y_1(-P y_2' - Q y_2) - y_2(-P y_1' - Q y_1) \\ &= -P y_1 y_2' - \cancel{Q y_1 y_2} + P y_1' y_2 + \cancel{Q y_1 y_2} \\ &= -P(\underbrace{y_1 y_2' - y_1' y_2}_{= W}) \end{aligned}$$

$\underline{W' + PW = 0}$

6. **(20 pts) Solve the initial-value problem**

$$y'' - y = \frac{1}{\cosh(x)} \quad \textbf{subject to} \quad y(0) = 0 \textbf{ and } y'(0) = 1.$$

It is recalled that $\cosh^2(x) - \sinh^2(x) = 1$ and that $\cosh(0) = 1$.

1/ Homogeneous equation : $y_c = c_1 \cosh(x) + c_2 \sinh(x)$

2/ Particular solution : variation of parameters

$$y_1 = \cosh(x), \quad y_2 = \sinh(x), \quad W = \begin{vmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{vmatrix} = 1, \quad W_1 = \begin{vmatrix} 0 & \sinh(x) \\ \frac{1}{\cosh(x)} & \cosh(x) \end{vmatrix} = -\frac{\sinh(x)}{\cosh(x)}$$

$$W_2 = \begin{vmatrix} \cosh(x) & 0 \\ \sinh(x) & \frac{1}{\cosh(x)} \end{vmatrix} = 1$$

$$u_1' = \frac{W_1}{W} = -\frac{\sinh(x)}{\cosh(x)} \Rightarrow u_1 = -\ln(\cosh x)$$

$$u_2' = \frac{W_2}{W} = 1 \Rightarrow u_2 = x$$

$$y_p = u_1 y_1 + u_2 y_2 = -\ln(\cosh x)\cosh(x) + x \sinh(x)$$

3/ General solution

$$y = c_1 \cosh(x) + c_2 \sinh(x) - \ln(\cosh x)\cosh(x) + x \sinh(x)$$

4/ Initial conditions

$$0 = y(0) = c_1 + 0 + 0 + 0$$
$$1 = y'(0) = 0 + c_2 + 0 + 0$$

, thus $c_1 = 0$, $c_2 = 1$

5/ Solution of the IVP

$$y = (x+1)\sinh(x) - \ln(\cosh x)\cosh(x)$$