Math 198 : Practice Sheet for Test 2

Pointers to the solutions for the practice test

1. Homogeneous equation $\Rightarrow$ $y_c = C_1 \exp(x)\cos(x) + C_2 \exp(x)\sin(x)$

Particular solution — via undetermined coefficients:

$$y_p = \exp(2x)\left(A\cos(x) + B\sin(x)\right) \quad , \text{ we get } \quad A = 7/5 \ , \ B = -1/5 .$$

2. $\left.\begin{array}{l} k = \frac{12}{2} = 6 \text{ lb/ft} \\ m = \frac{12}{32} = \frac{3}{8} \text{ slug} \end{array}\right\} \to \omega = \sqrt{\frac{k}{m}} = 4 \ s^{-1}$ $\quad x(0) = 1 \ , \ x'(0) = -4$ $\quad \left\{ \text{or faster: } mg = ks \Rightarrow \frac{k}{m} = \frac{g}{s} = \frac{32}{2} = 16 \right\}$

(a) Equation of motion:

$$x(t) = 1 \times \cos(4t) - 1 \times \sin(4t)$$

(b) Amplitude: $A = \sqrt{1^2 + 1^2} = \sqrt{2}$; Period $T = \frac{2\pi}{4} = \frac{\pi}{2}\ s$; Frequency $f = \frac{1}{T} = \frac{2}{\pi} s^{-1}$

(c) $x(t) = \sqrt{2}\cos\left(4t + \frac{\pi}{4}\right)$, smallest $t^* > 0$ for which $x(t^*) = 1$:

$$\cos\left(4t^* + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad , \quad 4t^* + \frac{\pi}{4} = 7\frac{\pi}{4} \quad , \quad t^* = \frac{3\pi}{8}\ s$$

(d) $x'(t) = -4\sqrt{2}\sin\left(4t + \frac{\pi}{4}\right) = 0 \iff 4t + \frac{\pi}{4} = k\pi \iff t = -\frac{\pi}{16} + k\frac{\pi}{4}$

(e) $x'\left(\frac{3\pi}{16}\right) = -4\sqrt{2}\sin(\ \pi\ ) = 0$

3. $$y(x) = \cosh(\sqrt{3}x) + \frac{5}{\sqrt{3}}\sinh(\sqrt{3}x) = \left(\frac{1}{2} + \frac{5}{2\sqrt{3}}\right)e^{\sqrt{3}x} + \left(\frac{1}{2} - \frac{5}{2\sqrt{3}}\right)e^{-\sqrt{3}x}$$

4. (a) Fundamental set of solutions: $\left(x\cos(\ln x), \underbrace{x\sin(\ln x)}_{=y_1}\right)$

(b) Set $\left.\begin{array}{l} y = u y_1 \\ y' = u y_1' + u' y_1 \\ y'' = u y_1'' + 2u' y_1' + u'' y_1 \end{array}\right\}$ $\underbrace{x^2 y'' - x y' + 2y}_{=0} = u(\underbrace{x^2 y_1'' - x y_1' + 2y_1}_{\to 0}) + u'(x^2 2y_1' - x y_1) + u''(x^2 y_1)$

With $y_1 = x\sin(\ln x)$, $y_1' = \sin(\ln x) + \cos(\ln x)$, we get

$$0 = u'\left(2x^2\sin(\ln x) + 2x^2\cos(\ln x) - x^2\sin(\ln x)\right) + u''\left(x^3\sin(\ln x)\right)$$

$$0 = u'' + \left(\frac{1}{x} + \frac{2}{x}\frac{\cos(\ln x)}{\sin(\ln x)}\right)u'$$

solution: $u' = \exp\left(-\int\left(\frac{1}{x} + \frac{2}{x}\frac{\cos(\ln x)}{\sin(\ln x)}\right)dx\right) = \exp\left(-\left(\ln x + 2\ln|\sin(\ln x)|\right)\right) = \frac{1}{x} * \frac{1}{\sin^2(\ln x)}$

$u = -\frac{\cos(\ln x)}{\sin(\ln x)}$, and $y = -\frac{\cos(\ln x)}{\sin(\ln x)} \times x\sin(\ln x) = -x\cos(\ln x)$.

5. Homogeneous equation $\Longrightarrow y_c = c_1 \exp(x) + c_2\, x \exp(x)$

Particular solution — via variation of parameters:

$$y_1 = \exp(x), \quad y_2 = x\exp(x), \qquad W = \begin{vmatrix} \exp(x) & x\exp(x) \\ \exp(x) & (x+1)\exp(x) \end{vmatrix} = \exp(2x)$$

$$W_1 = \begin{vmatrix} 0 & x\exp(x) \\ \exp(x)\,\mathrm{Arctan}(x) & \diagup \end{vmatrix} = -x\,\mathrm{Arctan}(x)\exp(2x), \quad u_1' = \frac{W_1}{W} = -x\,\mathrm{Arctan}(x)$$

$$W_2 = \begin{vmatrix} \exp(x) & 0 \\ \exp(x) & \exp(x)\,\mathrm{Arctan}(x) \end{vmatrix} = \mathrm{Arctan}(x)\exp(2x), \quad u_2' = \frac{W_2}{W} = \mathrm{Arctan}(x)$$

$$-u_1 = \int x\,\mathrm{Arctan}(x)\,dx = \frac{x^2}{2}\mathrm{Arctan}(x) - \int \frac{x^2}{2}\frac{1}{1+x^2}\,dx = \frac{x^2}{2}\mathrm{Arctan}(x) - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$u_1 = -\frac{x^2}{2}\mathrm{Arctan}(x) + \frac{x}{2} - \frac{1}{2}\mathrm{Arctan}(x) = -\frac{1}{2}(1+x^2)\,\mathrm{Arctan}(x) + \frac{x}{2}$$

$$u_2 = \int \mathrm{Arctan}(x)\,dx = x\,\mathrm{Arctan}(x) - \int \frac{x}{1+x^2}\,dx = x\,\mathrm{Arctan}(x) - \frac{1}{2}\ln(1+x^2)$$

$$y_p = u_1 y_1 + u_2 y_2 = \cdots$$

6. Auxiliary equation: $0 = m^4 - m^3 - 2m^2 = m^2(m-2)(m+1)$ has solutions: $0, 0, -1, 2$

$$y = c_1 + c_2 x + c_3 \exp(-x) + c_4 \exp(2x)$$

7. $$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0, \quad \text{with } \frac{\beta}{m} = 10\,s^{-1}, \ \frac{k}{m} = 16\,s^{-2}.$$

Auxiliary equation has roots -2 and -8, so $x = c_1 \exp(-2t) + c_2 \exp(-8t)$

(a) $\left.\begin{array}{l} x(0) = 1 \\ x'(0) = 0 \end{array}\right\} \Longrightarrow x(t) = \frac{4}{3}\exp(-2t) - \frac{1}{3}\exp(-8t)$

(b) $\left.\begin{array}{l} x(0) = 1 \\ x'(0) = -12 \end{array}\right\} \Longrightarrow x(t) = -\frac{2}{3}\exp(-2t) + \frac{5}{3}\exp(-8t)$

8. (a) $$y = C_1 \exp(x) + C_2 \exp(-x) + C_3\, x\exp(x) + C_4\, x\exp(-x)$$
$$= c_1 \cosh(x) + c_2 \sinh(x) + c_3\, x\cosh(x) + c_4\, x\sinh(x)$$

(b) $y_p = Ax^2\cosh(x) + Bx^2\sinh(x) \xrightarrow[\text{calculations}]{\text{nasty}} A = 0, \ B = \frac{1}{8}.$

9. Homogeneous equation $\Longrightarrow y_c = c_1 x + c_2 x^2$

Particular solution — via variation of parameters

$$\begin{array}{l} y_1 = x \\ y_2 = x^2 \end{array}, \quad W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & x^2 \exp(x) \end{vmatrix} = x^3 \exp(x), \quad W_1 = \begin{vmatrix} 0 & x^2 \\ x^2 \exp(x) & 2x \end{vmatrix} = -x^4 \exp(x)$$

$$u_2' = \frac{W_2}{W} = x \exp(x), \quad u_2 = \int x \exp(x) \overset{dx}{=} x \exp(x) - \int \exp(x)\, dx = (x-1)\exp(x)$$

$$u_1' = \frac{W_1}{W} = -x^2 \exp(x), \quad u_1 = -\int x^2 \exp(x)\, dx = -x^2 \exp(x) + 2 \underbrace{\int x \exp(x)\, dx}_{(x-1)\exp(x)} = (-x^2 + 2x - 2)\exp(x)$$

$$y_p = u_1 y_1 + u_2 y_2 = \cdots = (x^2 - 2x)\exp(x) \quad ; \quad y = y_c + y_p .$$

10. Homogeneous equation $\Longrightarrow \quad y_c = c_1 \exp(x) \cos(2x) + c_2 \exp(x) \sin(2x)$

Particular solution — via undetermined coefficients:

$$y_p = A x \exp(x) \cos(2x) + B x \exp(x) \sin(2x) \Longrightarrow A = 0, \quad B = \tfrac{1}{4}.$$

11. Variation of parameters: $\begin{array}{l} y_1 = \cos(\ln x) \\ y_2 = \sin(\ln x) \end{array}$, $\quad W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} = \frac{1}{x}$

$$W_1 = \begin{vmatrix} 0 & \sin(\ln x) \\ \frac{1}{x^2}\sec(\ln x) & \cdots \end{vmatrix} = -\frac{1}{x^2} \frac{\sin(\ln x)}{\cos(\ln x)} ; \quad W_2 = \begin{vmatrix} \cos(\ln x) & 0 \\ \cdots & \frac{1}{x^2}\sec(\ln x) \end{vmatrix} = \frac{1}{x^2} \cancel{\cos^2(\ln x)}$$

$$u_1' = \frac{W_1}{W} = \frac{-\frac{1}{x}\sin(\ln x)}{\cos(\ln x)} \Longrightarrow u_1 = \ln|\cos(\ln x)|$$

$$u_2' = \frac{W_2}{W} = \frac{1}{x} \cancel{\quad} \Longrightarrow u_2 = \cancel{\tan(\ln x)} \; \ln(x)$$

$$y_p = u_1 y_1 + u_2 y_2 \qquad\qquad y_p(1) = 0$$

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{=0} + u_1 y_1' + u_2 y_2' \qquad\qquad y_p'(1) = 0$$

$y = c_1 y_1 + c_2 y_2 + y_p$ satisfies $\begin{array}{l} y(1) = 1 \\ y'(1) = 1 \end{array}$, so that

$$\begin{array}{l} c_1 \times 1 + c_2 \times 0 + 0 = 1 \\ c_1 \times 0 + c_2 \times 1 + 0 = 1 \end{array}, \text{ i.e. } c_1 = c_2 = 1$$

$$y = \big(1 + \ln|\cos(\ln x)|\big) \cos(\ln x) + \big(1 + \ln(x)\big) \sin(\ln x)$$