Math 198, Section 2, Test 3

April 11, 2008

Name:

Pledged

Honor code: I have neither given nor received help on this test.

Note: the total number of points is 110 — anything above 100 is extra credit.

1. **(18pts) Find the singular point(s) of the differential equation**

$$(x+2)^2\, x^2\, y'' + x\,(x+12)\, y' + \frac{4}{(x-1)^3}\, y = 0$$

Determine the indicial roots corresponding to the regular singular point(s).

In standard form : $y'' + \frac{x+12}{(x+2)^2 x} y' + \frac{4}{(x+2)^2 x^2 (x-1)^3} y = 0$,

we see that $\underline{-2, 0, 1}$ are singular points

and that $\underline{0}$ is the only regular singular point.

From the equation $x^2 y'' + \frac{x+12}{(x+2)^2} x y' + \frac{4}{(x+2)^2 (x-1)^3} y = 0$,

we read the indicial equation : $r(r-1) + \frac{0+12}{(0+2)^2} r + \frac{4}{(0+2)^2 (0-1)^3} = 0$

that is

$$r^2 - r + 3r - 1 = 0$$

$$\underline{r^2 + 2r - 1 = 0}$$

The indicial roots are $\underline{-1-\sqrt{2}}$ and $\underline{-1+\sqrt{2}}$

2. (24pts) Calculate the following

(a) $\mathscr{L}^{-1}\Big(\frac{(s+1)^2}{s^3}\Big) = \mathscr{L}^{-1}\left(\frac{s^2+2s+1}{s^3}\right) = \mathscr{L}^{-1}\left(\frac{1}{s}+\frac{2}{s^2}+\frac{1}{s^3}\right)$

$$= 1 + 2t + \frac{1}{2}t^2$$

(b) $\mathscr{L}^{-1}\Big(\frac{1}{s^2+4s+5}\Big) = \mathscr{L}^{-1}\left(\frac{1}{(s+2)^2+1}\right) = \sin t \, e^{-2t}$

(c) $\mathscr{L}(t^2\cos(t)) = \frac{d^2}{ds^2}\left(\mathscr{L}(\cos t)\right) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{1\cdot(s^2+1)-2s\cdot s}{(s^2+1)^2}\right)$

$$= \frac{d}{ds}\left(\frac{-s^2+1}{(s^2+1)^2}\right) = \frac{-2s\cdot(s^2+1)^2 - 2\cdot 2s\cdot(s^2+1)(-s^2+1)}{(s^2+1)^4}$$

$$= \frac{-2s^3-2s+4s^3-4s}{(s^2+1)^3} = \frac{2s^3-6s}{(s^2+1)^3} = \frac{2s(s^2-3)}{(s^2+1)^3}$$

(d) $\mathscr{L}\Big(\int_0^t \sin(\tau)\cos(t-\tau)d\tau\Big) = \mathscr{L}(\sin t * \cos t) = \mathscr{L}(\sin t)\cdot\mathscr{L}(\cos t)$

$$= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$$

3. **(18pts) Solve the initial-value problem**

$$y'' + y = \delta(t - \pi/2) + \delta(t - 3\pi/2), \qquad y(0) = 0,\ y'(0) = 0.$$

Write your answer as a piecewise defined function involving only one basic trigonometric function.

Apply the Laplace transform to get:

$$s^2 Y(s) - 0 - 0 + Y(s) = e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}$$

$$Y(s) = \frac{1}{s^2+1} e^{-\frac{\pi}{2}s} + \frac{1}{s^2+1} e^{-\frac{3\pi}{2}s}$$

$$y(t) = \sin\left(t - \frac{\pi}{2}\right) \mathcal{U}\left(t - \frac{\pi}{2}\right) + \sin\left(t - \frac{3\pi}{2}\right) \mathcal{U}\left(t - \frac{3\pi}{2}\right)$$

$$= -\cos t\ \mathcal{U}\left(t - \frac{\pi}{2}\right) + \cos t\ \mathcal{U}\left(t - \frac{3\pi}{2}\right)$$

$$\underline{y(t) = \cos t\left(\mathcal{U}\left(t - \frac{3\pi}{2}\right) - \mathcal{U}\left(t - \frac{\pi}{2}\right)\right)}$$

$$y(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ -\cos t, & \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ 0, & \frac{3\pi}{2} \le t \end{cases}$$

4. **(20pts) Give the first four nonzero terms of the power series centered at 0 that solves the initial-value problem**

$$(x^2+1)\,y'' + 2x\,y' = 0, \quad y(0) = 0,\ y'(0) = 1.$$

Write $y = \sum_{n=0}^{\infty} c_n x^n$. Note that $c_0 = 0$ and $c_1 = 1$.

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

$$0 = x^2 y'' + y'' + 2x y' = \sum_{n=0}^{\infty} c_n n(n-1) x^n + \underbrace{\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}}_{=\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n} + \sum_{n=0}^{\infty} 2 c_n n x^n$$

$$0 = \sum_{n=0}^{\infty} \Big(c_n n(n+1) + c_{n+2}(n+2)(n+1) \Big) x^n$$

We get $c_n n \cancel{(n+1)} + c_{n+2}(n+2)\cancel{(n+1)} = 0$, all $n \geq 0$

$$\underline{c_{n+2} = -\frac{n}{n+2} c_n}$$

In view of $c_0 = 0$, we obtain $c_2 = 0$, $c_4 = 0$, $c_6 = 0, \dots$

Then, with $c_1 = 1$, we get $c_3 = -\frac{1}{3} c_1 = -\frac{1}{3}$, $c_5 = -\frac{3}{5} c_3 = \frac{1}{5}$, $c_7 = -\frac{5}{7} c_5 = -\frac{1}{7}, \dots$

Therefore, the solution is

$$\underline{y = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \dots}$$

5. (30pts) Consider the system of differential equations

$$\frac{dx}{dt} = x + y,$$
$$\frac{dy}{dt} = -x + y.$$

(a) Use the methods of Chapter 8 to find the general solution.

(b) Use the Laplace transform to find the solution that satisfies the initial conditions $x(0) = -2$ and $y(0) = 2$.

(a) With $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, we have $X' = AX$.

Eigenvalues of A: solve $0 = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+1$, $(\lambda-1)^2 = -1$, $\lambda - 1 = \pm i$, $\underline{\lambda = 1 \pm i}$

Eigenvector for $\lambda = 1+i$: solve the system:

$k_1 + k_2 = (1+i)k_1$, $-k_1 + k_2 = (1+i)k_2$, $-ik_1 + k_2 = 0$, $-k_1 - ik_2 = 0$. Take for example $k_1 = 1$, $k_2 = i$,

that is $\underline{K = \begin{pmatrix} 1 \\ i \end{pmatrix}}$. In particular, we have $Re(K) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Im(K) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General solution: $X = c_1 e^t \left[Re(K)\cos t - Im(K)\sin t \right] + c_2 e^t \left[Im(K)\cos t + Re(K)\sin t \right]$

$$X = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

(b) Apply the Laplace transform: $sX(s) + 2 = X(s) + Y(s)$, $sY(s) - 2 = -X(s) + Y(s)$, $(s-1)X(s) - Y(s) = -2$, $X(s) + (s-1)Y(s) = 2$

Eliminate Y: $((s-1)^2+1)X(s) = -2(s-1)+2$, $X(s) = -2\frac{s-1}{(s-1)^2+1} + 2\frac{1}{(s-1)^2+1}$

We get: $\underline{x(t) = -2\cos t\, e^t + 2\sin t\, e^t}$

Likewise: $((s-1)^2+1)Y(s) = 2(s-1)+2$, $Y(s) = 2\frac{s-1}{(s-1)^2+1} + 2\frac{1}{(s-1)^2+1}$,

and finally $\underline{y(t) = 2\cos t\, e^t + 2\sin t\, e^t}$