

# Math 198, Section 2, Test 2

March 12, 2008

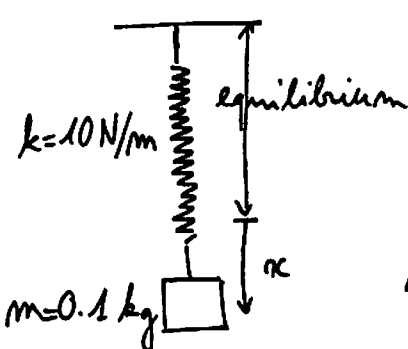
Name:

Pledged

Honor code: I have neither given nor received help on this test.

Note: the total number of points is 110 — anything above 100 is extra credit.

1. (15 pts) A mass  $m = 0.1$  kg attached to a spring whose constant is  $k = 10$  N/m is released from a point 30 cm below its equilibrium position with an upward velocity of 4 m/s.
  - (a) (5 pts) Write the initial-value problem describing the motion.
  - (b) (5 pts) Give the equation of the motion.
  - (c) (5 pts) Determine the amplitude of the oscillations.



(a) The initial-value problem is

$$\frac{d^2 x}{dt^2} + \frac{k}{m} x = 0 \quad \begin{matrix} x(0) = x_0 \\ x'(0) = x_1 \end{matrix}$$

with  $x_0 = 0.3$ ,  $x_1 = -4$ , and  $\frac{k}{m} = \frac{10}{0.1} = 100$ .

In other words:  $\frac{d^2 x}{dt^2} + 100 x = 0$ ,  $x(0) = 0.3$ ,  $x'(0) = -4$ .

(b) The equation of the motion is then:

$$x(t) = 0.3 \cos(10t) - 0.4 \sin(10t)$$

(c) The amplitude of the oscillations is given by

$$A = \sqrt{0.3^2 + 0.4^2} = \sqrt{(9+16) \cdot 10^{-2}} = 5 \cdot 10^{-2},$$

that is  $A = 0.05$  m

2. (20 pts) Consider the differential equation

$$4x^2y'' + y = 0.$$

Using the reduction of order — given that  $y_1 = x^{1/2} \ln x$  is a solution on  $(0, \infty)$  — or any other method at your disposal, find the general solution on  $(0, \infty)$ .

We won't use the reduction of the order, we just recognize a Cauchy-Euler equation whose auxiliary equation is

$$0 = 4m(m-1) + 1 = 4m^2 - 4m + 1 = (2m-1)^2.$$

There is a double root at  $m = 1/2$ .

It then follows that the general solution is

$$\underline{y = c_1 x^{1/2} + c_2 x^{1/2} \ln x.}$$

3. (20 pts) Solve the differential equation

$$y'' - 2y' + y = x \exp(x).$$

1/ Homogeneous equation

$$\text{Auxiliary equation: } 0 = m^2 - 2m + 1 = (m-1)^2$$

$$\text{Complementary solution: } y_c = C_1 \exp(x) + C_2 x \exp(x).$$

2/ Particular solution

We look for  $y_p$  in the form  $y_p = x^2 (A \exp(x) + Bx \exp(x))$

$$y_p = A x^2 \exp(x) + B x^3 \exp(x)$$

$$y_p' = 2A x \exp(x) + (A+3B) x^2 \exp(x) + B x^3 \exp(x)$$

$$y_p'' = 2A \exp(x) + (4A+6B) x \exp(x) + (A+6B) x^2 \exp(x) + B x^3 \exp(x)$$

$$y_p'' - 2y_p' + y_p = x \exp(x)$$

$$\begin{aligned} & (B - 2B + B) x^3 \exp(x) && / \\ & + (A + 6B - 2A - 6B + A) x^2 \exp(x) && / \\ & + (4A + 6B - 4A) x \exp(x) && 6B x \exp(x) \\ & + (2A) \exp(x) && + 2A \exp(x) \end{aligned}$$

So we take  $2A=0$  and  $6B=1$ , that is  $A=0$  and  $B=1/6$ .

$$y_p = \frac{1}{6} x^3 \exp(x)$$

3/ General solution

$$y = C_1 \exp(x) + C_2 x \exp(x) + \frac{1}{6} x^3 \exp(x)$$

4. (20 pts) Write the general solution on the interval  $(0, \infty)$  of the following differential equation:

$$x^2 y'' + xy' - y = \frac{x}{x+1}$$

1/ Homogeneous equation

Auxiliary equation:  $0 = m(m-1) + m - 1 = m^2 - 1 = (m-1)(m+1)$

$$y_c = C_1 x + C_2 \frac{1}{x}$$

2/ Particular solution : variation of parameters

$$y_1 = x, \quad y_2 = \frac{1}{x}, \quad W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}, \quad W_1 = \begin{vmatrix} 0 & \frac{1}{x} \\ \frac{1}{x(x+1)} & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x^2(x+1)}$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{1}{x(x+1)} \end{vmatrix} = \frac{1}{x+1}$$

$$u_1' = \frac{W_1}{W} = \frac{1}{2} \frac{1}{x(x+1)} = \frac{1}{2} \left( \frac{1}{x} - \frac{1}{x+1} \right)$$

$$u_1 = \frac{1}{2} (\ln|x| - \ln|x+1|) = \frac{1}{2} (\ln(x) - \ln(x+1)) = \ln \sqrt{\frac{x}{x+1}}$$

$$u_2' = \frac{W_2}{W} = -\frac{1}{2} \frac{x}{x+1} = -\frac{1}{2} \left( 1 - \frac{1}{x+1} \right)$$

$$u_2 = -\frac{1}{2} (x - \ln(x+1)) = -\frac{x}{2} + \ln \sqrt{x+1}$$

$$y_p = u_1 y_1 + u_2 y_2 = \ln \left( \sqrt{\frac{x}{x+1}} \right) \times x - \frac{1}{2} + \ln \sqrt{x+1} \times \frac{1}{x}$$

3/ General solution

$$y = C_1 x + C_2 \frac{1}{x} + x \ln \sqrt{\frac{x}{x+1}} - \frac{1}{2} + \frac{1}{x} \ln \sqrt{x+1}$$

5. (15 pts) Let  $y_1$  and  $y_2$  be two solutions of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

on an interval  $I$  where the functions  $P$  and  $Q$  are continuous.

- (a) (5 pts) Define the Wronskian of  $y_1$  and  $y_2$ .  
 (b) (5 pts) Give a criterion for  $y_1$  and  $y_2$  to form a fundamental set of solutions.  
 (c) (5 pts) Find a first-order linear differential equation satisfied by the Wronskian.

(a) Wronskian of  $y_1$  and  $y_2$  :  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

(b)  $[(y_1, y_2)$  fundamental set of solutions]  $\iff$  [for all  $x$ ,  $W \neq 0$ ]

(c) From  $W = y_1 y_2' - y_1' y_2$ , we get

$$\begin{aligned} W' &= \cancel{y_1' y_2'} + y_1 y_2'' - y_2'' y_1 - \cancel{y_1' y_2'} \\ &= y_1 (-P y_2' - Q y_2) - y_2 (-P y_1' - Q y_1) \\ &= -P y_1 y_2' - \cancel{Q y_2 y_1} + P y_2' y_1 + \cancel{Q y_1 y_2} \\ &= -P (y_1 y_2' - y_2' y_1) \\ &= -P W \end{aligned}$$

$W' + P W = 0$

6. (20 pts) Solve the initial-value problem

$$y'' - y = \frac{1}{\cosh(x)} \quad \text{subject to } y(0) = 0 \text{ and } y'(0) = 1.$$

It is recalled that  $\cosh^2(x) - \sinh^2(x) = 1$  and that  $\cosh(0) = 1$ .

1/ Homogeneous equation :  $y_c = C_1 \cosh(x) + C_2 \sinh(x)$

2/ Particular solution : variation of parameters

$$y_1 = \cosh(x) \quad y_2 = \sinh(x) \quad W = \begin{vmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{vmatrix} = 1, \quad W_1 = \begin{vmatrix} 0 & \sinh(x) \\ \frac{1}{\cosh(x)} & \cosh(x) \end{vmatrix} = -\frac{\sinh(x)}{\cosh(x)}$$

$$W_2 = \begin{vmatrix} \cosh(x) & 0 \\ \sinh(x) & \frac{1}{\cosh(x)} \end{vmatrix} = 1$$

$$u_1' = \frac{W_1}{W} = -\frac{\sinh(x)}{\cosh(x)} \Rightarrow u_1 = -\ln(\cosh(x))$$

$$u_2' = \frac{W_2}{W} = 1 \Rightarrow u_2 = x$$

$$y_p = u_1 y_1 + u_2 y_2 = -\ln(\cosh(x)) \cosh(x) + x \sinh(x)$$

3/ General solution

$$y = C_1 \cosh(x) + C_2 \sinh(x) + \ln(\cosh(x)) \cosh(x) + x \sinh(x)$$

4/ Initial conditions

$$0 = y(0) = C_1 + 0 + 0 + 0, \quad \text{thus } C_1 = 0$$

$$1 = y'(0) = 0 + C_2 + 0 + 0, \quad C_2 = 1$$

5/ Solution of the IVP

$$y = \underline{\underline{(x+1) \sinh(x) + \ln(\cosh(x)) \cosh(x)}}$$